PRAMANA
—journal of
physics

© Printed in India

Vol. 50, No. 4,
April 1998
pp. 307–314

Schwarzschild black hole with global monopole charge

NARESH DADHICH, K NARAYAN* and U A YAJNIK*
Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India
*Department of Physics, Indian Institute of Technology, Mumbai 400 076, India
E-mail: nkd@iucaa.ernet.in; narayan@tristan.TN.CORNELL.EDU;
yajnik@niharika.phy.iitb.ernet.in

MS received 2 September 1997

Abstract. We derive the metric for a Schwarzschild black hole with global monopole charge by relaxing asymptotic flatness of the Schwarzschild field. We then study the effect of global monopole charge on particle orbits and the Hawking radiation. It turns out that existence, boundedness and stability of circular orbits scale up by $(1 - 8\pi\eta^2)^{-1}$, and the perihelion shift and the light bending by $(1 - 8\pi\eta^2)^{-3/2}$, while the Hawking temperature scales down by $(1 - 8\pi\eta^2)^2$ the Schwarzschild values. Here η is the global charge.

Keywords. Black hole; Schwarzschild solution; global monopole; Hawking radiation; particle orbits; topological defect.

PACS Nos 04.20; 14.80; 04.50

1. Introduction

The Schwarzschild solution is the unique spherically symmetric solution of the vacuum Einstein equation. It represents a static black hole. It turns out [1] that a global monopole charge accompanies spontaneous breaking of global 0(3) symmetry into $U(1)$ in phase transitions in the Universe. Putting on global charge to the Schwarzschild black hole will amount to breaking the vacuum and asymptotic flatness of the Schwarzschild spacetime. It still represents a localized object with a horizon. Our main aim in this paper is to study the effect of global monopole charge on particle orbits and the Hawking radiation.

The spacetime of pure global monopole charge, when the Schwarzschild mass vanishes, can be regarded in some sense to be 'minimally' curved for gravitational charge density, $4\pi\rho_c = R_{ik}u^iu^k, u^iu_i = 1$, vanishes. That means the relativistic (active gravitational) mass is zero for the spacetime [2]. All radial trajectories will always remain parallel. Thus introduction of global monopole charge does not significantly alter the nature of the Schwarzschild field. We wish to investigate what effect does its introduction entail on physically measurable quantities. Recently a similar investigation has been carried out for a global monopole in the Kaluza–Klein spacetime [3].

In § 2 we shall briefly outline the derivation of the metric for a static black hole with a global monopole charge followed in § 3 by field theoretic considerations. The effect of global charge on particle orbits and the Hawking radiation process will be investigated in § 4 and § 5. We conclude with a discussion in § 6.

2. The metric

Let us write the general spherically symmetric metric

$$ds^2 = B dt^2 - A dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.1}$$

where A and B are functions of r and t.

Now the equations $R_{01} = 0$ and $R_0^0 = R_1^1$ will imply $B = A^{-1} = 1 + 2\phi$, where $\phi = \phi(r)$ [4]. Note that no boundary condition has been used. With this

$$R_0^0 = R_1^1 = -\nabla^2\phi \tag{2.2}$$

and

$$R_2^2 = R_3^3 = -\frac{2}{r^2}(r\phi)', \tag{2.3}$$

where a dash denotes derivative with respect to r. Then $R_0^0 = 0$ gives the good old Laplace equation which has the well-known general solution

$$\phi = k - \frac{M}{r}. \tag{2.4}$$

Now $R_2^2 = 0$ will determine $k = 0$ and we obtain the Schwarzschild solution. Retaining k will make $R_2^2 \neq 0$ and will give rise to stresses,

$$T_0^0 = T_1^1 = -\frac{k}{4\pi r^2}. \tag{2.5}$$

These are precisely the stresses at large r required for a global monopole as we shall see in the next section. Here k will be related to the global monopole charge which produces neither acceleration nor tidal acceleration for radially moving free particles. Note that the gravitational charge density, $4\pi\rho_c = R_{ik}u^i u^k$, continues to remain zero indicated by vanishing of R_{00}.

3. The gravitational field of a global monopole

Several authors [1, 5, 6, and the references therein] have discussed the gravitational field of a global monopole formed by spontaneous symmetry breaking of a triplet of scalar fields with a global symmetry group $O(3)$. The Lagrangian density of the isoscalar triplet ψ^a with $a = 1, 2, 3$ is

$$\mathcal{L} = \frac{1}{2}(\partial\psi^a)^2 - \frac{\lambda}{4}(\psi^a\psi^a - \eta^2)^2. \tag{3.1}$$

Topologically non-trivial self-supporting solutions to this system can be found. The ansatz describing a monopole is

$$\psi^a(\bar{\mathbf{x}}) = \eta f(r)\frac{x^a}{|\bar{\mathbf{x}}|}, \tag{3.2}$$

where

$$x^a x^a = r^2, \tag{3.3}$$

with x^a-meaning the corresponding cartesian component of $\bar{\mathbf{x}}$. Outside the monopole core we must have

$$f(r) \to 1 \text{ as } |\bar{\mathbf{x}}| \to \infty \tag{3.4}$$

for $V(\psi)$ to vanish asymptotically. The vanishing of energy density is however not fast enough because

$$|\nabla\psi|^2 \to \frac{\eta^2}{r^2}. \tag{3.5}$$

The stress tensor of the system outside the core can be approximated as

$$T_0^0 = T_1^1 = \frac{\eta^2}{r^2} \tag{3.6}$$

and the other components vanish. These are precisely the stresses generated (eq. (2.5)) by keeping k in (2.4).

Thus the most general metric for $T_0^0 = T_1^1 \neq 0$ and the remaining stress components being zero is

$$\mathrm{d}s^2 = \left(1 + 2k - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 + 2k - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2, \tag{3.7}$$

with $k = -4\pi\eta^2$ and M having the usual meaning of mass of the central gravitating object [1]. Here then central mass is that of the monopole which is usually negligible. However if we consider the spherically symmetric gravitational collapse of the matter around such a monopole, we find that a black hole is formed. The parameter M then corresponds to the mass of the black hole.

Even when $M = 0$, this spacetime has non-zero curvature:

$$R_0^0 = R_1^1 = R_{01} = 0 \tag{3.8}$$

but

$$R_2^2 = \frac{8\pi\eta^2}{r^2}. \tag{3.9}$$

That the total energy of such solutions is divergent makes them unrealistic except perhaps as fleeting entities in the course of a cosmic phase transition [1]. The core mass M is nevertheless finite. It is interesting to note that retention of k in (2.4) amounts to giving up asymptotic flatness and that exactly generates the stresses of a global monopole.

A remarkable conclusion that follows from this is that a black hole can possess 'hair' in the form of topological charge. How will this charge be detected? Topologically nontrivial solutions are known to have the peculiar property that their symmetry generators are linear combinations of spacetime symmetries and internal symmetries. In the present case, it would mean that a motion on a sphere of constant radius r has to be accompanied by a suitable internal symmetry transformation. An asymptotic observer looking at particle-like excitations will find that if he carries the latter to new angular coordinate values, he has to reassign the internal charge carried by them. For the simplest spherically symmetric case considered, a closed trajectory in the group space corresponding to a $2\pi O(3)$ rotation will be traversed upon orbiting the black hole. Similar conclusions should follow for topological sectors of higher winding number, except that the spacetime metric will not be in the simple form corresponding to spherical symmetry.

Another interesting manifestation of the topological charge can be obstruction to global definition of the wave function of certain particle species. For example, if the symmetry group is really the spin group $SU(2)$ instead of $O(3)$ (more generally, spin (N) instead of $O(N)$) then an isospinor species χ may exist. Now a 2π space rotation will induce a $2\pi O(3)$ rotation which will change the sign of the wave function which therefore cannot be globally defined.

Note that the fate of the global charge S_z of the $O(3)$ group will be the same as that of any other global charge such as the baryon number.

4. Particle orbits

In this section we wish to investigate the effect of global charge on the particle orbits; existence, boundedness and stability of circular orbits, light deflection and perihelion-shift.

4.1 *Circular orbits*

For the metric (3.7), we have

$$\Delta \dot{t} = E, \quad r^2 \dot{\varphi} = l, \tag{4.1}$$

where $\Delta = 1 + 2k - 2M/r, \dot{t} = \mathrm{d}t/\mathrm{d}s$; E and l are specific energy and angular momentum of a test particle. Substituting (4.1) in the metric (we have set $\theta = \pi/2$ as usual),

$$\dot{r}^2 = E^2 - \Delta\left(\frac{l^2}{r^2} + 1\right) = E^2 - V^2. \tag{4.2}$$

For existence of circular orbit, we should have both $\dot{r} = \ddot{r} = 0$. This will give the existence threshold

$$r_e \geq \frac{3M}{1+2k}, \tag{4.3}$$

where equality refers to the photon orbit. Of course $l \geq 2\sqrt{3}M/(1+2k)$. For boundedness, we should have $E^2 \geq (1+2k)$, which leads to

$$r_b \geq \frac{4M}{1+2k} \tag{4.4}$$

while stability of circular orbits will further require $V'' \geq 0$, giving

$$r_s \geq \frac{6M}{1+2k}. \tag{4.5}$$

Thus, these thresholds simply indicate scaling of $M \to M/(1+2k) = M/(1 - 8\pi\eta^2)$.

4.2 *Light deflection*

Following the standard calculation [7], we write

$$\varphi(r) - \varphi(\infty) = \int_r^\infty \Delta^{-1/2}(r)\left[\left(\frac{r}{r_0}\right)^2\frac{\Delta(r_0)}{\Delta(r)} - 1\right]^{-1/2}\frac{\mathrm{d}r}{r}, \tag{4.6}$$

where r_0 is the radius vector of the closest approach to the gravitating body. The deflection angle will be given by

$$\delta\varphi = 2|\varphi(r_0) - \varphi(\infty)| - \frac{\pi}{\sqrt{1+2k}}. \tag{4.7}$$

Note that when $M = 0, \varphi$ has the period $2\pi/\sqrt{1+2k}$ and not 2π, there is a deficit angle similar to the case of cosmic string. From (4.6) and (4.7) we get

$$\delta\varphi = \frac{4M}{r_0(1+2k)^{3/2}} \approx \frac{4M}{r_0}(1 + 12\pi\eta^2). \tag{4.8}$$

4.3 *Perihelion-shift*

Here the corresponding relation [7] is

$$\varphi(r)-\varphi(r_-)=\int_{r_-}^{r}\left[\frac{r_-^2(\Delta^{-1}(r)-\Delta^{-1}(r_-))-r_+^2(\Delta^{-1}(r)-\Delta^{-1}(r_+))}{r_+^2r_-^2(\Delta^{-1}(r_+)-\Delta^{-1}(r_-))}-\frac{1}{r^2}\right]^{-1/2} \times\frac{\Delta^{-1/2}}{r^2}\mathrm{d}r, \tag{4.9}$$

where $r_\pm$ refer to aphelion and perihelion points. The perihelion shift is given by

$$\delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - \frac{2\pi}{\sqrt{1+2k}} \tag{4.10}$$

and we thus obtain

$$\delta\varphi = \frac{6\pi M}{L(1+2k)^3/2} \approx \frac{6\pi M}{L}(1 + 12\pi\eta^2), \tag{4.11}$$

where

$$L = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right).$$

Thus effect of global charge is to scale the Schwarzschild values by $(1 - 8\pi\eta^2)^{-1}$ for existence, boundedness and stability thresholds of circular orbits and by $(1 - 8\pi\eta^2)^{-3/2}$ for light deflection and perihelion-shift. It means a slight enhancement in the Schwarzschild values.

5. Semiclassical effects

Next we investigate the question of Hawking radiation in this background metric. Since the thermal nature of the radiation is tied to the existence of the horizon, we do not expect this to change. However the temperature parameter may carry a signature of the global monopole.

Consider the equation satisfied by a massless scalar quantum field ψ

$$\Box\psi = 0. \tag{5.1}$$

The advanced and retarded coordinates for this spacetime are

$$v = t + r^*, \tag{5.2}$$

$$u = t - r^*, \tag{5.3}$$

where r^* is the tortoise coordinate defined in this case as

$$\mathrm{d}r^* = \frac{r\,\mathrm{d}r}{1 + 2k - (2M/r)}. \tag{5.4}$$

The wave equation can be separated in (t, r, θ, φ) coordinates. The mode functions relevant to this spacetime are of the form

$$\frac{R_{\omega l}(r)}{r} Y_{lm}(\theta, \varphi)\mathrm{e}^{-i\omega t}. \tag{5.5}$$

Y_{lm} is a spherical harmonic and $R_{\omega l}$ satisfies the equation

$$\frac{\mathrm{d}^2 R_{\omega l}}{\mathrm{d}r^{*2}} + (\omega^2 - [l(l+1)r^{-2} + 2Mr^{-3}][1 + 2k - 2Mr^{-1}])R_{\omega l} = 0. \tag{5.6}$$

Because of the 'potential' term in square brackets, the standard incoming waves will partially scatter back the gravitational field to become a superposition of incoming and outgoing waves. However, the effective potential vanishes as $r^* \to \pm\infty$ (i.e. for $r \to 2M, \infty$). Thus, in those regions the mode functions will be of the form

$$\left(\frac{1}{r}\,\mathrm{e}^{-i\omega u} Y_{lm}\right) \text{ and } \left(\frac{1}{r}\,\mathrm{e}^{-i\omega v} Y_{lm}\right). \tag{5.7}$$

The surface gravity of the black hole formed as a result of the gravitational collapse is

$$\kappa = \frac{(1 + 2k)^2}{4M} = \frac{(1 - 8\pi\eta^2)^2}{4M}. \tag{5.8}$$

This redefinition of the surface gravity reduces the calculation to that of the Schwarzschild case. With this redefinition, the Penrose diagram of the present system is identical to that of the Schwarzschild case.

As in the Schwarzschild case, assuming there are no bound states at late times, the ingoing and outgoing modes form a complete basis in terms of which the field at late times can be expanded. Then an outgoing mode of the form $r^{-1}\mathrm{e}^{-i\omega u} Y_{lm}$ at late times can be traced back to early times in the same fashion as in the Schwarzschild case. At early times, the ray, moving along constant phase v lines will have the form $\mathrm{e}^{-i\omega u}(v)$. The affine parameter on the past horizon can be chosen as

$$\lambda = -c\mathrm{e}^{-\kappa u}. \tag{5.9}$$

Then the function

$$u(v) = -\frac{1}{\kappa}\ln\left[\frac{v_0 - v}{\text{const}}\right]. \tag{5.10}$$

This can be Bogolubov transformed in the same manner as the Schwarzschild case to give the Planckian spectrum

$$N_\beta = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{5.11}$$

$$= \left(e^{8\pi M\omega/(1+2k)^2} - 1\right)^{-1}. \tag{5.12}$$

Thus the thermal spectrum is recovered but the temperature is now $(1 - 8\pi\eta^2)^2/8\pi M$.

Consider next the quanta of the ψ field responsible for the monopole. The equations satisfied by

$$\tilde{\psi}^a = \psi^a - \psi^a_{bg} \tag{5.13}$$

are, after linearization,

$$\Box\tilde{\psi}^a + 2\lambda\eta^2\psi^a_{bg}\left(\sum_b \psi^b_{bg}\tilde{\psi}^b\right) = 0. \tag{5.14}$$

Asymptotically the equation becomes

$$\Box\tilde{\psi}^a + 2\lambda\eta^2\frac{x^a}{r^2}\left(\sum_b x^b\tilde{\psi}^b\right) = 0. \tag{5.15}$$

Consider the $a=3$ mode and the asymptotic z direction, the other two (x and y) remaining finite. Then the equations become

$$\Box\tilde{\psi}^3 + 2\lambda\eta^2\tilde{\psi}^3 = 0, \tag{5.16}$$

$$\Box\tilde{\psi}^1 = 0, \tag{5.17}$$

$$\Box\tilde{\psi}^2 = 0. \tag{5.18}$$

This corresponds to the radial mode being massive but the two transverse modes being massless. Thus the preceding analysis will apply to the two transverse modes as well. We see that an imprint of the topological charge of the black hole is left on the Hawking radiation as well, namely under angular displacement, different components of the ψ triplet will be detected.

Thus the fate of our black hole is similar to that of the magnetically charged black holes [8]. While the question of its ultimate fate cannot be settled within the semiclassical approximation, the answer will be the same as for a Schwarzschild black hole. If it can in fact evaporate completely, and does so, then it leaves behind a monopole identical to the original one before collapse.

In concluding this section, we note that the persistence of the thermal radiation is not surprising. It is known from the work in axiomatic field theory [9] that when horizons exist, a unique nontrivial automorphism exists for the ground state. The automorphism works out to be periodicity in time, leading to the Kubo–Martin–Schwinger boundary condition on the Green function. There has also been extensive work on thermodynamics of black holes with gauge cosmic strings and global monopoles [10, 11].

6. Discussion

The main purpose of this investigation was to study the effect of global monopole charge on particle orbits and the Hawking radiation. It turns out that existence, boundedness and stability threshold for circular orbits scale up by $(1 - 8\pi\eta^2)^{-1}$, perihelion shift and light bending by $(1 - 8\pi\eta^2)^{-3/2}$, while the Hawking temperature scales down by $(1 - 8\pi\eta^2)^2$ the Schwarzschild values. This is how the global monopole charge η affects the particle orbits and the Hawking radiation thermal spectrum. It may be noted that the event horizon is given by $r = 2M(1 - 8\pi\eta^2)^{-1}$, while the red-shifted proper acceleration remains unchanged as M/r^2. It is the scaling up of the horizon that leads to scaling down of the surface gravity (temperature) of the hole.

Acknowledgements

One of the authors (KN) thanks Jawaharlal Nehru Centre for Advanced Research, Bangalore for award of a summer fellowship and IUCAA for providing the facilities.

References

[1] M Barriola and A Vilenkin, *Phys. Rev. Lett.* **63**, 341 (1989)
[2] N Dadhich (1995) GR-14 abstracts, A. 98
[3] A Banerjee, S Chatterjee and A A Sen, *Class. Quantum Gravit.* **13**, 3141 (1996)
[4] N Dadhich, *On the Schwarzschild field*, gr-qc/9704068
[5] G W Gibbons, M E Ortiz, F Ruiz and T M Samols, *Nucl. Phys.* **B385**, 127 (1992)
[6] D Harari and C Lousto, *Phys. Rev.* **D42**, 2626 (1990)
[7] S Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, New York, 1972) pp. 189–90, 194–96
[8] K Lee, V P Nair and E J Weinberg, *Gen. Relativ. Gravit.* **24**, 1203 (1992)
[9] J J Bisognano and E H Wichmann, *J. Math. Phys.* **16**, 985 (1975)
[10] Jiliang Jing, Hongwei Yu and Yongjiu Wang, *Phys. Lett.* **A178**, 59 (1993)
[11] B Jensen, *Nucl. Phys.* **B453**, 413 (1995)
[12] N Dadhich, A duality relation: global monopole and texture, gr-qc/9712021

Note added in the proof:

By resolving the Riemann curvature into electric and magnetic parts, it has now been shown [12] the Schwarzschild black hole with global monopole (3.7) is dual to the Schwarzschild black hole without global monopole. The solution (3.7) with $M = 0$ is dual to flat spacetime and hence it can justifiably be considered as "minimally" curved. By duality we mean the interchange between active and passive electric parts of the field.