PRAMANA
—journal of
physics

© Indian Academy of Sciences

Vol. 51, No. 6
December 1998
pp. 711–725

Mass and decay constant of the $(q\bar{q})$-pion in a relativistic-potential model of independent quarks

S N JENA, M R BEHERA and S PANDA[+]
Department of Physics, Berhampur University, Berhampur 760 007, Orissa, India
[+] Department of Physics, Science College, Hinjilicut 761 102, Orissa, India

MS received 10 March 1998; revised 15 July 1998

Abstract. Under the assumption that a meson is an assembly of independent quark and antiquark, confined in a first approximation by an effective potential $V_q(r) = \frac{1}{2}(1+\gamma^0)(a^2r + V_0)$ which presumably represents the non-perturbative gluon interactions, the mass and decay constant of the $(q\bar{q})$-pion together with the masses of ρ and ω-mesons are calculated by considering perturbatively the corrections due to the possible residual interaction such as quark-pion coupling arising out of the requirement of chiral symmetry and quark-gluon coupling arising out of one-gluon exchange and that due to the spurious motion of center of mass of the meson core.This results in the physical mass of the $(q\bar{q})$-pion in consistency with that of the PCAC-pion and the pion decay constant in a reasonable agreement with experiment.

Keywords. Independent-quark model; chiral symmetry; one-gluon exchange; center of mass correction; pion mass; pion decay constant.

PACS No. 12.40

1. Introduction

Study of hadrons made up of low-mass quarks through quark model approaches has been quite successful in reproducing the hadron masses and other properties [1] satisfactorily. Still the description of the most illusive π-meson has been found to be a notable exception to this success. It is because of the fact that the pion of the quark model appears to be quite different from the pion of PCAC. Quantum chromodynamics, the fundamental theory of hadrons has a chiral $SU(2) \times SU(2)$ symmetry in the limit of vanishing up- and down-quark masses. Spontaneous breaking of this symmetry gives rise to pions as the associated massless Goldstone bosons. But in case of small and non-zero quark masses, there is a slight departure from this description providing the pion a small mass $m_\pi = 140$ MeV and giving rise to the so-called PCAC. On the other hand the pion is described as a $(q\bar{q})$-bound state in quark model approaches. Extension of the non-relativistic two-body potential model studies of heavy meson spectra to the ordinary light meson sector realizes in certain cases [2] a pion mass between 160 and 180 MeV. Since a relativistic treatment is more appropriate in the light meson sector, such non-relativistic results do not have much quantitative significance.

A relativistic description of $(\rho - \pi)$ system is provided by the otherwise successful static bag model by considering a bag confinement of an independent quark and an antiquark in their ground states.

Although in this model where the spin-dependent forces due to one gluon exchange are taken into account perturbatively, pion emerges naturally as the lightest state with m_π between 175 and 280 MeV [3], it is found impossible to get a pion of $m_\pi = 140$ MeV. However, it can be possible to realize a zero mass pion in the chiral limit by treating the static cavity eigenstates as localized wave packets of true momentum eigenstates [4] in analogy with states in a non-relativistic shell model. But in the static-cavity approximation for the quark-confinement, the chiral symmetry which is lost at the bag surface can only be restored by introducing an external pion field. This leads to the formulation of different versions of chiral symmetric bag models. Such a model [5] describing pion regards the external pion field as an approximation to the amplitude for finding the center of mass of a composite $(q\bar{q})$-pion at a given spacetime point. In other words pion is preferred to be represented by $(q\bar{q})$ bound states for distances comparable to the bag size while an elementary field description of it is believed to be adequate at larger distances. Taking into account the mass shift due to the lowest order pionic self energy, such a picture generates a pion mass between 268 and 396 MeV [5] with a large quark gluon coupling constant $\alpha_c \sim 1.5$.

In the present work we make an attempt to resolve this apparent dichotomy between the $q\bar{q}$ pion and PCAC-pion in the frame-work of a relativistic chiral potential model of independent quarks. Such a model has been found to be quite successful for the ground state octet baryons in reproducing satisfactorily the mass spectrum [6] as well as the electromagnetic properties [7]. In this scheme hadrons are considered as an assembly of independent quarks confined, in a first approximation, by an effective potential

$$V_q(r) = \tfrac{1}{2}(1+\gamma^0)(a^2 r + V_0), \tag{1.1}$$

which presumably represents the non-perturbative gluon interactions. This provides the zeroth order quark dynamics inside the hadronic core through a Lagrangian formulation. The residual interactions, due to quark-pion coupling arising out of the restoration of chiral symmetry in PCAC limit in $SU(2)$ sector and that due to one gluon exchange at short distances, are treated perturbatively. The effect of the center of mass motion is also taken into account following the prescription of Wong [8]. We would like to extend the same model to the mesonic ground states with the particular motive to realize the controversial pion. The additional elementary pion field to be introduced over and above the confined independent quarks of non-zero mass for restoration of chiral symmetry is considered to have a small mass $\tilde{m}_\pi = 140$ MeV in PCAC limit. Then our purpose here would be to realize a $(q\bar{q})$-pion with the same mass as that of the PCAC-pion in the event of small but non-zero quark masses. In the process we would also obtain for a consistency check an estimate of the $(\rho - \pi)$ mass difference as well as the pion decay constant f_π.

In § 2,we provide a brief outline of the potential model giving the zeroth order confined quark dynamics. This also deals with the corrections to the unperturbed energy of the $(q\bar{q})$-bound state due to the lowest order colour electrostatic and magnetostatic energies arising out of one-gluon exchange at short distances. Here again we take into account the energy shift due to quark-pion coupling arising out of the requirement of chiral symmetry restoration in PCAC limit. This calculation in low order perturbation theory uses the experimental value of the pion-decay constant $f_\pi = 93$ MeV and the field pion mass $\tilde{m}_\pi = 140$ MeV. The prescription for taking into account the spurious centre of mass correction is also mentioned briefly. Now incorporating these corrections of § 2 to the unperturbed zeroth order energy of the $(q\bar{q})$-bound state system and considering the effect of the centre of the mass motion, one can realize the mass of the $(q\bar{q})$-bound state in its static limit. This section also deals with the derivation of a simple expression for the pion-decay matrix element $F_\pi(\mathbf{p}^2)$ in the present model. Here $F_\pi(0)$ is estimated in the static limit with m_π as obtained from the present model and is compared with f_π for a consistency check. Finally § 3 provides the results and discussion.

2. Theoretical framework

In the present model a meson in general is pictured as an assembly of a quark and an antiquark with appropriate interactions according to quantum chromo-dynamics (QCD). The quark-gluon interaction originating from one-gluon exchange at short distances and the quark-pion interaction in the non-strange flavour sector required to preserve chiral symmetry are presumed to be residual interactions compared to the dominant confining interaction. Therefore, to a first approximation, the confining part of the interaction is believed to provide the zeroth order quark dynamics inside the mesonic core leading to the zeroth order, energy of the $(q\bar{q})$-assembly of a light meson.

The confining interaction which is expected to be generated by the non-perturbative multi-gluon mechanism is impossible to calculate theoretically from the first principle. Therefore from a phenomenological point of view, the present model assumes that the quarks in a meson-core are independently confined by an average flavour independent potential of the form [6, 7]

$$V_q(r) = \tfrac{1}{2}(1+\gamma^0)V(r) \tag{2.1}$$

with $V(r) = a^2 r + V_0, a > 0$. The quark-Lagrangian density in zeroth order in such a model is written as

$$\mathcal{L}_q^0(x) = \bar{\psi}_q(x)\left[\frac{i}{2}\gamma^\mu\partial_\mu - m_q - V_q(r)\right]\psi_q(x). \tag{2.2}$$

Assuming all the quarks in a meson core in their ground states with $J^P = \frac{1}{2}^+$, the normalized quark wave function $\psi_q(r)$ satisfying the Dirac equation derivable from $\mathcal{L}_q^0(x)$ as

$$[\gamma^0 E_q - \mathbf{r}\cdot\mathbf{p} - m_q - V_q(r)]\psi_q(r) = 0 \tag{2.3}$$

can be written in a two-component form

$$\psi_{q^+}(\mathbf{r}) = N_q\begin{pmatrix}\phi_q(\mathbf{r})\\ -i\frac{\sigma\cdot\hat{r}}{\lambda_q}\phi_q'(\mathbf{r})\end{pmatrix}\chi\uparrow \tag{2.4}$$

for the positive energy solution and

$$\psi_{q^-}(\mathbf{r}) = N_q\begin{pmatrix}\frac{\sigma\cdot\hat{r}}{\lambda_q}\phi_q'(\mathbf{r})\\ -i\phi_q(\mathbf{r})\end{pmatrix}\chi\uparrow \tag{2.5}$$

for the negative energy solution, where $\lambda_q = E_q + m_q$ and

$$\phi_q(\mathbf{r}) = A_q \frac{f_q(r)}{r} Y_0^0(\theta, \phi) \tag{2.6}$$

is the normalized radial angular part of $\psi_q(\mathbf{r})$ with normalization constant A_q. Taking $E_q' = E_q - V_0/2$, $m_q' = m_q + V_0/2$, $\lambda_q = E_q' + m_q'$, the reduced radial part $f_q(r)$ can be found to satisfy a Schrödinger-type equation

$$f_q''(r) + \lambda_q [E_q' - m_q' - a^2 r] f_q(r) = 0, \tag{2.7}$$

which can be transformed into a convenient dimensionless form

$$f_q''(\rho) + (\epsilon_{ns} - \rho) f_q(\rho) = 0, \tag{2.8}$$

where $\rho = r/r_{0q}$ is a dimensionless variable with $r_{0q} = (a^2\lambda_q)^{-1/3}$ and

$$\epsilon_{ns} = \left(\frac{\lambda_q}{a^4}\right)^{1/3} (E_q' - m_q'). \tag{2.9}$$

Now with $z = \rho - \epsilon_{ns}$, the eq. (2.8) reduces to

$$f_q''(z) - z f_q(z) = 0 \tag{2.10}$$

whose solution $f_q(z)$ is the Airy function $Ai(z)$. Since at $r = 0$, $f_q(r) = 0$ we have $Ai(z) = 0$ at $z = -\epsilon_{ns}$. If z_n are the roots of the Airy function such that $Ai(z_n) = 0$ then we have $z_n = -\epsilon_{ns}$. For the ground state of quarks, the ϵ_{ns} value is given by the first root z_1 of the Airy function so that

$$\epsilon_{ns} = \epsilon_{1s} = \epsilon_q = -z_1, \tag{2.11}$$

the value of this root being $z_1 = -2.33811$. Now the ground state individual quark binding energy $E_q = E_q' + V_0/2$ is obtainable from the energy eigenvalue condition (2.9) through the relation

$$E_q' = (m_q' + a x_q), \tag{2.12}$$

where x_q is the root of the equation

$$x_q^4 + b x_q^3 - \epsilon_q^3 = 0. \tag{2.13}$$

with $b = 2m_q'/a$. The quark binding energy E_q thus obtained leads to the energy of the meson core in zeroth order as

$$E_M^0 = \sum_q E_q. \tag{2.14}$$

The overall normalization constant N_q of $\psi_q(\mathbf{r})$ appearing in eqs (2.4) and (2.5) can be obtained in a simplified form as

$$N_q^2 = \frac{3(E_q' + m_q')}{2(2E_q' + m_q')}. \tag{2.15}$$

2.1 *One-gluon-exchange correction*

The individual quarks in a meson-core are considered so far to be experiencing the only force coming from the average effective potential $V_q(r)$ in (1.1). All that remains inside

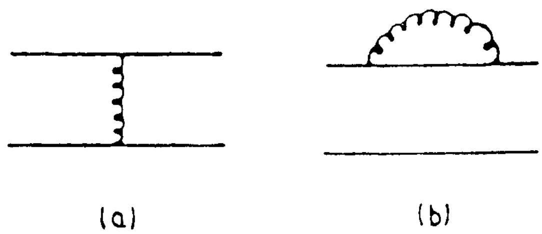

Figure 1. One-gluon exchange contribution to the energy of a $(q\bar{q})$-configuration.

the meson-core is the hopefully weak one-gluon exchange interaction provided by the interaction Lagrangian density

$$\mathcal{L}_I^g = \sum_a J_i^{\mu a}(x) A_\mu^a(x) \tag{2.16}$$

where $A_\mu^a(\mathrm{x})$ are the eight vector-gluon fields and $J_i^{\mu a}(x)$ is the ith-quark colour current. Since at small distances the quarks should be almost free, it is reasonable to calculate the shift in the energy of the meson-core (arising out of the quark interaction energy due to its coupling to the coloured gluons) using a first order perturbation theory. Such an approach leads to the colour-electric and colour-magnetic energy shifts (as shown in figures 1(a) and (b)) [6],

$$(\Delta E_M)_g = (\Delta E_M)_g^e + (\Delta E_M)_g^m \tag{2.17}$$

where

$$(\Delta E_M)_g^e = \frac{\alpha_c}{\pi} \sum_{i,j} \left\langle \sum_a \lambda_i^a \lambda_j^a \right\rangle N_i^2 N_j^2 I_{ij}^e,$$

$$(\Delta E_M)_g^m = \frac{-4\alpha_c}{3\pi} \sum_{i<j} \left\langle \sum_a \lambda_i^a \lambda_j^a (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \right\rangle \frac{N_i^2 N_j^2}{\lambda_i \lambda_j} I_{ij}^m. \tag{2.18}$$

Here λ_i^a are the usual Gellmann $SU(3)$ matrices, and α_c is the quark-gluon coupling constant.

$$I_{ij}^e = \int_0^\infty \mathrm{d}k F_i^e(k) F_j^e(k) \tag{2.19}$$

where

$$F_i^e(k) = \frac{1}{2\lambda_i^2} [(4E_i'\lambda_i - k^2)\langle\langle j_0(kr_i)\rangle\rangle - 2\lambda_i a^2 \langle\langle r_i j_0(kr_i)\rangle\rangle] \tag{2.20}$$

and

$$I_{ij}^m = \int_0^\infty \mathrm{d}k k^2 \langle\langle j_0(kr_i)\rangle\rangle \langle\langle j_0(kr_j)\rangle\rangle \tag{2.21}$$

where $j_0(kr_i)$ is the zeroth order spherical Bessel function and the double angular brackets represent the expectation values with respect to $\phi_q(\mathbf{r})$. Finally taking into account the

Table 1. Coefficients appearing in the calculation of the colour-electric and -magnetic energy corrections due to one-gluon exchange.

Mesons	a_{uu}	a_{us}	a_{ss}	b_{uu}	b_{us}	b_{ss}
ω	0	0	0	2	0	0
ρ	0	0	0	2	0	0
π	0	0	0	−6	0	0

specific quark flavour and spin configurations in various ground state mesons and using the relations $\langle\sum_a(\lambda_i^a)^2\rangle = 16/3$ and $\langle\sum_a \lambda_i^a\lambda_j^a\rangle_{i\neq j} = -8/3$, one can write in general the energy correction due to one-gluon exchange as

$$(\Delta E_M)_g^e = \alpha_c(a_{uu}T_{uu}^e + a_{us}T_{us}^e + a_{ss}T_{ss}^e), \tag{2.22}$$

$$(\Delta E_M)_g^m = \alpha_c(b_{uu}T_{uu}^m + b_{us}T_{us}^m + b_{ss}T_{ss}^m). \tag{2.23}$$

Here a_{ij} and b_{ij} are the numerical coefficients depending on each meson and are listed in table 1 for mesons (ω, ρ, π) considered here in the context of π-meson.The quantities $T_{i,j}^{e,m}$ are

$$T_{i,j}^e = \frac{12}{\pi}\frac{(E_i' + m_i')(E_j' + m_j')}{(2E_i' + m_i')(2E_j' + m_j')}I_{ij}^e, \tag{2.24}$$

$$T_{i,j}^m = \frac{8}{\pi(2E_i' + m_i')(2E_j' + m_j')}I_{ij}^m. \tag{2.25}$$

One can note from table 1 that the colour-electric contributions for the mesons vanish when the constituent quark and antiquark masses in a meson-core are equal. Therefore, the degeneracy among the meson like (ω,ρ and π) is essentially removed at this level through the strong spin-spin interaction in the colour-magnetic part only.

2.2 *Chiral symmetry and pionic correction*

Looking to the zeroth order Lagrangian density $\mathcal{L}_q^0$ which takes into account the non-perturbative multi-gluon-interactions including gluon-self coupling through the phenomenological potential $V_q(r)$, one can note that under a global infinitesimal chiral transformation at least in the non-strange flavour sector,

$$\psi(x) \longrightarrow \psi(x) - i\gamma^5\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\varepsilon})}{2}\psi(x) \tag{2.26}$$

the axial vector current of quarks is not conserved. This is because the scalar term in $\mathcal{L}_q^0$ proportional to $G(r) = (m_q + V(r)/2)$ is chirally odd. The vector part of the potential poses no problem in this respect. But in view of the experimental success of the partial conservation of axial-vector-current (PCAC) and hence the fact that the chiral $SU(2)\times SU(2)$ is one of the best symmetries of strong interaction, it is desirable to conserve the total axial vector current at least in the (u, d)-flavour sector. This is usually done at a phenomenological level [9] by introducing elementary pion field that also carries an axial current such that the four divergence of the total axial-vector current satisfies the PCAC-condition.

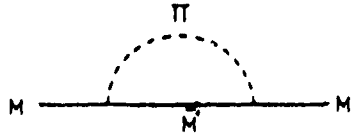

Figure 2. Pionic self-energy of a meson due to quark coupling with pions.

We therefore introduce in the usual manner, an elementary pion field $\phi(x)$ of small but finite mass $\tilde{m}_\pi = 140\,\text{MeV}$ with the quark-pion interaction Lagrangian density,

$$\mathcal{L}_I^\pi = \frac{-i}{f_\pi} G(r)\bar{\psi}(x)\gamma^5(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\psi(x) \tag{2.27}$$

which is linear in isovector pion field $\phi(x)$. Here $f_\pi = 93\,\text{MeV}$ is the phenomenological pion decay constant. Then the four divergence of the total axial-vector current becomes

$$\partial_\mu A^\mu(x) = -f_\pi \tilde{m}_\pi^2 \phi(x) \tag{2.28}$$

yielding the usual PCAC condition. Consequently, the pion field coupling to the non-strange quarks would give rise to pionic self-energy of the mesons which would ultimately contribute to the physical masses of the meson. This aspect can be studied in the usual perturbative approach [7, 10].

The coupling of the pion-field to the non-strange quarks, shown in a minimal way through the single-loop self-energy diagram (figure 2), causes a shift in the energy of the meson-core. The self-energy of the meson-core due to pionic interaction is usually obtained from second order perturbation calculation as

$$\sum_M (E_M^0) = \sum_k \sum_{M'} \frac{V_j^{MM'}(k) V_j^{MM'}(k)}{(E_M^0 - \omega_k - m_M^0)} \tag{2.29}$$

where $\sum_k = \sum_j \int \mathrm{d}^3k/(2\pi)^3$ and M' is the intermediate meson-core state. $V_j^{MM'}(k)$ is the meson-pion vertex function which in the present model is obtained [7, 10] as

$$V_j^{MM'}(k) = i\frac{3g_A}{10f_\pi}\frac{u(k)}{\sqrt{2\omega_k}}\left\langle M' \middle| \sum_q (\boldsymbol{\sigma}_q\cdot\mathbf{k})(\tau_q)_j \middle| M \right\rangle \tag{2.30}$$

where

$$g_A = \frac{5}{9}\left(\frac{4E'_q + 5m'_q}{2E'_q + m'_q}\right) \tag{2.31}$$

and form factor $u(k)$ is given by

$$u(k) = \frac{5N_q^2}{3\lambda_u g_A}[2m'_q\langle\langle j_0(kr)\rangle\rangle + a^2\langle\langle r j_0(kr)\rangle\rangle + a^2\langle\langle j_1(kr)/k\rangle\rangle] \tag{2.32}$$

where $\langle\langle j_0(kr)\rangle\rangle$ and $\langle\langle j_1(kr)\rangle\rangle$ are the expectation values of the spherical Bessel function of zeroth and first order respectively with respect to $\phi_q(r)$.

Using the familiar Goldberger–Treimann relation $(\sqrt{4\pi} f_{NN\pi}/\tilde{m}_\pi) = (g_A/2f_\pi)$, one can write (2.30) as

$$V_j^{MM'}(k) = i\sqrt{4\pi}\frac{3f_{NN\pi}}{5\tilde{m}_\pi}\frac{ku(k)}{\sqrt{2\omega_k}}\left\langle M'\left|\sum_q(\vec{\sigma}_q\cdot\hat{k})(\tau_q)_j\right|M\right\rangle. \tag{2.33}$$

Here $\omega_k = (\mathbf{k}^2 + m_\pi^2)^{1/2}$ is the pion-energy. For degenerate intermediate meson-states on mass shell with $m_M^0 = m_{M'}$, the self energy becomes

$$\delta m_M = \sum_M(E_M^0 = m_M^0 = m_{M'}) = -\sum_{k,M'}\frac{V_j^{MM'}(k)V_j^{MM'}}{\omega_k}. \tag{2.34}$$

Now using the explicit expression for $V_j^{MM'}(k)$ as in (2.33), one gets

$$\delta m_M = \frac{-3}{25}f_{NN\pi}^2 I_\pi \sum_{M'} C_{MM'} \tag{2.35}$$

when

$$C_{MM'} = \left\langle M'|\sum_{q,q'}(\sigma_q\cdot\sigma_{q'})(\tau_q\cdot\tau_{q'})|M\right\rangle \tag{2.36}$$

and

$$I_\pi = \frac{1}{\pi\tilde{m}_\pi^2}\int_0^\infty\frac{\mathrm{d}kk^4u^2(k)}{\omega_k^2}. \tag{2.37}$$

Now using the values of $C_{MM'}$ summarized in table 2 [5] with appropriate intermediate meson-states MM' shown in figure 3, the self-energy δm_M for mesons like ω, ρ and π can be computed as

$$(\delta m_\omega, \delta m_\rho, \delta m_\pi) = (-72, -48, -72)\,\frac{f_{NN\pi}^2 I_\pi}{25}. \tag{2.38}$$

The self-energy δm_M calculated here contains both the quark self-energy and one pion exchange contributions.

2.3 *Centre of mass correction and the meson mass m_M*

In this shell type relativistic independent-quark model, the independent motion of the quarks inside the hadron-core does not lead to a state of definite total momentum as it should, to represent the physical state of a meson.This problem appears in the same way in

Table 2. Contribution of $\langle\sum_{q,q'}(\sigma_q\cdot\sigma_{q'})(\tau_q\cdot\tau_{q'})\rangle = C_{MM'}$ from the various intermediate meson states.

Mesons	Intermediate meson states	$C_{MM'}$
ω	ρ	24
	ω	8
ρ		
	π	8
π	ρ	24

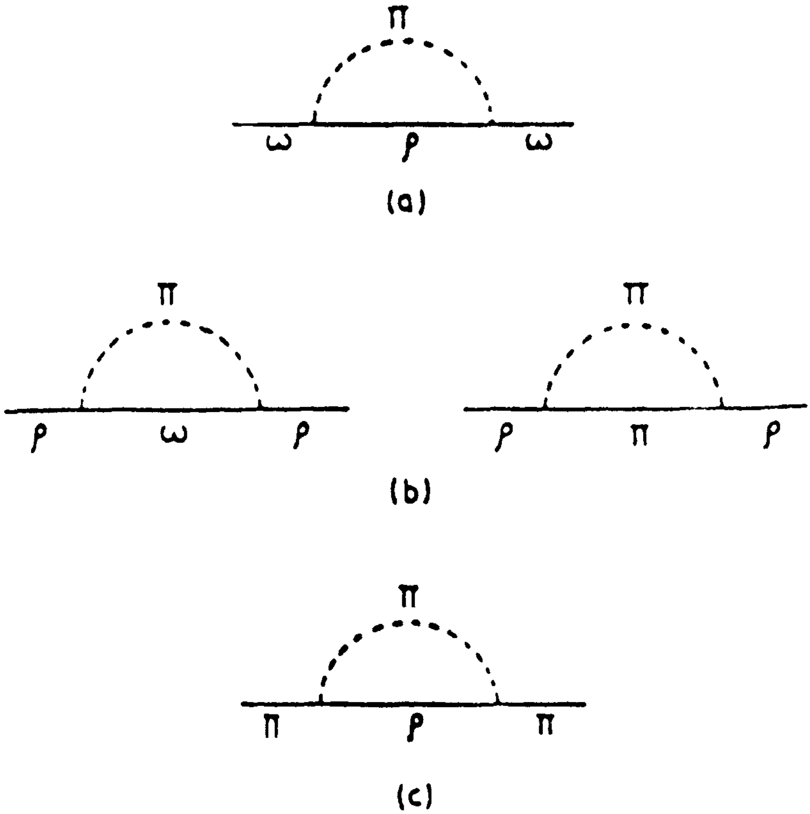

Figure 3. Relevant diagrams with appropriate intermediate states contributing to the pionic self-energies of mesons like (**a**) ω; (**b**) ρ, (**c**) π.

nuclear physics in the case of ^{3}He and also in bag models and therefore has to be resolved accordingly [11, 12]. The energy associated with the spurious centre of mass motion must provide a further correction to the meson energy obtained from the individual quark binding energy over and above the perturbative corrections discussed in § 2.1 and 2.2. This is accounted for following the prescriptions of Wong and other workers [13].

In such a prescription the static meson core state with the core centre at X is decomposed into components $\chi_M(\mathbf{P})$ of plane wave momentum eigen states as

$$|M(\mathbf{X})\rangle_c = \int \frac{\mathrm{d}^3\mathbf{P}}{W_M(\mathbf{P})} e^{(i\mathbf{P}\cdot\mathbf{X})} \chi_M(\mathbf{P}) |M(\mathbf{P})\rangle. \tag{2.39}$$

The inverse relation is

$$|M(\mathbf{P})\rangle = \frac{1}{(2\pi)^3} \frac{W_M(\mathbf{P})}{\chi_M(\mathbf{P})} \int \mathrm{d}^3\mathbf{X} e^{(i\mathbf{P}\cdot\mathbf{X})} |M(X)\rangle_c \tag{2.40}$$

where $|M(\mathbf{P})\rangle$ is normalised usually as

$$\langle M(\mathbf{P}')|M(\mathbf{P})\rangle = (2\pi)^3 W_M(\mathbf{P}) \delta^3(\mathbf{P} - \mathbf{P}') \tag{2.41}$$

with $W_M(\mathbf{P}) = 2\omega_p$. The momentum profile function $\chi_M(\mathbf{p})$ can be obtained as

$$|\chi_M(\mathbf{P})|^2 = \frac{W_M(\mathbf{P})}{(2\pi)^3}\tilde{I}_M(\mathbf{P}) \tag{2.42}$$

where $\tilde{I}(\mathbf{P})$ is the Fourier transform of the Hill–Wheeler overlap function [8]. In the present model $\tilde{I}(\mathbf{P})$ is chosen in Gaussian form as

$$\tilde{I}_M(\mathbf{P}) = C_M\left(\frac{r_M^2}{2\pi}\right)^{3/2} e^{(-\mathbf{P}^2 r_M^2/2)} \tag{2.43}$$

with

$$C_M = \frac{2r_M^2}{15}\frac{(E_q'^2 - m_q'^2)(4E_q' + m_q')}{(2E_q' + m_q')} \tag{2.44}$$

where r_M is the r.m.s charge radius $\langle r^2\rangle_M^{1/2}$ of the meson. This permits ready estimate of the centre-of-mass momentum $\mathbf{P}$ for the meson as

$$\langle\mathbf{P}^2\rangle_M = \int \mathrm{d}^3\mathbf{P} I_M(\mathbf{P})\mathbf{P}^2 = \sum_q \langle\mathbf{p}^2\rangle_q \tag{2.45}$$

where $\sum_q\langle\mathbf{p}^2\rangle_q$ is the average value of the square of the individual quark momentum taken over the $1s_{1/2}$ single quark states and is given in the present model as

$$\langle\mathbf{p}^2\rangle_q = \frac{(E_q'^2 - m_q'^2)(4E_q' + m_q')}{5(2E_q' + m_q')}. \tag{2.46}$$

Thus we find the zeroth order energy E_M^0 in (2.13) for a ground state meson M arising out of the binding energies of the constituent quark and antiquark confined independently by a phenomenological average potential $V_q(r)$ must be corrected for the energy shifts due to the residual quark-gluon [eqs 2.22 to 2.25] and quark-pion interaction eq. (2.38) discussed in § 2.1 and 2.2. This would give the total energy of the $(q\bar{q})$-system in its ground state as

$$E_M = E_M^0 + (\Delta E_M)_g^e + (\Delta E_M)_g^m + \delta m_M. \tag{2.47}$$

Finally taking into account the centre of mass motion of the $(q\bar{q})$-system with the c.m-momentum $\mathbf{P}$ given in eqs (2.45) and (2.46), one can obtain the physical mass of $(q\bar{q})$-meson in its ground state as

$$m_M = [E_M^2 - \langle\mathbf{P}^2\rangle_M]^{1/2}. \tag{2.48}$$

Since our main objective here is to obtain the mass of the $(q\bar{q})$-pion, we would confine ourselves only to the non-strange meson-sector. Then (2.48) can yield the mass of the $(q\bar{q})$-pion together with that of ρ and ω mesons.

2.4 *Decay constant of the pion*

In order to study the decay constant of the pion-core in the present model, we first of all calculate the pion decay matrix element F_π from vacuum to pion momentum eigenstates defined by [8, 11]

$$\langle 0|\psi_{q^-}(\mathbf{r})\gamma^{\mu}\gamma^5\psi_{q^+}(\mathbf{r})|\pi(\mathbf{P})\rangle = i\sqrt{2}F_{\pi}(\mathbf{P}^2)P^{\mu}e^{(i\mathbf{P}\cdot\mathbf{r})}. \tag{2.49}$$

Taking the time component and transforming to the core state according to eq. (2.39), we can have

$$\langle 0|\psi_{q^-}(\mathbf{r})\gamma^0\gamma^5\psi_{q^+}(\mathbf{r})|\pi(0)\rangle_c = i\sqrt{2}\int \mathrm{d}^3\mathbf{P}P^0\frac{\chi_{\pi}(\mathbf{P})}{W_{\pi}(\mathbf{P})}F_{\pi}(\mathbf{P}^2)e^{i\mathbf{P}\cdot\mathbf{r}}. \tag{2.50}$$

Then the pion-decay matrix element is obtained in the form

$$F_{\pi}(\mathbf{P}^2) = \left[\frac{3(2\pi)^3W_{\pi}(\mathbf{P})}{(P^0)^2\tilde{I}_{\pi}(\mathbf{P})}\right]^{1/2}\tilde{f}_A(\mathbf{P}) \tag{2.51}$$

where

$$\tilde{f}_A(\mathbf{P}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{r}e^{(-i\mathbf{P}\cdot\mathbf{r})}\langle 0|\psi_{q^-}(\mathbf{r})\gamma^0\gamma^5\psi_{q^+}(\mathbf{r})|\pi(0)\rangle_c i\sqrt{6}. \tag{2.52}$$

Using the forms of $\psi_{q^+}(\mathbf{r})$ as given in (2.4), one can further obtain the expression (2.51) as

$$\tilde{f}_A(\mathbf{P}) = \frac{N_q^2}{(2\pi)^3}\int \mathrm{d}^3\mathbf{r}e^{(-i\mathbf{P}\cdot\mathbf{r})}\left[\phi_q^2(\mathbf{r}) - \frac{\phi_q'^2(\mathbf{r})}{\lambda_q^2}\right] \tag{2.53}$$

which on further simplification becomes

$$\tilde{f}_A(\mathbf{P}) = \frac{3}{(2\pi)^3(2E_q' + m_q)}\left[\left(m_q' + \frac{\mathbf{P}^2}{4\lambda_q}\right)\langle\langle j_0(|\mathbf{P}|r)\rangle\rangle + \frac{a^2}{2}\langle\langle rj_0|\mathbf{P}|r\rangle\rangle\right]. \tag{2.54}$$

Now with $I_{\pi}(\mathbf{P})$ for π-meson from (2.43) and $f_A(\mathbf{P})$ from (2.54), it is straight-forward to obtain the pion decay matrix element $F_{\pi}(\mathbf{P}^2)$ from (2.51) in a more useful form as

$$F_{\pi}(\mathbf{P}^2) = F_{\pi}(0)\left[\frac{m_{\pi}W_{\pi}(\mathbf{P})}{2(P^0)^2}\right]^{1/2}\Lambda(\mathbf{P}^2) \tag{2.55}$$

where

$$F_{\pi}(0) = \frac{\sqrt{6}}{B_M}[(2\pi)^{3/2}m_{\pi}r_{\pi}^3C_u]^{-1/2} \tag{2.56}$$

and

$$\Lambda(\mathbf{P}^2) = \frac{3}{(E_u' + 2m_u')}\left[\left(m_u' + \frac{\mathbf{P}^2}{4\lambda_u}\right)\langle\langle j_0(|\mathbf{P}|r)\rangle\rangle + \frac{a^2}{2}\langle\langle rj_0|\mathbf{P}|r\rangle\rangle\right]e^{(\mathbf{P}^2r_{\pi}^2/4)}. \tag{2.57}$$

Here

$$B_0 = \frac{(2E_0' + m_0')}{(E_0' + 2m_0')}. \tag{2.58}$$

Now in the static limit if one identifies $F_{\pi}(0)$ as the pion-decay constant f_{π}, then it can be evaluated according to (2.56).

3. Results and discussion

For a quantitative evaluation of the $(q\bar{q})$-pion mass and the pion decay constant, one needs the potential parameters (a, V_0) and the non-strange quark mass $m_q(m_u = m_d)$ as model inputs.The potential parameters (a, V_0) describe the phenomenological average potential which is used in this model as a substitute for the long range part of the two-body interaction. In our earlier work on baryons [6] a suitable choice of (a, V_0) has been found which together with a quark-gluon coupling constant $\alpha_c = 0.576$, has led us to a reasonable estimate of the physical masses of the ground state octet baryons. But these parameters may be different in the present case since the quarks belonging to mesons and baryons may be acted upon by different long range potentials. For the short-range one-gluon exchange interaction, it is well known from the QCD scenario that the two-body quark-quark potential V_{qq} in a baryon is half of the quark antiquark potential $V_{q\bar{q}}$ in a meson. However for the long distance confining part of the interaction arising out of the non-perturbative multi-gluon exchange, no such straight-forward theoretical derivation exists within the first principle QCD applications. Limited study due to Dosch and Müller [14] in lattice gauge theory calculation with finite size lattice and without including vacuum polarization has shown that the three body potential for baryons may be written approximately as 0.54 times the two-body $(q\bar{q})$-potential. But the analysis does not clearly show the dependence of this relationship on the lattice size and it is also not clear how this relationship will turn out in the continuum limit. Therefore, it may not be totally unreasonable if we take the average central potential for quarks in a meson to be about the same as that in a baryon. The results so obtained may provide justification a posteriori for such an assumption.

Nevertheless we would not like to draw any analogy of our present potential with such two-body potentials. It is rather like a bag confinement where the delta function potential of a bag model taken to be the same for the confined quark/antiquark in a meson or baryon system is being replaced by an effective phenomenological potential of the form used in the present work. There is apparently no theoretical basis in taking the average potential to be identical for meson and baryon system except for the reason of phenomenological simplicity which may find a posteriori justification after adequate explanation of the available data in the meson and baryon sector.

Therefore, we retain the same set of parameters $(a, V_0, m_q$ and $\alpha_c)$ as obtained in our earlier work on baryons [6] for the present study in mesonic sector.

$$(a, V_0) = (386.05, -426.75)\,\text{MeV},$$
$$(m'_u = m'_d, \alpha_c) = (10\,\text{MeV},\ 0.576). \tag{3.1}$$

The energy eigenvalue condition (2.8) then yields $E'_u = E'_d = 735\,\text{MeV}$ and hence the energy E^0_M of the meson-core according to (2.14). Then coming to calculate the energy shift due to residual one-gluon exchange interaction according to eqs (2.22)–(2.25), one finds from table 1 that energy-shift $(\Delta E_M)^e_g$ due to colour-electrostatic interaction energy turns out to be zero here,while the one due to colour-magnetostatic interaction energy only contributes. For evaluation $(\Delta E_M)^m_g$ one needs T^m_{uu} as given in (2.25) which when computed gives a value $T^m_{uu} = 63.955\,\text{MeV}$. Then $(\Delta E_M)^m_g$ for various different mesonic systems like (ρ, ω, π) which are otherwise mass degenerate at the zeroth order, are

Table 3. Energy corrections and physical masses of ground state (ω, ρ, π) mesons (in MeV).

Mesons	E_M^0	$\langle \mathbf{P}^2 \rangle_M$	$(\Delta E_M)_g^m$		δm_M	m_M		
	MeV	MeV^2	$\alpha_c = 0.576$	$\alpha_c = 0.603$	MeV	$\alpha_c = 0.576$	$\alpha_c = 0.603$	Expt
ω	973.242	430.64	73.68	77.15	−70.78	722.64	727.32	783
ρ	973.242	430.64	73.68	77.15	−47.18	754.21	758.81	770
π	973.242	430.64	−221.03	−231.45	−70.78	183.62	140.02	140

calculated with $\alpha_c = 0.576$. However, if one believes that the quark gluon coupling constant α_c may have some dependence on the scale size, one can always make a different choice of α_c in the present case of mesons which may enable one to realize the $(q\bar{q})$-pion mass m_π consistent with the PCAC pion mass $m_\pi = 140\,\text{MeV}$. In view of this table 3 provides the calculated value of $(\Delta E_M)_g^m$ for two choices of α_c like $\alpha_c = 0.576$ and $\alpha_c = 0.603$. The integral expression I_π in (2.37) is calculated. The value of $I_\pi = 296.283\,\text{MeV}$, which enables one to obtain the pionic self-energies of mesons (ω, ρ, π) through (2.38). The values of δm_M is obtained with the model calculated value $f_{NN\pi} = 0.08$ and are presented in table 3. Finally the square momentum spread $\langle \mathbf{P}^2 \rangle_M$ of the meson system, calculated from (2.45) and (2.46) is also listed here together with E_M^0 and the resulting physical masses m_π, m_ρ and m_ω.

From the various energy corrections it is found that the degeneracy in mass due to $SU(2)$-symmetry between ω and ρ-mesons is removed through the spin-isospin pionic corrections, while between ω and π-mesons, it is lifted due to the gluonic correction. However, the mass degeneracy between ρ and π-meson is effectively removed through both gluonic and pionic corrections. If one retains $\alpha_c = 0.576$ found from the baryon sector, then the physical masses of ω and ρ are found in good agreement with the experimental ones. However, the mass of the pion is slightly greater than that of the PCAC pion with $m_\pi = 183\,\text{MeV}$. But on the other hand with slightly different choice $\alpha_c = 0.603$, it is possible to obtain $m_\pi = \tilde{m}_\pi = 140\,\text{MeV}$, when m_ω and m_ρ are not too much different from the experimental values. It must be noted here that the pion mass comes down from a value of something like 700 MeV to less than 200 MeV because of the large value of the colour magnetic interaction energy $(\Delta E_M)_g^m$ arising out of the one-gluon exchange giving a spin-spin contribution which is calculated using first order perturbation theory. The quark-gluon coupling constant $\alpha_c = 0.603$ taken in our calculation is quite consistent with the idea of treating one-gluon-exchange effects in lowest order perturbation. In estimating the $(q\bar{q})$-meson mass in the present chiral model the PCAC pion in terms of an elementary field with $\tilde{m}_\pi = 140\,\text{MeV}$ and $f_\pi = 93\,\text{MeV}$ has been taken as an input and in the process we have recovered the $(q\bar{q})$ pion mass to be of the same order as that of PCAC pion with a suitable choice of $\alpha_c = 0.603$. Now, for a consistency check the pion decay constant is estimated in this model through the expression (2.56) using the calculated value of $(q\bar{q})$-pion mass $m_\pi = 140\,\text{MeV}$. Identifying $F_\pi(O)$ as the pion decay constant f_π in the static limit, we find that it comes out to be 108.62 MeV, which is in good agreement with the experimental value 93 MeV. In this calculation we have used the value of r.m.s charge radius of the pion r_π as 0.66 fm which is its experimental value [15]. The value of $F_\pi(O)$ in this model is infinite in the limit of vanishing pion mass, which is unlike the observation of Donoghue and Johnson [11].

The pion decay matrix element $F\pi(\mathbf{P}^2)$ given by eq. (2.55) depends on the squared three momentum $\mathbf{P}^2$. This momentum dependence comes primarily from a kinematical factor which for one-shell pion $(\rho^0 - \omega_0)$ is $[m_\pi \omega_\pi(\mathbf{P})/2(P^0)^2]^{1/2} = (m_\pi/\omega_P)^{1/2}$ and also from a factor like $\Lambda(\mathbf{P}^2)$. The average value of the square of the centre-of-mass momentum $\langle \mathbf{P}^2 \rangle_\pi$ for the pion in the present model comes to be 0.4306 GeV^2 which is evident from table 3. At this finite value of $\langle \mathbf{P}^2 \rangle_\pi$, it is possible to calculate $\Lambda(\langle \mathbf{P}^2 \rangle_\pi) = 1.715$ from eq. (2.56). Consequently, $F_\pi(\langle \mathbf{P}^2 \rangle)$ is obtained as 85.03 MeV, which is in rough agreement with the experimental value. The significance of this agreement is, however, not clear since $F_\pi(\mathbf{P}^2)$ is valid only for the projected state of the good momentum $\mathbf{P}^2$, while $\langle \mathbf{P} \rangle = 0$ holds for the unprojected core state. It is likely that the usual pion decay matrix element containing a Lorentz-invariant decay constant F_π cannot be recovered completely unless the theoretical projected pion states also have a Lorentz-invariant internal structure. Here, the problem is concerned with centre-of-mass corrections of the pion core in the quark model.

The present model thus indirectly indicates that by incorporating the appropriate gluonic, pionic and centre-of-mass corrections, it is possible for the traditional quark model state of the pion to coincide with the PCAC pion to provide an effective way to reconcile these two seemingly very different physical pictures of the pion. Furthermore, in view of the reasonable agreement of the pion decay constant and the masses of the mesons like ω, ρ and π in the non-strange flavour sector, with the corresponding experimental results it is reasonable to expect that the phenomenological effective central potential $V_q(r)$ confining the quarks in a baryon is about the same as the potential confining quark and anti-quark in a meson.

It is true that the linear potential used here is one of the many forms of the confining potential studied in the literature. All these potentials have exhibited almost equal success in explaining available data in different sectors which does not throw any light on the uniqueness of any of these potentials. In view of this one may conclude that the potentials are only a method of parametrization of the bound quark eigenmodes inside hadrons which ultimately decide the hadronic properties. Therefore it is the bound quark eigen modes representing the internal dynamics of hadron which should be unique as per the dictates of quantum chromo-dynamics. Therefore the uniqueness of the effective potential does not matter as long as it can reproduce the shape of the quark orbitals as close as possible to its reality demanded by the experimental data.

Acknowledgements

The authors are thankful to N Barik, Mayurbhanj Professor of Physics, Utkal University, Orissa, India for his valuable suggestions and useful discussions on this work.

References

[1] A Chodos, R L Jaffe, K Johnson, C B Thorn and V F Weisskpof, *Phys. Rev.* **D9**, 3471 (1974)
A Chodos, R L Jaffe, K Johnson and C B Thorn, *Phys. Rev.* **D10**, 2599 (1974)
T De Grand, R L Jaffe, K Johnson and I Kiskis, *Phys. Rev.* **D12**, 2060 (1975)
[2] N Barik and S N Jena, *Phys. Rev.* **D24**, 680 (1981); **D12**, 618 (1982)
[3] T De Grand, R L Jaffe, K Johnson and I Kiskis, *Phys. Rev.* **D12**, 2060 (1975)

[4] J F Donoghue and K Johnson, *Phys. Rev.* **D21**, 1975 (1980)
[5] C E De Tar, *Phys. Rev.* **D24**, 752 (1981)
[6] S N Jena, M R Behera and S Panda, *Phys. Rev.* **D54**, 11 (1996)
P Leal Ferreira, *Lett. Nuovo Cimento; Phys. Rev.* **D20**, 157 (1977); **20**, 511 (1977)
[7] S N Jena and S Panda, *Pramana – J. Phys.* **35**, 21 (1990); *Int. J. Mod. Phys.* **A7**, 2841 (1992a); *J. Phys.* **G18**, 273 (1992b)
[8] C W Wong, *Phys. Rev.* **D24**, 1416 (1981)
[9] A Chodos and C B Thorn, *Phys. Rev.* **D12**, 2733 (1975)
G E Brown, M Rho and V Vento, *Phys. Lett.* **B84**, 383 (1979a)
G E Brown and M Rho, *Phys. Lett.* **82**, 177 (1979b)
V Vento, J C Jun, E M Nyman, M Rho and G E Brown, *Nucl. Phys.* **A345**, 413 (1980)
S Theberge, A W Thomas and G A Miller, *Phys. Rev.* **D22**, 2238 (1980)
S Theberge, A W Thomas and G A Miller, *Phys. Rev.* **D23**, 2106 (1981)
S Theberge, A W Thomas and G A Miller, *Phys. Rev.* **D23**, 216 (1981)
A W Thomas, *Adv. Nucl. Phys.* **13**, 1 (1983)
[10] A W Thomas, *Adv. Nucl. Phys.* **13**, 1 (1983)
S N Jena, M R Behera and S Panda, *Phys. Rev.* **D54**, 11 (1996)
[11] J F Donoghue and K Johnson, *Phys. Rev.* **D21**, 1975 (1980)
[12] E Peierls and J Yoccoz, *Proc. Phys. Soc. (London)* **70**, 381 (1957)
C E Carlson and M Chachkhunashvili, Nordita Rep. (1981) (Unpublished)
D L Hill and J A Wheeler, *Phys. Rev.* **D39**, 1102 (1953)
C W Wong, *Phys. Rep.* **C15**, 283 (1975)
[13] I Duck, *Phys. Lett.* **B77**, 283 (1977)
J Bertelski, A Szymacha, L Mankichriz and S Tatur, *Phys. Rev.* **D29**, 1035 (1984)
E Eich, D Rein and R Rodenberg, *Z. Phys.* **C28**, 225 (1985)
N Barik and B K Dash, *Phys. Rev.* **D33**, 1925 (1986)
[14] H G Dosch and V F Muller, *Nucl. Phys.* **B116**, 470 (1976)
[15] Amendalia *et al*, *Phys. Lett.* **B138**, 454 (1984)