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Hierarchies of non-classical states in quantum optics

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Abstract. The conventional separation of states of the quantised radiation field into "classical" and "nonclassical" types is expressed in a dual operator form and then refined. This is based on new features of the normal ordering rule for passage from classical to quantum dynamical variables. The cases of single and two-mode radiation fields are discussed.

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1. Introduction

The purpose of this presentation is to look at some old ideas in quantum optics from a new point of view, and to refine them in an interesting way that may lead to improved understanding and insight. We are concerned with the description and a new classification of states of the quantised electromagnetic field, and drawing out the experimental implications of such classification. In particular it is interesting to see how one can bring out as sharply as possible those features that show the nonclassical properties of radiation.

Given a quantum mechanical state of radiation produced in some way, imagine that a specific set of measurements is made. It may happen that these results are explainable within a classical statistical framework, in which case we may conclude that these measurements have not revealed the quantum nature of radiation. But there may be other measurements that can be carried out on the same state which are not explainable in this manner; and then we are entitled to call the state "genuinely nonclassical".

It must be emphasized that these ideas and categorisations of states are within the overall framework of quantum theory. This implies that they are ultimately based on some physically well-motivated convention. This convention is an old one, and is closely connected to the normal ordering rule for passage from classical to quantum dynamical variables. In the sequel we will recall the statement of the convention directly in terms of states, and then develop a dual operator form. It is the latter that suggests interesting refinements with attendant consequences.

2. Single mode radiation field states

We consider to begin with a single mode quantized radiation field. The photon creation and annihilation operators $\hat{a}^\dagger, \hat{a}$ obey the canonical commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1\cdot \tag{1}$$

Now the diagonal coherent state representation theorem [1] states that any density matrix $\hat{\rho}$ for this system can always be expanded in terms of projections onto the coherent states:

$$\hat{\rho} = \int \frac{d^2 z}{\pi} \phi(z) |z\rangle\langle z|\cdot \tag{2}$$

Here $\phi(z)$ is a real weight function which represents $\hat{\rho}$, and the coherent states are normalised eigenstates of $\hat{a}$ defined in the usual manner:

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad \hat{a}|z\rangle = z|z\rangle, \qquad z \in \mathcal{C}. \tag{3}$$

The object $\phi(z)$ is in general a distribution, and the extent to which it can be singular can be precisely characterized [2]. If we switch to the real and imaginary parts of z by $z = (x + iy)/\sqrt{2}$ and then to their Fourier conjugates σ and τ, we have the statement:

$$\text{Fourier transform of } \phi(z) = e^{(\sigma^2+\tau^2)/4} \times \text{(square integrable function of } \sigma \text{ and } \tau) \tag{4}$$

We will see examples later where this degree of singularity is indeed present.

The conventional distinction between "classical" and "nonclassical" $\hat{\rho}$ is stated in terms of $\phi(z)$ [3]:

$$\begin{aligned} &\hat{\rho} \text{ "classical"} \Leftrightarrow \phi(z) \geq 0, \text{ not more singular than a delta function;} \\ &\hat{\rho} \text{ "nonclassical"} \Leftrightarrow \phi(z) \not\geq 0, \text{ possibly more singular than a delta function.} \end{aligned} \tag{5}$$

The motivation, as is well known, is that if $\hat{\rho}$ is "classical" in this sense, then all the *normal ordered correlation functions* can be reproduced by a suitable *classical statistical ensemble*. This as emphasized earlier is a convention but a reasonable one.

For $\hat{\rho}$ to be "classical", an infinite hierarchy of independent inequalities have to be obeyed; failure of any one of them is evidence of $\hat{\rho}$ being "nonclassical". To establish that one has a "classical" state – short of knowing $\phi(z)$ explicitly, which can be quite difficult – is therefore quite hard. Some of the familiar independent consequences of being "classical" are the quadrature fluctuation conditions $\Delta q \geq \frac{1}{\sqrt{2}}$, $\Delta p \geq \frac{1}{\sqrt{2}}$; the superpoissonian condition $(\Delta N)^2 > \langle N \rangle$ on the photon number distribution; the recently discovered local conditions on the photon number probabilities; etc. [4] On the other hand, by any reasonable method of counting, the vast majority of quantum states $\hat{\rho}$ are "nonclassical" – but they are quite hard to produce.

3. A dual operator approach for classification of states

Now we present a formulation of the distinction (5) in an operator form, based on properties of the normal ordering rule. Within quantum mechanics, as is well known, any operator $\hat{F}$ is uniquely determined by its diagonal coherent state matrix elements $\langle z|\hat{F}|z\rangle$, and moreover hermiticity of the former corresponds precisely to reality of the latter. Now the normal ordering rule of correspondence begins with any classical function $f(z^*, z)$, replaces z by $\hat{a}$ and z^* by $\hat{a}^\dagger$, and by placing every factor $\hat{a}^\dagger$ to the left of every factor $\hat{a}$ arrives at a uniquely determined quantum mechanical operator $\hat{F}_N$:

$$\begin{aligned} f(z^*, z) \to \hat{F}_N &= f\left(\hat{a}^\dagger \text{ to left}, \hat{a} \text{ to right}\right), \\ \langle z|\hat{F}_N|z\rangle &= f(z^*, z)\cdot \end{aligned} \tag{6}$$

So in any state $\hat{\rho}$ for the expectation value of $\hat{F}_N$ we have

$$\langle \hat{F}_N\rangle = \text{Tr}\left(\hat{\rho}\hat{F}_N\right) = \int \frac{d^2z}{\pi}\phi(z)f(z^*, z)\cdot \tag{7}$$

The correspondence $f \leftrightarrow \hat{F}_N$ is clearly linear, and as stated earlier reality goes over into hermiticity. But the key point for our purposes is that positivity is not preserved in the passage from classical f to quantum $\hat{F}_N$:

$$\begin{aligned} \hat{F}_N \geq 0 &\Rightarrow f \geq 0, \\ f \geq 0 &\not\Rightarrow \hat{F}_N \geq 0 \end{aligned} \tag{8}$$

Elementary examples showing this are the following:

$$\begin{aligned} f &= (z^* + z)^2 \leftrightarrow \hat{F}_n = \left(\hat{a}^\dagger + \hat{a}\right)^2 - 1, \\ f &= (z^* + z)^4 \leftrightarrow \hat{F}_N \left(\left(\hat{a}^\dagger + \hat{a}\right)^2 - 3\right)^2 - 6, \\ f &= e^{-z^* z}\sum_{n=0}^{\infty} \frac{C_n}{n!} z^{*n} z^n \leftrightarrow \hat{F}_N = \sum_{n=0}^{\infty} C_n |n\rangle\langle n|\cdot \end{aligned} \tag{9}$$

In the last case we can easily construct examples where f is nonnegative even though some coefficients C_n are negative. This means that (when the normal ordering rule is used) every positive operator $\hat{F}_N$ definitely arises from a positive classical function f, but some positive classical functions f do lead to indefinite $\hat{F}_N$: we may refer to this as "quantum negativity" permitted by the normal ordering rule – so in a state $\hat{\rho}$, the $\hat{F}_N$ corresponding to some classical nonnegative $f(z^*, z)$ may well have a negative expectation value. Combining eqs (5,7) we now see: a state $\hat{\rho}$ is "classical" if this permitted quantum negativity never shows up in expectation values, "nonclassical" if it does. This dual statement is exactly the same in content as the conventional statement (5), but now the focus is on expectation values of observables and not on $\phi(z)$.

Purely for purposes of comparison, we describe how positivity behaves in two other familiar rules of correspondence. With the antinormal ordering rule [5], we have

$$\begin{aligned} f(z^*, z) \to \hat{F}_A &= f\left(\hat{a}^\dagger \text{ to right}, \hat{a} \text{ to left}\right), \\ f \geq 0 \Rightarrow \hat{F}_A &= \int \frac{d^2z}{\pi} f(z^*, z)|z\rangle\langle z| \geq 0, \\ \hat{F}_A \geq 0 &\not\Rightarrow f \geq 0\cdot \end{aligned} \tag{10}$$

In the Weyl rule of correspondence [6], which in an algebraic sense is midway between the above two rules, positivity fails in both directions:

$$f(z^*,z) \to \hat{F}_W = f\left(\frac{\hat{q}-i\hat{p}}{\sqrt{2}}, \frac{\hat{q}+i\hat{p}}{\sqrt{2}}\right)\Bigg|_{\text{symmetrized in } \hat{q} \text{ and } \hat{p}};$$

$$f = \delta^{(2)}(z) \to \hat{F}_W = \text{parity operator},$$

$$\hat{F}_W = |1\rangle\langle 1| \to f(q,p) = \frac{2}{\pi}\left(q^2+p^2-\frac{1}{2}\right)e^{-q^2-p^2}. \tag{11}$$

(For the Weyl rule, the hermitian quadrature components $\hat{q},\hat{p}$ are the natural variables).

4. A refinement of the classification scheme

Now we return to the normal ordering rule characterised by eqs (7,8), and motivate a refinement of the distinction between "classical" and "nonclassical" $\hat{\rho}$. Suppose we limit the class of operators $\hat{F}$ (we omit hereafter the subscript N) being measured in some well-defined way. Specifically, consider all classical $f(z^*,z)$ which are phase invariant:

$$f\left(z^* e^{-i\alpha}, z\, e^{i\alpha}\right) = f(z^*,z)\cdot \tag{12}$$

A complete independent basis for such f, and the corresponding $\hat{F}$'s, is given by

$$f_n(z^*,z) = e^{-z^*z}\frac{z^{*n}z^n}{n!} \Longleftrightarrow \hat{F}^{(n)} = |n\rangle\langle n|\cdot \tag{13}$$

For expectation values of these $\hat{F}^{(n)}$ all the information contained in $\phi(z)$ is not needed as an angular average will suffice:

$$\langle \hat{F}^{(n)}\rangle = \langle n|\hat{\rho}|n\rangle = \int_0^\infty dI\cdot P(I)\cdot e^{-I}\cdot I^n/n!,$$

$$P(I) = \int_0^{2\pi} \frac{d\theta}{2\pi}\phi\left(I^{1/2}e^{i\theta}\right)\cdot \tag{14}$$

In general, of course, $P(I)$ is also a distribution which can be characterized in quite precise terms. This can be indicated in terms of the Fourier Bessel integral theorem for square integrable functions on the positive real line [7]:

$$f(I) = \int_0^\infty dK\cdot g(K)J_0(2\sqrt{IK}) \Leftrightarrow$$

$$g(K) = \int_0^\infty dI\cdot f(I)J_0(2\sqrt{KI}), \quad \int_0^\infty dI\cdot|f(I)|^2 = \int_0^\infty dK|g(K)|^2 \tag{15}$$

Then the consequence of eq. (4) for $\phi(z)$ reads as follows for $P(I)$:

$$P(I) = \text{Fourier Bessel transform of } e^{K/2} \times \text{(square integrable function of } K)\cdot \tag{16}$$

The situation is marginally better than for $\phi(z)$ as only the angular average of the latter is involved. Now while positivity of $\phi(z)$ implies that of $P(I)$, the converse does not hold. This motivates the introduction of a three fold classification, a refinement of (5):

$$\begin{aligned} \hat{\rho} \text{ ``classical''} &\Leftrightarrow \phi(z) \geq 0,\ P(I) \geq 0, \\ \hat{\rho} \text{ ``weakly nonclassical''} &\Leftrightarrow \phi(z) \ngeq 0, P(I) \geq 0, \\ \hat{\rho} \text{ ``strongly nonclassical''} &\Leftrightarrow \phi(z) \ngeq 0, P(I) \ngeq 0 \end{aligned} \tag{17}$$

In comparing eqs (5) and (16), we see that the "classical" case remains unchanged. However the previous "nonclassical" is split up into the "weakly nonclassical", in which all phase insensitive measurements seem classical, while some phase sensitive ones reveal nonclassicality; and the "strongly nonclassical" where even some phase insensitive measurements show up "quantum negativity".

The motivation is always the question: does the "quantum negativity" permitted by normal ordering show up in expectation values or not? And the answer, naturally, depends on the class of measurements contemplated.

An instructive example of a system which causes a transition from the "classical" to the "weakly nonclassical" regime is given by the Kerr Hamiltonian,

$$H_{\text{Kerr}} = \alpha\, \hat{a}^\dagger \hat{a} + \beta \left(\hat{a}^\dagger \hat{a}\right)^2 \tag{18}$$

which conserves photon number. We start with a coherent state $|z_0\rangle$ which has $\phi(z) = \delta^{(2)}(z - z_0),\ P(I) = \delta(I - |z_0|^2)$ and so is "classical". Evolution under the Hamiltonian (18) conserves photon number probabilities and so $P(I)$ as well. However $\phi(z)$ goes into the weakly nonclassical regime as the following arguments show. For any state, positivity of $\phi(z)$ trivially implies that of the associated Wigner function $W(q,p)$. For a pure state, Hudson's Theorem [8] shows that a positive Wigner function implies that both $W(q,p)$ and the wave function $\psi(q)$ are Gaussian. Now the initial coherent state $|z_0\rangle$ has a Gaussian $\psi_0(q)$, but this Gaussian nature is destroyed by evolution under H_{Kerr}. Since however the state remains pure, we conclude that $W(q,p)$ cannot be positive, hence $\phi(z)$ has lost its positivity as well.

Another useful example of seeing how our classification works uses the family of states with Gaussian Wigner distributions [9]. Here analytic results can be presented. We limit ourselves to centred Gaussians separable in the quadrature variables q and p. These form a two-parameter family, the parameters being the spreads in q and p:

$$\begin{aligned} W(q,p) &= \frac{1}{2\pi\alpha\beta} \exp\left(-\frac{q^2}{2\alpha^2} - \frac{p^2}{2\beta^2}\right), \\ \alpha &= \Delta q\ ,\ \beta = \Delta p; \quad \alpha\,\beta \geq 1/2 \end{aligned} \tag{19}$$

The inequality on the product $\alpha\beta$ is the uncertainty principle; for nonsqueezed states we have individually $\alpha, \beta > \frac{1}{\sqrt{2}}$, while if one of them is less than $\frac{1}{\sqrt{2}}$ we have squeezing.

Purely formally in the spirit of eq. (4) we have:

$$\phi(z) = \int\int \frac{d\sigma d\tau}{2\pi} e^{i(\sigma x - \tau y)} e^{-\frac{1}{4}(2\alpha^2-1)\sigma^2 - \frac{1}{4}(2\beta^2-1)\tau^2},$$
$$z = (x+iy)/\sqrt{2}. \tag{20}$$

In the nonsqueezed case, $\alpha, \beta > 1/\sqrt{2}$, $\phi(z)$ exists as a finite nonnegative function,

$$\phi(z) = 2[(2\alpha^2-1)(2\beta^2-1)]^{-1/2} \times \exp\left(-\frac{x^2}{2\alpha^2-1} - \frac{y^2}{2\beta^2-1}\right), \alpha, \beta > 1/\sqrt{2} \tag{21}$$

and one has $P(I) > 0$ as well. So in our classification the state is "classical". However the moment one of α and β dips below $1/\sqrt{2}$ and the state is squeezed, we see that the Fourier transform of $\phi(z)$ is an exploding Gaussian in one variable, so it is a distribution of the kind allowed by eq. (4). A detailed study shows that $P(I)$ also becomes a distribution, so with the onset of squeezing we have a sudden jump from the "classical" to the "strongly nonclassical" case, completely omitting the "weakly nonclassical" option.

5. Generalisation to the two-mode case

Finally we indicate briefly the generalisation of these ideas to the two-mode case. Here naturally there is a richer classification. The diagonal representation involves an object $\phi(z_1, z_2)$ dependent on two complex variables. Depending on the class of observables being measured, the amount of information needed varies. We have this situation:

To measure	Need
All operators	$\phi(z_1, z_2)$
Operators conserving total photon number $\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$	$\mathcal{P}(I_1, I_2, \theta) = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \phi\left(I_1^{1/2} e^{i\theta_1}, I_2^{1/2} e^{i(\theta_1+\theta)}\right)$
Operators conserving individual photon numbers $\hat{a}_1^\dagger \hat{a}_1$ and $\hat{a}_2^\dagger \hat{a}_2$	$P(I_1, I_2) = \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{P}(I_1, I_2, \theta)$

This motivates a four-fold classification of states as follows:

$\hat{\rho}$	ϕ	$\mathcal{P}$	P
Classical	≥ 0	≥ 0	≥ 0
Weakly nonclassical I	$\not\geq 0$	≥ 0	≥ 0
Weakly nonclassical II	$\not\geq 0$	$\not\geq$	≥ 0
Strongly nonclassical	$\not\geq 0$	$\not\geq 0$	$\not\geq 0$

These are exhaustive and mutually exclusive – as one goes down the table, the states become progressively more nonclassical as more and more operators are available to show the "quantum negativity".

The pair coherent states [10] can be shown to be neither classical nor even weakly nonclassical-I. So also for the squeezed thermal state for strong enough squeezing.

In conclusion, we have presented a new look at old things in quantum optics, a way to relate the extent of nonclassicality of the state to the set of measurements being made. This highlights the "quantum negativity" idea implicit in the normal ordering rule, and helps us classify and discriminate among states in a more detailed and richer way than before.

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References

[1] E C G Sudarshan, *Phys. Rev. Lett.* **10**, 277 (1963)
R J Glauber, *Phys. Rev.* **131**, 2766 (1963)

[2] See, for instance, J R Klauder and E C G Sudarshan, *Fundamentals of quantum optics* (Benjamin, New York, 1968)

[3] M C Teich and B E A Saleh, in *Progress in Optics*, edited by E Wolf (North-Holland, Amsterdam, 1988) Vol. 26
D F Walls, *Nature* **280**, 451 (1979)

[4] D F Walls in ref.(3) above
L Mandel, *Opt. Lett.* **4**, 205 (1979)
G S Agarwal and K Tara, *Phys. Rev.* **A46**, 485 (1992)
Arvind, N Mukunda and R Simon, *J. Phys.* **A31** 565 (1998)

[5] C L Mehta and E C G Sudarshan, *Phys. Rev.* **B138**, 275 (1965)

[6] H Weyl, *The theory of groups and quantum mechanics* (Dover, New York, 1931), p.275

[7] N N Lebedev, *Special functions and their applications* (Dover, New York, 1972), p.130

[8] R L Hudson, *Rep. Math. Phys.* **6**, 249 (1974)

[9] Arvind, N Mukunda and R Simon, *Phys. Rev.* **A56** 5042 (1997)

[10] D Bhaumik, K Bhaumik and B Dutta-Roy, *J. Phys.* **A9**, 1507 (1976)
G S Agarwal, *Phys. Rev. Lett.* **57**, 827 (1986) and JOSA **B5**, 1940 (1988)