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Gravitational energy in Brans–Dicke cosmological models

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Abstract. The behaviour of gravitational energy and scalar field during the evolution of the universe within the framework of Brans–Dicke theory has been discussed. With help of the Landau–Lifshitz pseudo-tensor for the flat Friedmann–Robertson–Walker model, it is found that (i) the total energy of the universe is always zero, (ii) the Brans–Dicke scalar field for all $\omega \geq 0$ contributes energy to the negative energy of gravitational field and this gets transferred to the vacuum energy which accelerates the expansion of the universe.

Keywords. Brans–Dicke theory; gravitational energy; scalar field; expanding universe.

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1. Introduction

The fundamental conservation laws in physics are related to the universal invariance properties of physical laws. The problem of energy conservation has plagued the theory of general relativity since its inception. Passing from Minkowskian flat spacetime to the pseudo-Riemannian curved spacetime, the vanishing of the ordinary divergence of T^{ik} is replaced, according to the principle of general covariance, by covariant divergencelessness. The source of the problem lies in the fact that unlike ordinary divergencelessness, the vanishing of the covariant divergence does not in general constitute a continuity equation and consequently is not a manifestation of a conservation law in the usual sense. Conservation laws in general relativity were first formulated by Einstein [1]. Landau and Lifshitz [2] postulated a symmetric pseudo-tensor which has preference over those of Einstein [1] and Mϕller [3] as it can give angular momentum in asymptotically flat spacetimes and obeys a set of equations of continuity. Chandrasekhar [4] has shown that the conservation laws of general relativity, expressed in terms of the Landau–Lifshitz [2] symmetric energy-momentum pseudo-tensor, can be used to determine the various conserved quantities in the different post-Newtonian approximations. A number of authors proposed energy-momentum conservation in general relativistic system (refer [5] and references therein).

The general energy-momentum conservation is required to hold in all coordinate systems. For mathematical reasons, a symmetric tensor is precluded from satisfying a global ordinary divergencelessness condition valid in all coordinate systems. Landau–Lifshitz pseudo-tensor which is symmetric has the property that by adding it to the matter energy-momentum tensor density one obtains a physical quantity which is conserved in any coordinate system. Recently Cooperstock [6] and Rosen [7], using Einstein's canonical energy-momentum complex to represent gravitational energy and matter energy, have shown that the total energy of the closed Friedmann universe is zero. Johri *et al* [8] have obtained similar results for Friedmann flat and closed universe using Landau–Lifschitz energy-momentum complex. They have also shown that during inflation the vacuum energy driving the accelerated expansion, and ultimately responsible for the creation of matter (radiation) in the universe have been drawn from the energy of the gravitational field.

In view of the interest of authors for more than two decades in scalar-tensor theories, especially the Brans–Dicke scalar-tensor theory which is a very useful tool in understanding inflationary models of universe [9–10] it is worthwhile to present the counterpart of results of [8] in Brans–Dicke theory for studying effects of scalar field.

2. The equations

In the absence of gravitational field the law of conservation of energy-momentum of the material (and electromagnetic) field is given by

$$\frac{\partial T^{ij}}{\partial x^j} = 0, \tag{1}$$

whereas the general relativistic generalization of this equation in the presence of gravitational field is expressed by

$$T^j_{i;j} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^j}[\sqrt{-g}T^j_i] - \frac{1}{2}T^{jl}\frac{\partial g_{jl}}{\partial x^i} = 0. \tag{2}$$

However (2) does not generally express conservation law because the four-momentum

$$P^i = \int \sqrt{-g}T^{ij}\mathrm{d}S_j \tag{3}$$

is conserved only when

$$\frac{\partial\sqrt{-g}T^{ij}}{\partial x^j} = 0. \tag{4}$$

For a general relativistic conservation law we shall consider a geodesic coordinate system in which at some particular point all the first derivatives of the metric tensor g_{ij} vanish [2]. In this coordinate system (2) reduces to (4). In geodesic coordinate system Einstein tensor G^{ij} takes the form

$$G^{ij} = \frac{1}{2}\frac{\partial}{\partial x^l}\left[\frac{1}{-g}\frac{\partial}{\partial x^m}\left((-g)(g^{ij}g^{lm} - g^{il}g^{jm}\right)\right]. \tag{5}$$

The gravitational field equations with usual notations in Brans–Dicke theory may be written in the form

$$G^{ij} = \frac{8\pi}{\phi}(T^{ij} + B^{ij}), \tag{6}$$

$$\Box\phi = \phi^{;i}_{;i} = \sqrt{-g}[\sqrt{-g}\phi^{,i}]_{,i} = \frac{8\phi}{(3+2\omega)}T. \tag{7}$$

Here B^{ij} is the energy-momentum tensor of the scalar field ϕ and defined as

$$B^{ij} = \frac{\omega}{8\pi\phi}\left(\phi^{,i}\phi^{,j} - \frac{1}{2}g^{ij}\phi_{,m}\phi^{,m}\right) + \frac{1}{8\phi}\left(\phi^{i;j} - g^{ij}\Box\phi\right). \tag{8}$$

Now, by using (5), in geodesic coordinate system, we can rewrite the Brans–Dicke field equation (6) as

$$(-g)\left(\frac{T^{\,ij} + b^{ij}}{\phi}\right) = \frac{\partial h^{ijl}}{\partial x^l}, \tag{9}$$

where b^{ij} is the component of B^{ij} without the first derivative of the metric tensor, g_{ij} and h^{ijl} have the definition

$$h^{ijl} = \frac{1}{16\pi}\frac{\partial}{\partial x^m}\left[(-g)(g^{ij}g^{lm} - g^{il}g^{jm})\right]. \tag{10}$$

Moving from geodesic coordinate system to an arbitrate coordinate system, effect of the gravitational field appears, symbolized by t^{ij}(symmetric) and the term b^{ij} will be replaced by its original counterpart B^{ij}. Equation (9) generalizes to

$$(-g)\left(\frac{T^{\,ij} + B^{ij} + t^{ij}}{\phi}\right) = \frac{\partial h^{ijl}}{\partial x^l} \tag{11}$$

Furthermore from (11), it follows that

$$\frac{\partial}{\partial x^j}\left(\frac{(-g)(T^{\,ij} + B^{ij} + t^{ij})}{\phi}\right) = 0. \tag{12}$$

This suggests that the four-momentum of the physical system defined by

$$P^{\,i} = \int (-g)\frac{T^{\,ij} + B^{ij} + t^{ij}}{\phi}\mathrm{d}S_j \tag{13}$$

provides a conservation law in Brans–Dicke theory.

If we consider $x^0 =$ const. hypersurface, then (13) reduces to

$$P^{\,i} = \int (-g)\frac{T^{i0} + B^{i0} + t^{i0}}{\phi}\mathrm{d}V = \int \frac{\partial h^{i0l}}{\partial x^l}\mathrm{d}V. \tag{14}$$

3. Evaluation of total energy of the universe

In this section we evaluate the total energy of the universe represented by flat Friedmann–Robertson–Walker (FRW) metric

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2), \tag{15}$$

containing a perfect cosmological fluid with energy-momentum tensor

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{16}$$

and equation of state

$$p = \gamma\rho; \qquad 0 \leq \gamma \leq 1. \tag{17}$$

Using (10) after a straightforward calculation for metric (15), we obtain

$$\frac{\partial h^{i0l}}{\partial x^l} = 0. \tag{18}$$

From (11), (14) and (18), we get

$$P^i = 0. \tag{19}$$

Equations (11) and (18) yield

$$T^{00} + B^{00} + t^{00} = 0, \tag{20}$$

or

$$\rho_{\text{mat}} + \rho_\phi + \rho_{\text{grav.}} = 0. \tag{21}$$

With the help of (8), one can obtain the value of ρ_ϕ for the metric (15) as

$$\rho_\phi = B^{00} = \frac{\omega}{16\pi}\left(\frac{\dot{\phi}^2}{\phi} - \frac{6\dot{\phi}}{\omega}\frac{\dot{R}}{R}\right). \tag{22}$$

By a simple, but lengthy calculation one can find the expressions in terms of time coordinate for scale factor $R(t)$, scalar field ϕ, energy densities of material, scalar and gravitational fields as given in table 1.

4. Concluding remarks

It is well-known that the energy of the gravitational field cannot be localised, as such the pseudo-tensor representation of the gravitational energy at any point is not covariant for general coordinate transformation; however in certain situations, the use of pseudo-tensors for gravitational energy leads to definite and meaningful results. For instance (a) Landau and Lifshitz [2] have shown that their pseudo-tensor t^{ij} may be expressed in terms of Christoffel symbols which behave like tensors with respect to linear transformation of coordinates; in this sense t^{ij} has tensorial character so far as linear transformations are concerned. This applies to formulation of gravitational energy in the Friedmann models in co-moving coordinates connected by linear transformations. (b) It is noteworthy that (12) is identically satisfied in any arbitrary coordinate system; consequently, there is a conservation law for the quantities P^i in (14). This implies that the 4-momentum of the

γ	$R(t)$	ϕ	$\rho_{\text{mat.}}$	ρ_ϕ	$\rho_{\text{grav.}}$
-1	$t^{\frac{2\omega+1}{2}}$	$\frac{32\pi\rho_{\text{vac.}}}{(3+2\omega)(5+6\omega)}t^2$	$\rho_{\text{vac.}}$	$-\frac{4(4\omega+3)}{(3+2\omega)(5+6\omega)}\rho_{\text{vac.}}$	$-\frac{3(2\omega+1)^2}{(3+2\omega)(5+6\omega)}\rho_{\text{vac.}}$
0	$t^{\frac{2\omega+2}{3\omega+4}}$	$\frac{4\pi(3\omega+4)}{(3+2\omega)}\rho_0 t^{\frac{2}{3\omega+4}}$	$\rho_0 t^{-\frac{6(\omega+1)}{(3\omega+4)}}$	$-\frac{(5\omega+6)}{(3+2\omega)(4+3\omega)}\rho_0 t^{-\frac{6(\omega+1))}{(3\omega+4)}}$	$-\frac{6(\omega+1)^2}{(3+2\omega)(4+3\omega)}\rho_0 t^{-\frac{6(\omega+1)}{(4+3\omega)}}$
$\frac{1}{3}$	$t^{\frac{1}{2}}$	$12\pi\rho_0$	$\rho_0 t^{-2}$	0	$-\rho_0 t^{-2}$
1	$t^{\frac{1}{2}}$	$-\frac{36\pi\rho_0}{(3+2\omega)}t^{-1}$	$\rho_0 t^{-3}$	$-\frac{9\omega}{(3+2\omega)}\rho_0 t^{-3}$	$-\frac{(3-7\omega)}{(3+2\omega)}\rho_0 t^{-3}$

gravitational field, BD scalar field and the material field taken together, has got a definite meaning and is independent of the frame of reference, inspite of the fact that the pseudo-tensor t^{ij} is involved in (14). In the case of Friedmann models, the right hand side of (14), corresponding to P^0, vanishes (please refer [8]). This leads to the important conclusion that the total energy (material + scalar + gravitational) in any co-moving volume of the universe is identically zero.

It may be clarified that we do not seek to evaluate the total gravitational energy of the Friedmann models for which the asymptotically flat spacetime condition is required. Our aim is simply to co-relate the material energy density to the gravitational energy-density and the scalar energy-density in the universe. It may be relevant to mention here that the seminal result of cosmology "the total energy of the universe is zero" can also be established without using pseudo-tensors as done by Cooperstock [6]; we have deliberately used Landau–Lifshitz pseudo-tensor in our treatment in order to stress the negative character of gravitational energy and its implication for the transfer of energy from the gravitational field to matter field during inflation.

In the Brans–Dicke theory, unlike general relativity, energy of scalar and gravitational fields both balance the energy of material field. From the table we can see that for $\rho_{\text{vac.}} > 0$ the energy density of scalar field (ρ_ϕ) is always negative for all $\omega > 0$. Here the only difference from general relativistic case is that ρ_ϕ is also contributing energy for inflation. The energy of the inflationary universe is actually not only drawn from the negative energy of the gravitational field but also drawn from negative energy of the scalar field.

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References

[1] A Einstein, *Berlin Ber*, **42**, 111 (1916)
[2] L Landau and E Lifshitz, *The classical theory of fields*, Vol. 2 (Addision–Wesley Press, Cambridge) (1951)
[3] C Mϕller, *Ann. Phys.*, **4**, 347 (1958)
[4] S Chandrasekhar, *Astrophys. J.*, **158**, 45 (1969)
[5] N Nissani and E Leibowitz, *Int. J. Theor. Phys.*, **28**, 235 (1989)
[6] F I Cooperstock, *Gen. Relativ. Gravit.*, **26**, 323 (1994)
[7] N Rosen, *Gen. Relativ. Gravit.*, **26**, 319 (1994)
[8] V B Johri, D Kalligas, G P Singh and C W F Everitt, *Gen. Relativ. Gravit.*, **27**, 313 (1995)
[9] C Mathiazhagan and V B Johri, *Class. Quant. Gravit.* **1**, L29 (1984)
[10] D La and P J Steinhardt, *Phys. Rev. Lett.* **62**, 376 (1989)