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Anisotropic fluid distributions on pseudo-spheroidal spacetimes

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Abstract. An exact solution of Einstein's field equations for anisotropic fluid distribution on the background of a pseudo-spheroidal spacetime has been reported. The models based on this solution are found to accommodate density variation of high degree from the centre to the boundary of the distribution and admit a subclass for which both the radial and tangential pressures vanish at the boundary of the configuration.

Keywords. General relativity; anisotropic fluid sphere.

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1. Introduction

Theoretical investigations of Ruderman [1] and Canuto [2] suggest that matter distributions in superdense state with densities exceeding their values corresponding to nuclear matter regime are likely to develop anisotropy in pressure. Subsequently associating a spherical distribution of matter in the form of perfect fluid with interiors of stellar configurations is an idealization far from reality especially when situations involving high densities of matter are involved. The pressure anisotropy usually may arise due to various causes such as – the existence of a solid core, the presence of type-P super fluid, the complexity of interactions or the existence of certain external fields. These observations provide motivation for studying models of spherical distributions of matter with radial pressure differing from tangential pressure and of the superdense star models based on them.

The studies of Bowers and Liang [3] showed that anisotropy may have non-negligible effect on maximum equilibrium mass and surface red shift of the distribution. Bayin [4] studied solutions for stationary as well as for slowly rotating anisotropic fluid spheres. Maharaj and Maartens [5] have solved Einstein's field equations for anisotropic distributions with uniform density. Interior solutions of Einstein's field equations for anisotropic spheres with variable energy density have been reported by Gokhroo and Mehra [6]. Patel and Mehta [7] have discussed anisotropic distributions on a pseudo-spheroidal spacetime.

In this paper we have considered distributions of matter with pressure anisotropy on the background of pseudospheroidal spacetimes. In §2, we have set up the field equations governing anisotropic matter distribution and solved them in §3, to obtain a two-parameter family of solutions. A specific solution of this class, obtained for a particular choice of the parameter K, is discussed in §4, and its physical viability is also critically examined.

2. The field equations

We shall assume that the spacetime of a static spherical distribution of anisotropic matter inside a stellar configuration is described by the spacetime metric

$$\mathrm{d}s^2 = e^{\nu(r)}\mathrm{d}t^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{1}$$

whose associated physical 3-space has the geometry of a 3-pseudospheroid immersed in a 4-dimensional Euclidean space. The geometrical aspects of this metric and its suitability to describe superdense matter distributions has already been investigated by us [8].

The metric potentials of the spacetime and the physical variables of the distribution are related through Einstein's field equations:

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij}. \tag{2}$$

Following Maharaj and Maartens [5], we adopt the expression

$$T_{ij} = (\rho + P)u_i u_j - Pg_{ij} + \pi_{ij}, \tag{3}$$

for the energy-momentum tensor for the anisotropic fluid distribution in equilibrium where u_i denotes unit four velocity field of matter and ρ, P respectively denote the energy density and isotropic pressure. The anisotropic stress tensor π_{ij} is given by the expression:

$$\pi^{ij} = \sqrt{3}S\left[C^iC^j - \frac{1}{3}(u^iu^j - g^{ij})\right]. \tag{4}$$

For radially symmetric anisotropic distributions of matter, $S = S(r)$ denotes the magnitude of the anisotropic stress tensor and $C^i = (0, e^{-\lambda/2}, 0, 0)$ being a radial vector. The energy-momentum tensor of (3) has the non-vanishing components:

$$T_0^0 = \rho,\ T_1^1 = -\left(P + \frac{2S}{\sqrt{3}}\right),\ T_2^2 = T_3^3 = -\left(P - \frac{S}{\sqrt{3}}\right). \tag{5}$$

The pressure along the radial direction

$$P_r = -T_1^1 = P + \frac{2S}{\sqrt{3}}, \tag{6a}$$

will be different from the pressure

$$P_{\perp} = -T_2^2 = P - \frac{S}{\sqrt{3}}, \tag{6b}$$

along the tangential direction. The difference between these pressures

$$S = \frac{P_r - P_{\perp}}{\sqrt{3}} \tag{7}$$

measures the anisotropy of the fluid.

The field equation (2) is equivalent to the following set of three equations:

$$8\pi\rho = \frac{3(K-1)}{R^2}\left(1 + \frac{K}{3}\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2} > 0, \tag{8}$$

$$8\pi P_r = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{(K-1)}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1}, \tag{9}$$

$$\begin{aligned} 8\pi\sqrt{3}S = &- \left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right)\left(1 + \frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-1} \\ &- \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right)^{-1} + \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right)^{-2} \\ &+ \frac{K-1}{2R^2}\left(1 + K\frac{r^2}{R^2}\right)^{-2} r\nu'. \end{aligned} \tag{10}$$

Equation (8) provides the law of variation of density of matter from which it follows that the density gradient

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{2K(K-1)r}{8\pi R^4}\left(5 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-3} \tag{11}$$

is negative. Equations (9) and (10) relate the pressure P_r and anisotropy $S(r)$ with the metric potential $\nu(r)$.

3. Solution of field equations

Equations (8), (9) and (10) constitute a set of three equations in four unknown variables ν, ρ, P_r, and S. Specific solutions of this system can be obtained when one more relation connecting them is specified a priori. Maharaj and Maartens solved field equations by assuming uniform density for matter distribution. Gokhroo and Mehra solved them by considering matter distribution with preassigned expression for the density. Patel and Mehta assumed an expression for $S(r)$ to solve the system of equations (8) through (10). We have shown here that the system of equations (8) through (10) admits a class of easily surveyable solutions for a suitable choice of $S(r)$ and investigated the physical plausibility and appropriateness of this new class of solutions to describe interiors of superdense stars.

A noteworthy aspect of this class of solutions is that it includes a particular model with both the radial and tangential pressures vanishing at the boundary of the distribution.

We introduce new variables z and ψ defined by

$$z = \sqrt{1+\frac{r^2}{R^2}}, \quad \psi = \frac{e^{\nu/2}}{(1-K+Kz^2)^{1/4}}, \tag{12}$$

in terms of which (10) assumes the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \left[\frac{2K(2K-1)(1-K+Kz^2)-5K^2z^2}{4(1-K+Kz^2)} + \frac{8\sqrt{3}\pi R^2 S(1-K+Kz^2)}{z^2-1}\right]\psi = 0. \tag{13}$$

On prescribing

$$8\pi\sqrt{3}S = -\frac{(z^2-1)[2K(2K-1)(1-K+Kz^2)-5K^2z^2]}{4R^2(1-K+Kz^2)^3}, \tag{14}$$

the second term of (13) vanishes and the resulting equation admits the general solution

$$\psi = Cz + D, \tag{15}$$

with C and D as arbitrary constants of integration leading to

$$e^{\nu/2} = (1-K+Kz^2)^{1/4}(Cz+D), \tag{16}$$

as the solution of (10) with $S(r)$ is given by (14).

Subsequently the metric

$$\mathrm{d}s^2 = \sqrt{1+K\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2 \mathrm{d}t^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{17}$$

describes spacetime of an anisotropic fluid distribution in equilibrium with (8) providing the law of variation of density and radial pressure P_r and the anisotropy $S(r)$ having the explicit expressions

$$8\pi P_r = \frac{C\sqrt{1+\frac{r^2}{R^2}}\left[3+2K\frac{r^2}{R^2}+K(2-K)\frac{r^2}{R^2}\right]+D\left[1+K(2-K)\frac{r^2}{R^2}\right]}{R^2\left(1+K\frac{r^2}{R^2}\right)^2\left[C\sqrt{1+\frac{r^2}{R^2}}+D\right]}, \tag{18}$$

$$8\pi\sqrt{3}S = -\frac{\frac{r^2}{R^2}\left[2K(2K-1)\left(1+K\frac{r^2}{R^2}\right)-5K^2\left(1+\frac{r^2}{R^2}\right)\right]}{4R^2\left(1+K\frac{r^2}{R^2}\right)^3}. \tag{19}$$

The assumption (14) which describes the variation of anisotropy parameter $S(r)$ throughout the fluid distribution and is equivalent to (19) has the following desired features:

1. $S(r)$ vanishes at the centre. This ensures the regularity of the distribution at the centre.
2. $S(r)$ increases with r in the neighbourhood of the centre, reaches a maximum value and then subsequently decreases as r increases.
3. The form of $S(r)$ is suitable to describe a distribution for which both the radial and the transverse pressures vanish at a suitably chosen boundary separating the interior from the empty exterior.

In the absence of any precise information about the growth of anisotropy with r in the interior of a stellar configuration it is pertinent to investigate the implications of such considerations. Since this approach does not assume any equation of state for matter it is necessary to examine the physical plausibility of the solution in the light of the energy conditions. A physically plausible solution is expected to fulfill the following requirements throughout its region of validity.

(i) $$\rho > 0,\ P_r \geq 0,\ P_\perp \geq 0 \tag{20a}$$

(ii) $$\frac{\mathrm{d}\rho}{\mathrm{d}r} < 0,\quad \frac{\mathrm{d}P_r}{\mathrm{d}r} < 0 \tag{20b}$$

(iii) $$\rho - P_r - 2P_\perp \geq 0 \tag{20c}$$

(iv) $$\frac{\mathrm{d}P_r}{\mathrm{d}\rho} \leq 1, \frac{\mathrm{d}P_\perp}{\mathrm{d}\rho} \leq 1 \tag{20d}$$

(v) If the metric (17) is to be suitable to describe spacetime inside an anisotropic fluid sphere, it should continuously match with the Schwarzschild exterior metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2, \tag{21}$$

which uniquely describes the empty spacetime outside the fluid distribution, across its boundary $r = a$ where $P_r(a) = 0$.

The continuity of metric coefficients and the pressure along radial direction across $r = a$ imply the following relations:

$$e^{\nu(a)} = \frac{1 + \frac{a^2}{R^2}}{1 + K\frac{a^2}{R^2}} = 1 - \frac{2m}{a} \tag{22}$$

$P_r(a) = 0$. i.e.

$$C\sqrt{1 + \frac{a^2}{R^2}}\left[3 + 2K\frac{a^2}{R^2} + K(2-K)\frac{a^2}{R^2}\right] + D\left[1 + K(2-K)\frac{a^2}{R^2}\right] = 0 \tag{23}$$

Equations (22) and (23) determine the constants m, C and D as

$$m = \frac{(K-1)a^3}{2R^2\left(1+K\frac{a^2}{R^2}\right)}, \tag{24}$$

$$C = -\frac{1+K(2-K)\frac{a^2}{R^2}}{2\left(1+K\frac{a^2}{R^2}\right)^{7/4}}, \tag{25}$$

$$D = \frac{\sqrt{1+\frac{a^2}{R^2}}\left[3+K(4-K)\frac{a^2}{R^2}\right]}{2\left(1+K\frac{a^2}{R^2}\right)^{7/4}}. \tag{26}$$

4. Particular case $K = 2$

We shall now discuss in detail the anisotropic fluid sphere model based on the above class of solutions for the particular choice $K = 2$. When $K = 2$ the expressions for radial and tangential pressures become relatively simple rendering the corresponding model as easily surveyable. Further this case provides anisotropic counterparts of our perfect fluid solutions on the background of pseudospheroidal spacetimes [8]. When $K = 2$, $\rho, P_r, S, P_\perp, C$ and D have the following explicit expressions:

$$8\pi\rho = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}, \tag{27}$$

$$8\pi P_r = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\right]}, \tag{28}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}, \tag{29}$$

$$8\pi P_\perp = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\right]} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}, \tag{30}$$

$$C = -\frac{1}{2\left(1+2\frac{a^2}{R^2}\right)^{7/4}}, \tag{31}$$

$$D = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)}{2\left(1+2\frac{a^2}{R^2}\right)^{7/4}}. \tag{32}$$

It is evident from (28) that P_r is positive throughout the distribution. The positivity of $P_\perp$ requires that $a \geq \sqrt{2}R$. After a lengthy but straightforward computation one finds that $\rho - P_r - 2P_\perp > 0$ throughout the distribution.

The casuality condition demands that $\mathrm{d}P_r/\mathrm{d}\rho < 1$ and $\mathrm{d}P_\perp/\mathrm{d}\rho < 1$, throughout the distribution. Owing to the complexity of expressions $\mathrm{d}P_r/\mathrm{d}\rho$ and $\mathrm{d}P_\perp/\mathrm{d}\rho$ it is difficult to verify the fulfilment of these conditions throughout the distribution. However we verified these conditions at the centre and on the boundary and found that they are satisfied there.

5. Discussion

The scheme given by Vaidya and Tikekar [9] for estimating the mass and size of fluid spheres on the background of spheroidal spacetimes can be used to determine the mass and size of anisotropic fluid spheres of the models of this class as well. Following this scheme we adopt $\rho(a) = 2 \times 10^{14}$ gm/cm^3 and introduce density variation parameter

$$\frac{\rho(a)}{\rho(0)} = \lambda$$

The condition $P_\perp \geq 0$.

$$\text{i.e. } \frac{a^2}{R^2} = \frac{1 - 6\lambda + \sqrt{24\lambda + 1}}{12\lambda} \geq 2, \tag{33}$$

then restricts λ to comply with the requirement $\lambda \leq 0.093$. Thus the introduction of anisotropy results in a high degree of density variation as one moves from the centre to the boundary. The perfect fluid models with pseudospheroidal geometry for its physical space

Table 1. Masses and radii of superdense star models corresponding to $K = 2$, ρ (a) = 2×10^{14} gm/cm^3 and $\lambda \leq 0.093$.

λ	R (km)	a(km)	m(km)	C	D
0.093	8.670	12.263	2.453	−0.120	2.285
0.090	8.516	12.257	2.469	−0.119	2.359
0.080	8.028	12.233	2.516	−0.110	2.413
0.070	7.510	12.201	2.564	−0.100	2.591
0.060	6.953	12.159	2.613	−0.089	2.755
0.050	6.347	12.107	2.661	−0.078	2.976

do not admit high density variation. We have given in table 1 the results of computation of mass, size and other relevant quantities for the anisotropic fluid models with $K = 2$. These estimates indicate that the models with $\lambda \leq 0.093$ may be suitable to describe the interiors of superdense spherical distributions of matter like neutron stars.

A common feature of the static anisotropic fluid models in literature is that their transverse pressure $P_\perp$ is discontinuous at the boundary. The above class of models contains a subclass with $P_\perp$ continuous across the boundary. For fluid spheres with $K = 2$ and $a = \sqrt{2}R$, both P_r and $P_\perp$ are decreasing functions of r and $P_r \geq P_\perp$ throughout, the equality holds at the centre and at the boundary. A noteworthy feature of this model is that the tangential pressure $P_\perp$ also vanishes at the boundary of the distribution.

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