PRAMANA
—journal of
physics

Vol. 52, No. 1
January 1999
pp. 39–47

Modified Kadomtsev–Petviashvili equation in cold collisionless plasma

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MS received 15 June 1998; revised 31 October 1998

Abstract. Nonlinear electromagnetic wave propagation through cold collisionless plasma in (2+1) dimensions is studied using the nonlinear reductive perturbation method. It is shown that to the lowest order of perturbation, the system of equations can be reduced to modified Kadomtsev–Petviashvili equation.

Keywords. Nonlinear waves; plasma dynamics; solitons; Kadomtsev–Petviashvili equation.

PACS Nos 51.50; 51.70; 52; 03.40; 03.50

1. Introduction

An exciting and extremely active area of research investigation during the past twenty years has been the study of solitons and the related issues of the construction of solutions to a wide class of nonlinear equations. The concept of solitons has now become ubiquitous in modern nonlinear science and indeed can be found in various branches of physics. Exciting and important discoveries have been made in the nonlinear dynamics of dissipative and conservative systems. There are different methods to study nonlinear systems.

The reductive perturbation method for the propagation of a slow modulation of a quasi-monochromatic whistler wave in a cold plasma was first established by Taniuti and Washimi [1]. This method was generalized to a wide class of nonlinear wave systems by Taniuti and Yajima [2]. Kakutani and Ono [3], Kawahara and Taniuti [4] and Taniuti and Wei [5] have investigated the propagation of hydromagnetic waves through a cold collision free plasma. The reductive perturbation method has been recently applied to various other problems [6–8].

In the present work, we have studied the propagation of electromagnetic waves through a cold collisionless plasma by using the nonlinear reductive perturbation method in (2+1) dimensions. It is found that to the lowest order of perturbation, the system of equations can be reduced to modified Kadomtsev–Petviashvilli equation (mKP) [9]. When this problem is studied in (1+1) dimensions, one can arrive at modified Korteweg–deVries (mKdV) equation [10–13].

2. Mathematical formulation of the problem

When electromagnetic waves pass through a medium the system gets perturbed. Since electrons are lighter than ions, electrons respond much more rapidly to the fields and ion motion can be neglected. In the equation for momentum of a cold plasma, pressure term is absent. Basic equations relevant to the present problem are the following

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{e}{m}\left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right], \tag{2}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j}, \tag{3}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{4}$$

$$\nabla \cdot \vec{E} = \frac{4\pi e}{m}(\rho - \rho_0). \tag{5}$$

For convenience we introduce a displacement vector field $\vec{S}$ [14], which describes the direction and distance that the plasma has moved away from the equilibrium as:

$$\vec{v} = \frac{\partial \vec{S}}{\partial t} + (\vec{v} \cdot \nabla)\vec{S}. \tag{6}$$

Now substituting (6) in (2) and(3) and simplifying we can write (2)and(3) in the form of two coupled partial differential equations

$$\frac{\partial^2 \vec{S}}{\partial t^2} + \frac{\partial}{\partial t}(\vec{v} \cdot \nabla)\vec{S} + \left[\left[\frac{\partial \vec{S}}{\partial t} + [(\vec{v} \cdot \nabla)\vec{S}] \cdot \nabla\right] \vec{S}\right] \vec{v}$$
$$= -\frac{e}{m}\left[\vec{E} + \left(\frac{\partial}{\partial t}\vec{S}\right) \times \vec{B} + (\vec{v} \cdot \nabla)\vec{S} \times \vec{B}\right] \tag{7}$$

$$c^2 \nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\partial^2 \vec{S}}{\partial t^2} + \frac{\partial}{\partial t}[(\vec{v} \cdot \nabla)\vec{S}]. \tag{8}$$

The electromagnetic waves are transverse in nature. Assuming that $\vec{B}$ is a linear function of $\vec{S}$ so that we can write

$$\left(\frac{\partial}{\partial t}\vec{S}\right) \times \vec{B} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{S} \times \vec{B}) \tag{9}$$

$$= \frac{1}{2}\frac{\partial \vec{S}}{\partial t} \times \vec{B} + \frac{1}{2}\frac{\partial B}{\partial t} \times \vec{S}. \tag{10}$$

Substituting (10) and (4) in (7) we can write

$$\frac{\partial^2 \vec{S}}{\partial t^2} + \frac{\partial}{\partial t}(\vec{v}\cdot\nabla)\vec{S} + \left[\left[\frac{\partial \vec{S}}{\partial t} + [(\vec{v}\cdot\nabla)\vec{S}]\cdot\nabla\right]\vec{S}\right]\vec{v}$$

$$= -\frac{e}{m}\left[\vec{E} + \frac{1}{2}\frac{\partial \vec{S}}{\partial t}\times\vec{B} - \frac{1}{2}\nabla\times\vec{E}\times\vec{S} + (\vec{v}\cdot\nabla)\vec{S}\times\vec{E}\right]. \quad (11)$$

Equations (8) and (11) are a system of complicated nonlinear partial differential equations and from these equations we can study the type of fluctuations that arise when an electromagnetic wave passes through the plasma. Here we consider an equilibrium state defined by $\rho = \rho_0$, $v = 0$, $E = 0$ where the electron density ρ, the electron velocity v and the ion density ρ_0 are considered as constants. For convenience we put $e = m = 1$. Let us now consider one dimensional plane wave propagating along the x-direction. All the physical quantities are assumed to be functions of the space coordinates x and y and time t. We will now introduce the stretching variables ξ, ζ and τ as

$$\xi = \varepsilon(x - Vt), \quad (12)$$

$$\zeta = \varepsilon^2 y, \quad (13)$$

$$\tau = \varepsilon^3 t. \quad (14)$$

The expression for ζ represents weak dependence of the field parameters on the coordinate y. E and S satisfy the following boundary conditions,

$$E_x^i \to 0, \quad \text{except that} \quad E_x^0 \to E_0 \cos\phi \quad (15)$$

$$E_y^i \to 0, \quad \text{except that} \quad E_y^0 \to E_0 \sin\phi \quad (16)$$

$$E_z^i \to 0 \quad (17)$$

$$\text{as } \xi \to -\infty, \; i = 0, 1, 2, 3, \ldots \quad (18)$$

$$S_x^i \to 0, \quad \text{except that} \quad S_x^0 \to S_0 \cos\phi \quad (19)$$

$$S_y^i \to 0, \quad \text{except that} \quad S_y^0 \to S_0 \sin\phi \quad (20)$$

$$S_z^i \to 0 \quad (21)$$

$$\text{as } \xi \to -\infty, \; i = 0, 1, 2, 3 \ldots \quad (22)$$

$B_y = 0$ as $\xi \to \infty$, where S_0, E_0 and ϕ are positive constants.

For an appropriate choice of the coordinate system we can write $S_0 = (S_{0,x}, S_{0,y}, 0)$. It may be rather difficult to solve the coupled equations with general geometrical considerations. Therefore we restrict ourselves to the case in which we are considering the parallel propagation of a wave along the magnetic field and in this case we choose $B_z = 0, B_x =$ constant. Let us seek a solution of (8) and (11) under the form of a Fourier expansion in harmonics of the fundamental $E^n \approx \exp\, in(kx - \omega t)$

$$E = \sum_{n=-\infty}^{+\infty} \vec{E}^n E^n, \quad (23)$$

$$S = \sum_{n=-\infty}^{+\infty} \vec{S}^n E^n. \tag{24}$$

Expressing the Fourier components of E and S in powers of a small parameter ε, we can write:

$$S^n = \sum_{j=0}^{\infty} \varepsilon^j S_J^n(x,t), \quad E^n = \sum_{j=0}^{\infty} \varepsilon^j E_j^n(x,t). \tag{25}$$

Here we are considering the second order approximation of the field variables, that is, we are neglecting terms which contains ε^3 and higher powers of ε. Applying the boundary conditions given by (14–22) and the expansions given by (23–24), (11) and (8) can be cast in the following forms:

$$\left[\frac{\partial^2}{\partial t^2} - 2in\omega\frac{\partial}{\partial t} - n^2\omega^2\right] S_0^n = \frac{\partial}{\partial t} - ink\left[\sum_{p+q=n} S^p E^q\right]\frac{1}{2} + E_0^n, \tag{26}$$

$$\left[\frac{\partial^2}{\partial t^2} + 2in\omega\frac{\partial}{\partial t} - n^2\omega^2\right][E_j^n + S_j^n] = c^2\left[\frac{\partial^2}{\partial x^2} + 2ink\frac{\partial}{\partial x} - n^2k^2\right] \times E_j^n(1-\delta_j, x). \tag{27}$$

Taking the leading order terms for the order $(1,n)$ from both equations, we can write the components of $S_{1,p}^n$ as functions of $E_{1,p}^n$, we find a linear homogeneous system for $E_{1,x}^n, E_{1,y}^n$ and $E_{1,z}^n$. The determinant of this system, $\Delta(n)$ is

$$\begin{aligned}\Delta(n) = n^2\omega^2 & \left[-n^2\gamma^2\omega^4 + \beta^2(k^2/4)S_x^2 + \gamma\beta n^2(K^2/2)(1+\alpha)S_y^2 \right.\\ & \left. + \gamma(k^2/2)n^2\beta(1+\alpha)S_z^2\right],\end{aligned} \tag{28}$$

where

$$\beta = (1+\alpha\gamma), \tag{29}$$

$$\gamma = \left(1 - \frac{k^2}{\omega^2}\right), \tag{30}$$

$$\alpha = \frac{E_0}{S_0}. \tag{31}$$

For $n = 1, \Delta(1)$ is zero if ω satisfies the dispersion relation

$$-\gamma^2\omega^4 + (\beta^2k^2/4)S_x^2 + \gamma\beta(k^2/2)(1+\alpha)S_y^2 + \gamma\beta k^2(1+\alpha)S_z^2 = 0, \tag{32}$$

where $S = (S_x, S_y, S_z)$ and $E = (E_x, E_y, E_z)$. Writing γ and β in terms of α, k, ω, we can obtain $V = \sqrt{\frac{\alpha}{(1+\alpha)}}c$. We assume that $S_0^0 = s$ and $E_0^0 = \alpha s$ are constants and that

$$S_0^n = E_0^n = 0 \quad \text{for} \quad n \neq 0. \tag{33}$$

The assumed conditions at infinity are, $E_j^n, S_j^n \to 0$ for $j = 0$ for all 'n' except for $(j,|n|) = (1,1)$, where the limit is assumed to be a finite constant. For $n = 1, \Delta(1) = 0$ for $j = 1$. Under this condition the system has a nontrivial solution. But for $n = 2,3,4\ldots$

$\Delta n \neq 0$, we have the trivial solution. That is for $j = 1$ and $n = 1$, we get $E_1^n = S_1^n = 0$. For $n = 0, \Delta(0) = 0$, we can choose $E_1^0 = S_1^0 = 0$. This completes the solution at order $(1, n)$.

For the next order, we can proceed in the same manner. The system will have a solution only if the determinant of the augmented matrix is zero.

Now expanding the dependent variables as,

$$S_x = S_0 + \varepsilon^1 S_x^1 + \varepsilon^2 S_x^2 + \dots \tag{34}$$
$$S_y = S_y^0 + \varepsilon^1 S_y^1 + \varepsilon^2 S_y^2 + \dots \tag{35}$$
$$S_z = S_z^0 + \varepsilon^1 S_z^1 + \varepsilon^2 S_z^2 + \dots \tag{36}$$
$$E_x = E_x^0 + \varepsilon^1 E_x^1 + \varepsilon^2 E_x^2 + \dots \tag{37}$$
$$E_y = E_y^0 + \varepsilon^1 E_y^1 + \varepsilon^2 E_y^2 + \dots \tag{38}$$
$$E_z = E_z^0 + \varepsilon^1 E_z^1 + \varepsilon^2 E_z^2 + \dots \tag{39}$$

Substituting these expansions in (8) and (11) and then collecting and solving coefficients of different orders of ϵ^j for $n = 1$. We get

at order ε^0

$$S_y^0 E_z^0 - S_z^0 E_y^0 = 0 \tag{40}$$
$$S_z^0 E_x^0 - S_x^0 E_z^0 = 0 \tag{41}$$
$$S_x^0 E_y^0 - S_y^0 E_x^0 = 0 \tag{42}$$
$$\frac{\partial^2 [E_x^0 + S_x^0]}{\partial \xi^2} = 0 \tag{43}$$
$$V^2 \frac{\partial^2}{\partial \xi^2} (\gamma E_y^0 + S_y^0) = 0, \tag{44}$$

at order ε^1

$$(1 + \alpha) S_z^0 S_x^1 = 2V^2 \frac{\partial S_y^0}{\partial \xi} \tag{45}$$
$$(1 + \alpha) S_y^0 S_x^1 = 2V^2 \frac{\partial S_z^0}{\partial \xi} \tag{46}$$
$$(E_x^1 + S_x^1) = 0 \tag{47}$$
$$(E_y^1 - \alpha S_y^1) = 0 \tag{48}$$
$$(E_z^1 - \alpha S_z^1) = 0, \tag{49}$$

at order ε^2

$$V^2 \frac{\partial S_x^1}{\partial \xi} = 1/2 [S_y^0 (E_z^2 - \alpha S_z^2) - S_z^0 (E_y^2 - \alpha S_y^2)] \tag{50}$$
$$V^2 \frac{\partial S_y^1}{\partial \xi} = 1/2 [S_z^0 (E_x^2 - \alpha S_x^2) - S_x^0 (E_z^2 - \alpha S_z^2)] \tag{51}$$

$$V^2 \frac{\partial S_z^1}{\partial \xi} = 1/2[S_x^0(E_y^2 - \alpha S_y^2) - S_y^0(E_x^2 - \alpha S_x^2) + (1+\alpha) S_x^1 S_y^1] \tag{52}$$

$$(E_x^2 + S_x^2) = 0 \tag{53}$$

$$\frac{\partial (E_y^2 - \alpha S_y^2)}{\partial \xi} = -2V \frac{(1+\alpha)^2}{c^2} \frac{\partial S_y^0}{\partial \tau} - (1+\alpha) \frac{\partial^2 E_y^0}{\partial \zeta^2} \tag{54}$$

$$\frac{\partial (E_z^2 - \alpha S_z^2)}{\partial \xi} = -2V \frac{(1+\alpha)^2}{c^2} \frac{\partial S_z^0}{\partial \tau} - (1+\alpha) \frac{\partial^2 E_z^0}{\partial \zeta^2}. \tag{55}$$

Solving for E_y^2, S_y^2 and E_z^2, S_z^2, from (54) and (55) we can get

$$\begin{aligned}(E_y^2 - \alpha S_y^2) = & \int_{-\infty}^{\xi} -2V(1+\alpha)^2/c^2 \frac{\partial S_y^0}{\partial \tau} \mathrm{d}\xi \\ & - \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \alpha(1+\alpha) \frac{\partial^2 S_y^0}{\partial \zeta^2} (\mathrm{d}\xi)^2\end{aligned} \tag{56}$$

$$\begin{aligned}(E_z^2 - \alpha S_z^2) = & \int_{-\infty}^{\xi} -2V(1+\alpha)^2/c^2 \frac{\partial S_z^0}{\partial \tau} \mathrm{d}\xi \\ & - \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \alpha(1+\alpha) \frac{\partial^2 S_z^0}{\partial \zeta^2} (\mathrm{d}\xi)^2\end{aligned} \tag{57}$$

From (50) we know that

$$2V^2 \frac{\partial S_x^1}{\partial \xi} = S_y^0(E_z^2 - \alpha S_z^2) - S_z^0(E_y^2 - \alpha S_y^2). \tag{58}$$

Substituting for

$$(E_y^2 - \alpha S_y^2) \text{ and } (E_z^2 - \alpha S_z^2) \tag{59}$$

in (58) we get

$$\begin{aligned}2V^2 \frac{\partial S_x^1}{\partial \xi} = & S_y^0 \left[\int_{-\infty}^{\xi} \frac{-2V(1+\alpha)^2}{c^2} \frac{\partial}{\partial \tau} S_z^0 \mathrm{d}\xi \right. \\ & \left. - \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \alpha(1+\alpha) \frac{\partial^2 S_y^0}{\partial \zeta^2} (\mathrm{d}\xi)^2 \right] \\ & - S_z^0 \left[\int_{-\infty}^{\xi} \frac{-2V(1+\alpha)^2}{c^2} \frac{\partial}{\partial \tau} S_y^0 \mathrm{d}\xi \right. \\ & \left. - \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} \alpha(1+\alpha) \frac{\partial^2 S_y^0}{\partial \zeta^2} (\mathrm{d}\xi)^2 \right].\end{aligned} \tag{60}$$

Introducing two new variables A and θ defined by

$$S_y^0 = A \cos\theta, \quad S_z^0 = A \sin\theta, \quad A = S_0 \sin\phi, \quad \theta \to 0 \text{ as } \xi \to -\infty \tag{61}$$

and substituting (61) in (45) we have

$$S_x^1 = \frac{2V^2}{(1+\alpha)} \frac{\partial \theta}{\partial \xi}. \tag{62}$$

Now substituting the value of S_x^1 and using the new variables, (60) can be written as,

$$-\mu\frac{\partial^2\theta}{\partial\xi^2} = \cos\theta\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}\sin\theta\mathrm{d}\xi + \sigma\cos\theta\frac{\partial^2}{\partial\zeta^2}\int_{-\infty}^{\xi}\int_{-\infty}^{\xi}\sin\theta(\mathrm{d}\xi)^2$$
$$-\sin\theta\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}\cos\theta\mathrm{d}\xi - \sigma\sin\theta\frac{\partial^2}{\partial\zeta^2}\int_{-\infty}^{\xi}\int_{-\infty}^{\xi}\cos\theta(\mathrm{d}\xi)^2. \quad (63)$$

$$\mu = \frac{2V^3c^2}{(1+\alpha)^3}$$
$$\sigma = \frac{\alpha c^2}{2V(1+\alpha)}.$$

Differentiating (63) with respect to ξ and simplifying we obtain,

$$\frac{\partial\theta}{\partial\tau} + \mu\frac{\partial^3\theta}{\partial\xi^3} + \sigma\left[\cos\theta\frac{\partial^2}{\partial\zeta^2}\int\sin\theta\mathrm{d}\xi - \sin\theta\frac{\partial^2}{\partial\zeta^2}\int_{\infty}^{\xi}\cos\theta\mathrm{d}\xi\right]$$
$$= -\mu\frac{\partial^2\theta}{\partial\xi^2}\left[\frac{\partial\theta}{\partial\xi}\right]^2 + \sigma\left[\cos\theta\frac{\partial^2}{\partial\zeta^2}\int\cos\theta\mathrm{d}\xi\right.$$
$$\left.+\sin\theta\frac{\partial^2}{\partial\zeta^2}\int\sin\theta\mathrm{d}\xi\right]\frac{\partial\theta}{\partial\xi}. \quad (64)$$

Equation (64) can be integrated with respect to ξ and simplified to give

$$\frac{\partial\theta}{\partial\tau} + \mu\frac{\partial^3\theta}{\partial\xi^3} - \sigma\int_{-\infty}^{\xi}\frac{\partial^2\theta}{\partial\zeta^2}\mathrm{d}\xi = -\mu\frac{1}{2}[\frac{\partial\theta}{\partial\xi}]^2\frac{\partial\theta}{\partial\xi}. \quad (65)$$

Putting $f = \frac{\partial\theta}{\partial\eta}$, the above equation becomes

$$\frac{\partial f}{\partial\tau} + \frac{3}{2}\mu f^2\frac{\partial f}{\partial\xi} + \mu\frac{\partial^3 f}{\partial\xi^3} = \sigma\int_{-\infty}^{\xi}\frac{\partial^2 f}{\partial\zeta^2}\mathrm{d}\xi, \quad (66)$$

where f is a function of ξ, ζ and τ.

Differentiating (66) with respect to ξ we obtain,

$$\left[\frac{\partial f}{\partial\tau} + \frac{3}{2}\mu f^2\frac{\partial f}{\partial\xi} + \mu\frac{\partial^3 f}{\partial\xi^3}\right]_{\xi} = \sigma\frac{\partial^2\theta}{\partial\zeta^2}. \quad (67)$$

This equation is the modified Kadomtsev–Petviashvili (mKP) equation. In steady propagation of the solution, this equation gives rise to a soliton solution. The steady state solution of this equation can be obtained by using elliptic integrals. Thus we find

$$f = 6k^2\mathrm{sech}^2 k\eta$$

where $\eta = \xi + \zeta - \lambda\tau$, and $k^2 = \mu(\sigma+\mu)$, λ is a constant. Since $f = \frac{\partial\theta}{\partial\eta}$, θ is obtained as $\theta = \int_{-\infty}^{\eta} f\mathrm{d}\eta \cdot \theta$ increases from 0 to 2π or decreases from 0 to -2π according as $k > 0$ or $k < 0$ as η goes from $-\infty$ to $+\infty$. We can obtain explicit form of solutions for S_y^0, S_z^0, E_y^0, E_z^0, E_x^1 and S_x^1: Thus we can write,

$$S_y^0 = S_0\sin\phi\cos(6k^2\sqrt{[1-\mathrm{sech}^2 k\eta]}),$$

$$
\begin{aligned}
S_z^0 &= S_0 \sin\phi \sin(6k^2\sqrt{[1-\mathrm{sech}^2 k\eta]}),\\
E_y^0 &= \alpha S_0 \sin\phi \cos(6k^2\sqrt{[1-\mathrm{sech}^2 k\eta)]}),\\
E_z^0 &= \alpha S_0 \sin\phi \sin(6k^2\sqrt{[1-(\mathrm{sech}^2 k\eta)]}),\\
E_x^1 &= \frac{6Vk^2\mathrm{sech}^2 k\eta}{(1+\alpha)}\\
S_x^1 &= -\frac{6Vk^2\mathrm{sech}^2 k\eta}{(1+\alpha)}.
\end{aligned}
$$

We have found that S_x^1, E_x^1, S_y^0 and E_y^0 components give soliton solutions and S_z^0 and E_z^0 components give kinks. We can see that the displacement vector field and the electric field are spatially localized. The stability analysis of this solution using small k-perturbation expansion method [15] has been studied and is left for future communication.

3. Conclusion

In this paper we have considered the propagation of an electromagnetic wave in a cold collisionless plasma We have attempted to find what happens to the dynamics of the system when the electromagnetic wave passes through the medium. We have studied the nature of fluctuations due to the interaction of the electric field associated with the eletromagnetic wave and the displacement vector field. A perturbation analysis is carried out to understand the nature of fluctuations due to the interaction between the electric field and the displacement vector field. The above fluctuations take place in the form of solitons and kinks.

Acknowledgements

The authors are thankful to K Babu Joseph and M Lakshmanan for many valuable discussions. One of us (VCK) is thankful to DST, Government of India for financial support under a research project and also wishes to thank IUCAA, Pune for library facilities extended to him.

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