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Aspects of Planckian scattering beyond the eikonal

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Abstract. We discuss an approach to compute two-particle scattering amplitudes for spinless light particles colliding at Planckian centre-of-mass energies, with increasing momentum transfer away from the eikonal limit. The leading corrections to the eikonal amplitude, in our 'external metric' approach, are shown to be vanishingly small in the limit of the source particle mass going to zero. For massless charged particles, the electromagnetic and gravitational interactions decouple in the eikonal limit, but mix non-trivially for the leading order corrections.

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1. Introduction

The efficacy of the shock wave picture [1] in the computation of two-particle scattering amplitudes [2] for large s (squared centre-of-mass energy) and small, fixed t (squared momentum transfer), in the eikonal limit $s/t \to \infty$, is now well-established, both for gravitational and electromagnetic interactions [3–8]. The graviton (photon) exchange ladder graphs neatly sum in this kinematical limit to reproduce exactly the semiclassical amplitude of the relatively slower test particle scattering off the gravitational (electromagnetic) shock wave due to the ultrarelativistic 'source' particle. Phenomena beyond this highly restrictive kinematical regime (e.g., for higher values of t) entail, for their analysis, a calculational scheme for systematic corrections to the eikonal. For electrodynamics, with values of t still sub-Planckian, this is afforded easily by the usual perturbative formulation of quantum electrodynamics. For gravity and electrodynamics of electric *and* magnetic charges, the lack of a proper local quantum field theory is a major setback to this programme. On the other hand, a determination of corrections to the eikonal is essential to unravel certain features of eikonal scattering itself, like the analytic structure (in complex s-plane) of the eikonal amplitude [2], or the possible interplay between electromagnetic and gravitational effects for charged particle eikonal scattering [9]. One approach which has probed the first of these features with some success is the one based on reggeized string exchange amplitudes with subsequent reduction to the gravitational eikonal limit including the leading order corrections [3]. In this letter, we follow a somewhat different approach: the scattering amplitude is calculated quantum *mechanically* by solving the Klein–Gordon equation of the ultrarelativistic particle in the classical Schwarzschild background of the slower 'target' particle in the appropriate Lorentz frame. Recall that the role of the scattering particles is the opposite to that in the shock wave picture [2] where the slower particle scatters off the shock wave due to the luminal one. But this switching allows us to investigate leading corrections to the eikonal. In solving for the wave function of the ultrarelativistic test particle, we include only up to the leading order non-linear term in the Schwarzschild metric; the subleading non-linearities do not affect the leading corrections to the eikonal amplitude. This means, of course, that our approach is restricted to impact parameters that are still very large compared to the Planck scale. Smaller impact parameter scattering lies beyond this quantum mechanical approach, necessitating a full-fledged quantum gravity theory:

2. Purely gravitational scattering

In the classical Schwarzschild background of the slow target particle (of mass M also considered small in comparison with $\sqrt{s}$)

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \qquad (1)$$

the radial part of the Klein–Gordon equation for an ultrarelativistic test particle is given by,

$$\frac{d^2f}{dr^2} + \frac{2GM}{r^2}\left(1-\frac{2GM}{r^2}\right)^{-1}\frac{df}{dr} - \left(1-\frac{2GM}{r}\right)^{-1}\left[\frac{l(l+1)}{r^2} + \frac{2GM}{r^3} - \left(1-\frac{2GM}{2}\right)^{-1}E^2\right]f = 0, \qquad (2)$$

where, E is the energy of the test particle measured in the frame of reference of the 'source' particle. Next, we linearize the metric owing to the assumed smallness of M, substitute $s = 2ME$ and finally take the limit $M \to 0$. Since, as already stated, our interest lies in the limit of large impact parameter b and hence large $l, l \equiv bE$, the relevant terms in the radial equation are

$$\frac{d^2f}{dr^2} - \left[\frac{l(l+1)}{r^2} - \frac{2GsE}{r} - E^2\right]f - \left[\frac{2GMl(l+1)}{r^3} - \frac{3(Gs)^2}{r^2}\right]f = 0. \qquad (3)$$

Here, only terms $O(r^{-3})$ or longer in range have been included in the effective potential, because shorter range terms are smaller in comparison. This is seen trivially by considering ratios of various terms and using the inequality $r \geq b$ and also the smallness of M. Thus, terms in the first square brackets are the only relevant ones in the eikonal limit, and those in the second square bracket constitute the leading corrections. In fact, the second term in the second square bracket is much smaller compared to the r^{-3} term, and the only reason it is retained is because its radial dependence (r^{-2}) allows it to be included in an exact calculation of the phase shift. In contrast, the $1/r^3$ term cannot be handled exactly, and we perform a Born approximation estimate of the phase shift correction due to it.

As mentioned, the eikonal amplitude is obtained from an exact solution of the radial equation with the second curly bracket dropped; it is a purely Coulomb scattering problem which has the well-known solution for the phase shift [12, 13]

$$\delta_l^{\text{eik}}(s) = \arg\Gamma(l+1-iGs) \tag{4}$$

in the partial wave expansion

$$f(\theta) = \frac{1}{2i\sqrt{s}}\sum_{l=0}^{\infty}(2l+1)[e^{2i\delta_l}-1]P_l(\cos\theta) \tag{5}$$

for the scattering amplitude. The eikonal amplitude can actually be expressed succinctly as [2]

$$f_{\text{eik}} = \frac{Gs^{3/2}}{2t}\frac{\Gamma(1-iGs)}{\Gamma(1+iGs)}\left(\frac{-t}{s}\right)^{iGs} \tag{6}$$

As already stated, the leading order correction to the eikonal amplitude comes from the first term in the second curly bracket in eq. (3). We estimate the correction to the phase shift due to this term, using the standard Born approximation formula for the phase shift for short ranged (i.e., falling off faster than $1/r^2$ at $r \to \infty$) potentials $V(r)$ [10]

$$\delta_l \approx 2ME\int_0^{\infty}\mathrm{d}r\, r^2 V(r) j_l(kr)\frac{R_l(r)}{r}, \tag{7}$$

where, $j_l(r)$ is the usual spherical Bessel function; for $R_l(r)$ here we must use the solution to the radial equation in the eikonal limit, given by [13]

$$\frac{R_l(r)}{r} = f(l,\lambda)(2kr)^l e^{ikr}F(-i\lambda+l+1;2l+2;-2ikr), \tag{8}$$

with $\lambda \equiv Gs$ and

$$f(l,\lambda) \equiv \frac{\exp(\pi\lambda/2)|\Gamma(l+1-i\lambda)|}{(2l+1)!}. \tag{9}$$

Using relations between hypergeometric and Bessel functions [14] eq. (8) can be rewritten as

$$\frac{R_l(r)}{r} = [f(l,\lambda)\Gamma(l+3/2)]2^{2l+1/2}\frac{J_{l+1/2}(kr)}{\sqrt{kr}}, \tag{10}$$

where $J_{l+1/2}(kr)$ is the standard Bessel function. Substituting this in eq. (7), using standard formulas involving Bessel functions [14] and evaluating the integral in the limit of large l (adequate for use of the Stirling formula for estimating factorials), we obtain the leading correction to the eikonal phase shift as

$$\delta_l^{\text{Born}} \approx 2e^{((\pi\lambda/2)-(3/2))}GM^2. \tag{11}$$

Clearly, this is vanishingly small in the limit $M \to 0$. Thus, the leading correction to the eikonal approximation yields a null result in our external metric approach [14].

The effect of the $(Gs)^2/r^2$ term in eq. (3) is accounted for simply by combining it with the centrifugal term, and yields the phase shift

$$\delta_l(s) = \arg \Gamma(p_l(s) + 1 - iGs), \tag{12}$$

where, $p_l(s) \equiv l(l+1) - 3(Gs)^2$. The eikonal limit corresponds to $p_l = l$, so that the effect of this correction term in the effective potential can be determined by a Taylor expansion of the argument of the gamma function around the eikonal. This yields a correction to the phase shift of $O((Gs)^3/l^2)$. The effect of this, together with the correction in eq. (11) summarises into the net phase shift

$$\delta_l(s) \approx \delta_l^{\mathrm{eik}}(s) + C_1 GM^2 + C_2 G_s \left(\frac{GM}{b}\right)^2. \tag{13}$$

Here, C_1 and C_2 are known constants of $O(1)$, and we have used the relation $l = bE$. It is easy to see that in the limit of $M \to 0$ and $b \gg G^{(1/2)}$, both corrections to the eikonal phase shift vanish smoothly. For finite (but small) M, the leading mass corrections to the eikonal are determined as above. The first of these agrees with the leading order mass correction term in [7] when their result for the scattering amplitude is expanded in powers of M.

The above analysis re-emphasises the strength of the eikonal approximation and also the difficulties to go beyond it in the external metric formalism. Thus, we have very little to add to the extant wisdom [3] on the issue of singularities of the eikonal amplitude. However, as advertised in the introduction, our approach does have the merit of clarifying and elucidating certain features of the eikonal approximation itself, as we now proceed to discuss.

3. Inclusion of electromagnetism

A key issue in Planckian scattering, and one which has not received too much attention, is that of the mixing of gravitational and electromagnetic effects in the eikonal approximation. In the earlier literature [2, 8], it was *assumed* that in the eikonal limit, the gravitational and electromagnetic shock waves acted quite independently, producing a net phase factor in the wave function of the test particle that was a sum of the individual phase factors. Since generically gravity couples to everything including electromagnetism, it becomes important to ascertain whether the assumed independence of the two interactions in the special kinematics of the eikonal limit really holds. This issue was first addressed in [9] where heuristic arguments were advanced to show that the assumed decoupling did indeed take place, thus vindicating results obtained using this crucial assumption. The present framework provides a less heuristic avenue to re-examine this question, and allows us to establish the earlier conclusions on a sounder footing.

As in [9], one begins by considering first the scattering of a (luminal) neutral test particle off the Reissner–Nordström metric due to a static point charge Q,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{2} + \frac{GQ^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{14}$$

where $d\Omega^2$ is the metric on the unit two-sphere. The radial part of the Klein–Gordon equation for a neutral ultrarelativisitic test particle is given by

$$\frac{d^2f}{dr^2} - \left(1 - \frac{2GM}{2} + \frac{GQ^2}{r^2}\right)^{-1}\left[\frac{l(l+1)}{r^2} - E^2\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}\right] f = 0. \tag{15}$$

As in the last section, we use the approximations $b \gg 2GM$ with the source particle mass M being infinitesimally small, and drop terms with a range shorter than $1/r^3$ in the radial equation; it is easy to convince oneself that these terms are smaller than the ones retained by considering ratios of various terms in our approximation. The resulting radial equation reduces to

$$\frac{d^2f}{dr^2} - \left[\frac{l(l+1)}{r^2} - \frac{2GsE}{r} - E^2\right] f - \left[\left\{\frac{2G(QE)^2}{r^2} + \frac{2GMl(l+1)}{r^3}\right\} - 3\frac{(Gs)^2}{r^2} - 12\frac{M(GQE)^2}{r^3}\right] f = 0. \tag{16}$$

The terms in the effective potential (the term in square brackets) have been written in *decreasing* order of strength. The terms in the first set of square brackets correspond to the potential to be used in the eikonal approximation, implying that the phase shift in the eikonal approximation is identical to that obtained in the Schwarzschild case. This means that, in the eikonal limit, the charge of the source particle plays no role, and hence there is no interplay between gravity and electromagnetism in this limit, a result that was established earlier [9] on a somewhat heuristic basis.

The curly brackets within the second square brackets in eq. (16) include corrections to the eikonal. *Unlike* in the Schwarzschild case, the leading correction to the 'eikonal' effective potential in this case comes from a term of type $G(QE)^2/r^2$, which can be used in an exact calculation of the phase shift. Before we proceed to estimate the correction to the phase shift due to this term, we observe that this correction *indeed involves coupling of gravitational and electromagnetic effects*. The calculation itself is a repeat of that performed in the last section to include the $(Gs)^2/r^2$ term; one obtains, in this case

$$\delta_l(s) \approx \arg\Gamma(p_l(s) + 1 - iGs), \tag{17}$$

where, $p_l(s)(p_l(s) + 1) \equiv l(l+1) + G(QE)^2$. As before, the eikonal limit corresponds to $p_l = l$. Taylor expanding around this limit, we obtain the leading correction to the phase shift to be of $O(GsQ^2(G/b^2))$. Although this is a small correction for generic values of the charge ($Q^2 \approx 1/137$) and large b, it does not vanish when $M \to 0$. Thus, in this case, our external metric approach does lead to a non-vanishing correction even for the massless limit of the source particle. Of course, the subleading corrections, corresponding to the other terms in the effective potential, resemble the corrections in the Schwarzschild case, needing no further discussion.

It can be shown that, the effect of a charge (Q' say) on the luminal test particle on the eikonal phase shift is simply accounted for by the shift $Gs \to Gs - QQ'$, as anticipated in ref. [2] and expected from the decoupling of gravity and electromagnetism observed above. Of course, the leading corrections in this case will involve the coupling of these two forces, just as we saw earlier.

4. Conclusions

While our (semi-classical) method is adequate to deal with the scattering process at Planckian centre-of-mass energies in the eikonal regime, strictly speaking, the corrections to the eikonal phase shifts that we have calculated are expected to be affected non-trivially under true quantum gravitational effects, similar to the inevitable necessity of field theoretic quantum electrodynamics for a proper calculation of the Lamb shift. The difference here is the lack of an appropriate quantum 'gravidynamics' which can be reliably used for computation. Since the issue at hand seemingly entails uncontrollable ultraviolet behaviour of a local field theoretic formulation of gravity, starting from Einsteinian general relativity, the use of string theory to tame these divergences is certainly an attractive option. On the other hand, the robustness of the eikonal amplitude may indicate certain non-perturbative aspects of spacetime geometry at short distances which may not be analyzable in terms of perturbative string theory.

It is satisfying to note that our earlier heuristic analysis on non-mixing of electromagnetic and gravitational effects for eikonal scattering [9] can indeed be placed on firmer footing. The correction terms are, however, affected by the charges in a non-trivial way, and once again, to have a proper appreciation of these terms, field theoretic effects must be accounted for.

Finally, a word about dilaton gravity. The same heuristic arguments which enable us to show the decoupling of gravity and electromagnetism in general relativity, leads to a non-trivial mixing of these interactions even in the eikonal approximation for the case of dilaton gravity [9]. This also seems to be the case when the technique of this paper is applied to dilaton gravity. One is left with the disturbing possibility that the inclusion of the dilaton might actually make the eikonal limit non-existent! We hope to report on this elsewhere in the near future.

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