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Particle production in curved spacetime

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Abstract. Particle production in curved spacetime has been discussed through the method of complex time WKB approximation. We consider Dirac equation in non-flat spacetime to understand particle production as particle–antiparticle rotation. The method is also generalized to understand particle production through parametric resonance. To understand the method of CWKB we consider particle production in Kasner spacetime as an example.

Keywords. Particle production; Robertson–Walker spacetime; parametric resonance.

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1. Introduction

The Dirac equation in curved spacetime is of considerable interest in astrophysics and cosmology. Previous investigations in this direction are due to Fock [1], Tetrode [2], Schrödinger [3] and McVittie [4]. A general discussion of the $m = 0$ neutrino case was given by Brill and Wheeler [5].

In astrophysics and cosmology the pair creation in expanding universes has been of considerable interest for several reasons. The choice of quantum vacuum is a very delicate object in curved spacetime and it greatly affects the calculations leading to pair creation. The works in this direction are due Chimento and Mollerach [6], Audretsch and Schäfer [7], Castagnino and Mazzitelli [8], Biswas and Guha [9], Biswas *et al* [10], Guha *et al* [11] and Biswas *et al* [12]. Calculation of pair creation and others related to has also been carried by Lotze [13], Barut and Duru [14], Biswas *et al* [15], Sahni [16], Barut and Singh [17] and many others.

In astrophysics the study of Dirac equation has gained importance in strange matter calculations as well as in neutron star calculations [18], with and without magnetic field. In cosmology the effect of particle production has some physical considerations, mainly in the study of damping and avoidance of initial singularity. A study of pair creation of spin $\frac{1}{2}$ particles (massive and massless) in Robertson–Walker universes was made by Parker [19], Lotze [13] and recently by Biswas *et al* [11, 12]. In the so called back-reaction problem, the effect of particle production back on the metric has not been incorporated successfully so far. Fischetti, Hartle and Hu [20], Anderson [21] tried to solve the back-reaction problem but the effect of particle production is not directly involved in the calculations. Ford [22] tried to stress the importance of particle production in solving the back-reaction problem, but the calculations, though interesting, lack to project the dynamical evolution of spacetime that can only be solved through the solution of Einstein field equations with a term for particle production density in the right-hand side of Einstein equations.

Various authors have studied particle production both in flat or non-flat spacetime. The common method is to define in and out vacuum and find the Bogolubov coefficients. The temporal equation resulting from Dirac equation in a Robertson–Walker framework has been found to differ [16, 17]. It has therefore been necessary to restudy Dirac equation in curved spacetime and find the correct temporal equation. It should be pointed out that the form of temporal equation is crucial in determining the particle production in expanding spacetime. In our previous works we followed a different procedure to calculate the pair production amplitude both in flat or non-flat R-W spacetime. In our approach [9–12], the turning points determined from the second order temporal Dirac equation are crucial in calculating the pair production amplitude. The turning points are fixed by the temporal equation in a given spacetime. To find out the correct turning points we investigate the derivation again without taking any results from other works. In this work we find that the Dirac equation in diagonal tetrad differs significantly from that of Barut and Singh [17]. We carry out the explicit derivation and find that our calculation is consistent with Sahni [16].

The present work, though a recheck of some previous calculations, also deals with some new results e.g., use of two dimensional temporal Dirac equation to understand particle production as particle–antiparticle rotation and parametric resonance particle production as a generalization of our result.

In § 2 and § 3 we consider Dirac equation in expanding non-flat spacetime and reduce it to two dimensional form to study particle–antiparticle rotation. Section 4 deals with parametric resonance particle production and § 5 deals with particle production in Kasner spacetime. We end up with discussion in § 6.

Recently particle production has been an active area of research [23–26] in cosmology to understand reheating mechanism after inflation, and the origin of cold dark matter. The method used [23–26] is parametric resonance. Our approach is an alternative one very similar to parametric resonance particle production. It is the aim of this work to highlight our approach with some specific examples. Though much work has been done on particle production, the present attempt sees the subject from a different approach with applicability in other directions.

2. The Dirac equation in expanding non-flat spacetime

The Dirac equation in curved spacetime is

$$\left[i\gamma^{\mu}(x)\frac{\partial}{\partial x^{\mu}} - i\gamma^{\mu}(x)\Gamma_{\mu}(x)\right]\psi = m\psi. \tag{1}$$

Here $\gamma^{\mu}(x)$ are curvature dependent Dirac matrices, $\Gamma_{\mu}(x)$ the spin connections and m the mass of the particle.

2.1 *The metric*

We start with Robertson–Walker type metric that describes a homogeneous isotropic non-flat universe given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right], \tag{2}$$

where $k = +1, 0, -1$, and for $k = 0$ this metric reduces to the conformally flat case. In order to understand the difference with flat-space equation we convert eq. (2) using

$$\begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{pmatrix}, \tag{3}$$

to the form

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\bigg[&\frac{1-k(x_2^2+x_3^2)}{\rho^2}\mathrm{d}x_1^2 + \frac{1-k(x_1^2+x_3^2)}{\rho^2}\mathrm{d}x_2^2 + \frac{1-k(x_1^2+x_2^2)}{\rho^2}\mathrm{d}x_3^2 \\ &+ \frac{2kx_1x_2}{\rho^2}\mathrm{d}x_1\mathrm{d}x_2 + \frac{2kx_1x_3}{\rho^2}\mathrm{d}x_1\mathrm{d}x_3 + \frac{2kx_2x_3}{\rho^2}\mathrm{d}x_2\mathrm{d}x_3\bigg] \\ = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \end{aligned} \tag{4}$$

with $\rho^2 = 1 - kr^2$ and

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2\frac{1-k(x_2^2+x_3^2)}{\rho^2} & -a^2\frac{kx_1x_2}{\rho^2} & -a^2\frac{kx_1x_3}{\rho^2} \\ 0 & -a^2\frac{kx_1x_2}{\rho^2} & -a^2\frac{1-k(x_1^2+x_3^2)}{\rho^2} & -a^2\frac{kx_2x_3}{\rho^2} \\ 0 & -a^2\frac{kx_1x_3}{\rho^2} & -a^2\frac{kx_2x_3}{\rho^2} & -a^2\frac{1-k(x_1^2+x_2^2)}{\rho^2} \end{pmatrix}. \tag{5}$$

The inverse of $g_{\mu\nu}$ is given by

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kx_1^2}{a^2} & \frac{kx_1x_2}{a^2} & \frac{kx_1x_3}{a^2} \\ 0 & \frac{kx_1x_2}{a^2} & -\frac{1-kx_2^2}{a^2} & \frac{kx_2x_3}{a^2} \\ 0 & \frac{kx_1x_3}{a^2} & \frac{kx_2x_3}{a^2} & -\frac{1-kx_3^2}{a^2} \end{pmatrix}. \tag{6}$$

2.2 *Affine connections*

Using $g_{\mu\nu}$ and $g^{\mu\nu}$ as given in equations (5) and (6) we calculate the affine connections $\Gamma^\alpha_{\mu\nu}$ with $\alpha, \mu, \nu = 0, 1, 2, 3$ to take the form

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\dot{a}\left(1+\frac{kx_1^2}{\rho^2}\right) & a\dot{a}\frac{kx_1x_2}{\rho^2} & a\dot{a}\frac{kx_1x_3}{\rho^2} \\ 0 & a\dot{a}\frac{kx_1x_2}{\rho^2} & a\dot{a}\left(1+\frac{kx_2^2}{\rho^2}\right) & a\dot{a}\frac{kx_2x_3}{\rho^2} \\ 0 & a\dot{a}\frac{kx_1x_3}{\rho^2} & a\dot{a}\frac{kx_2x_3}{\rho^2} & a\dot{a}\left(1+\frac{kx_3^2}{\rho^2}\right) \end{pmatrix}, \tag{7}$$

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & kx_1\left(1+\frac{kx_1^2}{\rho^2}\right) & \frac{k^2x_1^2x_2}{\rho^2} & \frac{k^2x_1^2x_3}{\rho^2} \\ 0 & \frac{k^2x_1^2x_2}{\rho^2} & kx_1\left(1+\frac{kx_2^2}{\rho^2}\right) & \frac{k^2x_1x_2x_3}{\rho^2} \\ 0 & \frac{k^2x_1^2x_3}{\rho^2} & \frac{k^2x_1x_2x_3}{\rho^2} & kx_1\left(1+\frac{kx_3^2}{\rho^2}\right) \end{pmatrix}, \tag{8}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & kx_2\left(1+\frac{kx_1^2}{\rho^2}\right) & \frac{k^2x_1x_2^2}{\rho^2} & \frac{k^2x_1x_2x_3}{\rho^2} \\ \dot{a}/a & \frac{k^2x_1x_2^2}{\rho^2} & kx_2\left(1+\frac{kx_2^2}{\rho^2}\right) & \frac{k^2x_2^2x_3}{\rho^2} \\ 0 & \frac{k^2x_1x_2x_3}{\rho^2} & \frac{k^2x_2^2x_3}{\rho^2} & kx_2\left(1+\frac{kx_3^2}{\rho^2}\right) \end{pmatrix}, \tag{9}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & kx_3\left(1+\frac{kx_1^2}{\rho^2}\right) & \frac{k^2x_1x_2x_3}{\rho^2} & \frac{k^2x_1x_3^2}{\rho^2} \\ 0 & \frac{k^2x_1x_2x_3}{\rho^2} & kx_3\left(1+\frac{kx_2^2}{\rho^2}\right) & \frac{k^2x_2x_3^2}{\rho^2} \\ \dot{a}/a & \frac{k^2x_1x_3^2}{\rho^2} & \frac{k^2x_2x_3^2}{\rho^2} & kx_3\left(1+\frac{kx_3^2}{\rho^2}\right) \end{pmatrix}. \tag{10}$$

2.3 *Curvature dependent Dirac matrices*

Henceforth curvature dependent Dirac matrices are written as $\gamma_\mu(x)$ and satisfy the anti-commutation rule $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$. The curvature independent ordinary Dirac matrices are denoted by γ_a, where $\mu, a = 0,1,2,3$; $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$; $\eta_{00} = 1$, $\eta_{ii} = -1$, $\eta_{0i} = 0$. The relation between $\gamma_\mu(x)$ and γ_a is given by vierbein-components L^a_μ defined through the relation

$$\gamma_\mu(x) = L^a_\mu(x)\gamma_a. \tag{11}$$

For R-W spacetime the vierbein are fixed through

$$ds^2 = L^a_\mu L_{\nu a} dx^\mu dx^\nu. \tag{12}$$

Here $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \varphi$ and ds^2 is given by (2). We find

$$L^a_\mu = \begin{matrix} \mu \longrightarrow \\ a \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a/\rho & 0 & 0 \\ 0 & 0 & ar & 0 \\ 0 & 0 & 0 & ar\sin\theta \end{pmatrix}. \tag{13}$$

In order to find the vierbein in cartesian coordinates denoted by $\widetilde{L}^a_\mu$, we use relation (3) to get

$$\widetilde{L}^a_\mu = \begin{matrix} \mu \rightarrow \\ a \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{ax_1}{r\rho} & \frac{ax_2}{r\rho} & \frac{ax_3}{r\rho} \\ 0 & \frac{ax_1x_3}{r\sqrt{x_1^2+x_2^2}} & \frac{ax_2x_3}{r\sqrt{x_1^2+x_2^2}} & -\frac{a\sqrt{x_1^2+x_2^2}}{r} \\ 0 & -\frac{ax_2}{r\sqrt{x_1^2+x_2^2}} & \frac{ax_1}{r\sqrt{x_1^2+x_2^2}} & 0 \end{pmatrix}. \tag{14}$$

In case of flat spacetime $k = 0$ (i.e. $\rho = 1$), $\widetilde{L}^a_\mu$ should reduce to a diagonal form. But equation (4) does not reduce to diagonal form for $k = 0$ as is expected. This is due to the fact that the vierbein is defined up to a Lorentz transformation in the flat tangent space. As Lorentz transformation for the 3-space dimensions can be viewed as a rotation, we rotate the vierbein (14) by multiplying it with the rotation matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ 0 & -\sin\varphi & \cos\varphi & 0 \end{pmatrix}, \tag{15}$$

or

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x_1}{r} & \frac{x_2}{r} & \frac{x_3}{r} \\ 0 & \frac{x_1x_3}{r\sqrt{x_1^2+x_2^2}} & \frac{x_2x_3}{r\sqrt{x_1^2+x_2^2}} & -\frac{\sqrt{x_1^2+x_2^2}}{r} \\ 0 & -\frac{x_2}{r\sqrt{x_1^2+x_2^2}} & \frac{x_1}{r\sqrt{x_1^2+x_2^2}} & 0 \end{pmatrix}, \tag{15a}$$

to obtain the new vierbein as,

$$\widetilde{L}^a_\mu = \begin{matrix} \mu \rightarrow \\ a \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a\frac{x_1^2+\rho(x_2^2+x_3^2)}{r^2\rho} & \frac{ax_1x_2(1-\rho)}{r^2\rho} & \frac{ax_1x_3(1-\rho)}{r^2\rho} \\ 0 & \frac{ax_1x_2(1-\rho)}{r^2\rho} & a\frac{x_2^2+\rho(x_1^2+x_3^2)}{r^2\rho} & \frac{ax_2x_3(1-\rho)}{r^2\rho} \\ 0 & \frac{ax_1x_3(1-\rho)}{r^2\rho} & \frac{ax_2x_3(1-\rho)}{r^2\rho} & a\frac{x_3^2+\rho(x_1^2+x_2^2)}{r^2\rho} \end{pmatrix}. \tag{16}$$

For $k = 0$ i.e. $\rho = 1$, this matrix turns out to be a diagonal one and moreover, it is also symmetric.

Thus we can calculate the covariant curvature dependent Dirac matrices $\gamma_\mu(x)$ in terms of curvature independent ordinary Dirac matrices γ_a by using equations (11) and (16) which are as follows:

$$\begin{aligned} \gamma_0(x) &= \gamma_0, \\ \gamma_1(x) &= \frac{a}{r^2\rho}[\{x_1^2+\rho(x_2^2+x_3^2)\}\gamma_1+x_1x_2(1-\rho)\gamma_2+x_1x_3(1-\rho)\gamma_3], \\ \gamma_2(x) &= \frac{a}{r^2\rho}[x_1x_2(1-\rho)\gamma_1+\{x_2^2+\rho(x_1^2+x_3^2)\}\gamma_2+x_2x_3(1-\rho)\gamma_3], \\ \gamma_3(x) &= \frac{a}{r^2\rho}[x_1x_3(1-\rho)\gamma_1+x_2x_3(1-\rho)\gamma_2+\{x_3^2+\rho(x_1^2+x_2^2)\}\gamma_3], \end{aligned} \tag{17}$$

and finally the contravariant curvature dependent Dirac matrices $\gamma^\mu(x)$, using the relations $\gamma^\mu(x) = g^{\mu\nu}\gamma_\nu(x)$ and equations (6) and (17), are obtained as

$$\begin{aligned}
\gamma^0(x) &= \gamma_0, \\
\gamma^1(x) &= \frac{1}{ar^2}[-(\rho x_1^2 + x_2^2 + x_3^2)\gamma_1 + x_1x_2(1-\rho)\gamma_2 + x_1x_3(1-\rho)\gamma_3], \\
\gamma^2(x) &= \frac{1}{ar^2}[x_1x_2(1-\rho)\gamma_1 - (\rho x_2^2 + x_1^2 + x_3^2)\gamma_2 + x_2x_3(1-\rho)\gamma_3], \\
\gamma^3(x) &= \frac{1}{ar^2}[x_1x_3(1-\rho)\gamma_1 + x_2x_3(1-\rho)\gamma_2 - (\rho x_3^2 + x_1^2 + x_2^2)\gamma_3].
\end{aligned} \tag{18}$$

2.4 *Spin connections*

The spin connections are given by the equation

$$\Gamma_\mu = -\tfrac{1}{8}[\gamma^\nu(x), \gamma_\nu(x)_{||\mu}], \tag{19}$$

where the covariant derivative is given by

$$\gamma_\nu(x)_{||\mu} = \gamma_\nu(x)_{|\mu} - \Gamma^\alpha_{\mu\nu}\gamma_\alpha(x). \tag{20}$$

We find Γ_μ by using equations (7), (8), (9), (10), (17) and (18) as

$$\begin{aligned}
\Gamma_0 &= 0, \\
\Gamma_1 &= -\frac{1}{2}\Bigg[\gamma_1\gamma_0\dot{a}\frac{x_1^2 + \rho(x_2^2 + x_3^2)}{r^2\rho} + \gamma_2\gamma_0\dot{a}\frac{x_1x_2(1-\rho)}{r^2\rho} \\
&\quad + \gamma_3\gamma_0\dot{a}\frac{x_1x_3(1-\rho)}{r^2\rho} + \gamma_1\gamma_2\frac{x_2(1-\rho)}{r^2} + \gamma_1\gamma_3\frac{x_3(1-\rho)}{r^2}\Bigg], \\
\Gamma_2 &= -\frac{1}{2}\Bigg[\gamma_1\gamma_0\dot{a}\frac{x_1x_2(1-\rho)}{r^2\rho} + \gamma_2\gamma_0\dot{a}\frac{x_2^2 + \rho(x_1^2 + x_3^2)}{r^2\rho} \\
&\quad + \gamma_3\gamma_0\dot{a}\frac{x_2x_3(1-\rho)}{r^2\rho} + \gamma_2\gamma_1\frac{x_1(1-\rho)}{r^2} + \gamma_2\gamma_3\frac{x_3(1-\rho)}{r^2}\Bigg], \\
\Gamma_3 &= -\frac{1}{2}\Bigg[\gamma_1\gamma_0\dot{a}\frac{x_1x_3(1-\rho)}{r^2\rho} + \gamma_2\gamma_0\dot{a}\frac{x_2x_3(1-\rho)}{r^2\rho} \\
&\quad + \gamma_3\gamma_0\dot{a}\frac{x_3^2 + \rho(x_1^2 + x_2^2)}{r^2\rho} + \gamma_3\gamma_1\frac{x_1(1-\rho)}{r^2} + \gamma_3\gamma_2\frac{x_2(1-\rho)}{r^2}\Bigg].
\end{aligned} \tag{21}$$

2.5 *The Dirac equation*

From equations (18) and (21), we get

$$\gamma^\mu(x)\Gamma_\mu(x) = -\left[\frac{3}{2}\gamma_0\frac{\dot{a}}{a} + \gamma_1\frac{x_1(1-\rho)}{ar^2} + \gamma_2\frac{x_2(1-\rho)}{ar^2} + \gamma_3\frac{x_3(1-\rho)}{ar^2}\right], \tag{22}$$

and also using equations (3) and (18) we may write

$$\gamma^{\mu}(x)\partial_{\mu} = \gamma_0\partial_t - \frac{1}{a}\left[\rho\left(\gamma_1\frac{x_1}{r} + \gamma_2\frac{x_2}{r} + \gamma_3\frac{x_3}{r}\right)\partial_r \right.$$
$$+ \left(\gamma_1\frac{x_1x_3}{r^2\sqrt{x_1^2+x_2^2}} + \gamma_2\frac{x_2x_3}{r^2\sqrt{x_1^2+x_2^2}} - \gamma_3\frac{\sqrt{x_1^2+x_2^2}}{r^2}\right)\partial_\theta$$
$$\left. + \left(-\gamma_1\frac{x_2}{x_1^2+x_2^2} + \gamma_2\frac{x_1}{x_1^2+x_2^2}\right)\partial_\varphi\right]. \qquad (23)$$

Thus, finally, Dirac equation (1) takes the form through (22) and (23) as

$$\left[i\gamma_0\partial_t + i\frac{3}{2}\gamma_0\frac{\dot a}{a} - \frac{i}{a}\left\{\rho\left(\gamma_1\frac{x_1}{r} + \gamma_2\frac{x_2}{r} + \gamma_3\frac{x_3}{r}\right)\partial_r\right.\right.$$
$$+ \left(\gamma_1\frac{x_1x_3}{r^2\sqrt{x_1^2+x_2^2}} + \gamma_2\frac{x_2x_3}{r^2\sqrt{x_1^2+x_2^2}} - \gamma_3\frac{\sqrt{x_1^2+x_2^2}}{r^2}\right)\partial_\theta$$
$$+ \left(-\gamma_1\frac{x_2}{x_1^2+x_2^2} + \gamma_2\frac{x_1}{x_1^2+x_2^2}\right)\partial_\varphi$$
$$\left.\left. - \gamma_1\frac{x_1(1-\rho)}{r^2} - \gamma_2\frac{x_2(1-\rho)}{r^2} - \gamma_3\frac{x_3(1-\rho)}{r^2}\right\} - m\right]\psi = 0. \qquad (24)$$

To give a more compact form of equation (24), we use the transformation of the curvature independent Dirac matrices in spherical polar coordinates by

$$\begin{pmatrix}\gamma_r\\ \gamma_\theta\\ \gamma_\varphi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta\\ -\sin\varphi & \cos\varphi & 0\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}, \qquad (25)$$

as

$$\left[i\gamma_0\partial_t + i\,\frac{3}{2}\gamma_0\frac{\dot a}{a} - \frac{i}{a}\left\{\rho\gamma_r\partial_r + \gamma_\theta\frac{1}{r}\partial_\theta + \gamma_\varphi\frac{1}{r\sin\theta}\partial_\varphi - (1-\rho)\frac{\gamma\cdot\mathbf{r}}{r^2}\right\} - m\right]\psi = 0. \qquad (26)$$

Premultiplying equation (26) by $-i\gamma_0$ and using the relation $\gamma_i = \gamma_0\alpha_i$, $\alpha_i = \gamma_0\gamma_i$, it becomes

$$\left[\partial_t + \frac{3}{2}\frac{\dot a}{a} + im\gamma_0 - \frac{1}{a}\left\{\alpha\cdot\nabla + (\rho-1)\alpha_r\left(\partial_r + \frac{1}{r}\right)\right\}\right]\psi = 0, \qquad (27)$$

which reduces to the flat spacetime equation for $\rho = 1$ i.e.,

$$\left[\partial_t + \frac{3}{2}\frac{\dot a}{a} + im\gamma_0 - \frac{1}{a}\alpha\cdot\nabla\right]\psi = 0. \qquad (28)$$

3. Solution of the Dirac equation

It is better to find out a diagonal tetrad gauge than the cartesian tetrad equation, because it is not so simple to find out the solution of eq. (27) for $\rho = 1$, as the flat spacetime equation for $\rho = 0$. On the other hand there is a study on the necessary and sufficient condition for the separability of equations obtained from the diagonal tetrad gauge. Moreover, there is a unitary transformation that maps the Dirac equation in the diagonal tetrad gauge into the equation in cartesian tetrad gauge.

In this case, metric (2) can be written by changing the variables

$$\begin{aligned} \mathrm{d}t &= \exp\left(\tfrac{1}{2}\alpha(\tau)\right)\mathrm{d}\tau, \\ a(t) &= \exp\left(\tfrac{1}{2}\alpha(\tau)\right), \\ r &= \xi(\chi), \end{aligned} \tag{29}$$

as,

$$\mathrm{d}s^2 = \mathrm{e}^{\alpha(\tau)}[\mathrm{d}\tau^2 - \{\mathrm{d}\chi^2 + \xi^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\}] \tag{30}$$

such that

$$\xi(\chi) = \begin{cases} \sin\chi, & \text{for } k = +1, \\ \chi, & \text{for } k = 0, \\ \sinh\chi, & \text{for } k = -1. \end{cases} \tag{31}$$

Using metric (30), the Dirac equation (1) takes the form

$$\begin{aligned} &\left[\mathrm{e}^{-\frac{1}{2}\alpha(\tau)}\left\{i\gamma_0\partial_\tau + i\frac{3}{4}\gamma_0\dot{\alpha} - i\left(\gamma_1\left(\partial_\chi + \frac{\xi'}{\xi}\right)\right.\right.\right. \\ &\left.\left.\left. + \gamma_2\frac{1}{\xi}\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \gamma_3\frac{1}{\xi\sin\theta}\partial_\varphi\right)\right\} - m\right]\bar{\psi} = 0, \end{aligned} \tag{32}$$

or simply

$$\left[\partial_\tau + \frac{3}{4}\dot{\alpha} - \left\{\alpha_1\left(\partial_\chi + \frac{\xi'}{\xi}\right) + \alpha_2\frac{1}{\xi}\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \alpha_3\frac{1}{\xi\sin\theta}\partial_\varphi\right\} + im\gamma_0\mathrm{e}^{\frac{1}{2}\alpha}\right]\bar{\psi} = 0, \tag{32a}$$

where $\dot{\alpha} = \partial\alpha/\partial\tau$ and $\xi' = \partial\xi/\partial\chi$. The unitary transfomnation S that maps equation (32a) into equation (27) is

$$\bar{\psi} = S\psi, \tag{33}$$

where

$$S = \frac{1}{2}(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 + 1)\exp\left(-\frac{\theta}{2}\gamma_3\gamma_1\right)\exp\left(-\frac{\varphi}{2}\gamma_1\gamma_2\right), \tag{34}$$

and its inverse is

$$S^{-1} = \exp\left(\frac{\varphi}{2}\gamma_1\gamma_2\right)\exp\left(\frac{\theta}{2}\gamma_3\gamma_1\right)\frac{1}{2}(1 - \gamma_1\gamma_2 - \gamma_2\gamma_3 - \gamma_3\gamma_1). \tag{35}$$

Here γ_i are the ordinary curvature independent Dirac matrices. In carrying out the transformation some of the identities (see Appendix A in which one of the identities is given in detail) are given below:

$$
\begin{aligned}
&S^{-1}\gamma_1\left(\partial_\chi+\frac{\xi'}{\xi}\right)(S\psi)=(\gamma_1\sin\theta\cos\varphi+\gamma_2\sin\theta\sin\varphi+\gamma_3\cos\theta)\left(\partial_\chi+\frac{\xi'}{\xi}\right)\psi,\\
&S^{-1}\gamma_2(\partial_\theta+\tfrac{1}{2}\cot\theta)(S\psi)=-\tfrac{1}{2}(\gamma_1\sin\theta\cos\varphi+\gamma_2\sin\theta\sin\varphi+\gamma_3\cos\theta)\psi\\
&\quad+\tfrac{1}{2}\cot\theta(\gamma_1\cos\theta\cos\varphi+\gamma_2\cos\theta\sin\varphi-\gamma_3\sin\theta)\psi\\
&\quad+(\gamma_1\cos\theta\cos\varphi+\gamma_2\cos\theta\sin\varphi-\gamma_3\sin\theta)\partial_\theta\psi,\\
&S^{-1}\gamma_3\partial_\varphi(S\psi)=-\tfrac{1}{2}(\gamma_1\cos\varphi+\gamma_2\sin\varphi)\psi+(\gamma_2\cos\varphi-\gamma_1\sin\varphi)\partial_\varphi\psi.
\end{aligned}
\tag{36}
$$

From equations (29) and (31), we get

$$
\dot{a}=\frac{1}{2}\dot{\alpha},\ \partial_\tau+\frac{3}{4}\dot{\alpha}=\mathrm{e}^{\alpha/2}\left(\partial_t+\frac{3}{2}\frac{\dot{a}}{a}\right),\quad \partial_\chi+\frac{\xi'}{\xi}=\rho\partial_r+\frac{1}{r}\rho.
$$

Using these results and the identities (36), it is possible to show that equation (32a) is exactly mapped into equation (27) (see Appendix B)

In order to separate the variables of (32), we take

$$
\bar{\psi}=\xi^{-1}(\chi)\sin^{-(1/2)}\theta\mathrm{e}^{-(3/4)\alpha(\tau)}\Phi, \tag{37}
$$

with $\Phi=\frac{1}{2}(\gamma_0\gamma_1+\gamma_2\gamma_0+\gamma_0\gamma_3-\gamma_0\gamma_1\gamma_2\gamma_3)\Sigma$ and

$$
\Sigma=\begin{pmatrix} f(\tau)b(\chi)\Theta_1(\theta)\\ f(\tau)b(\chi)\Theta_2(\theta)\\ g(\tau)a(\chi)\Theta_1(\theta)\\ -g(\tau)a(\chi)\Theta_2(\theta)\end{pmatrix}\mathrm{e}^{in\varphi}, \tag{38}
$$

where n takes the values $\pm1/2$, $\pm3/2$, $\pm5/2,\ldots$. We get the following four equations:

$$
\frac{1}{f}(i\partial_\tau+m\mathrm{e}^{\alpha/2})g-\frac{1}{a}\partial_\chi b+\frac{i}{\xi}\frac{b}{a}\frac{\partial_\theta(\Theta_1-\Theta_2)}{\Theta_1-\Theta_2}+\frac{n}{\xi\sin\theta}\frac{b}{a}\frac{\Theta_1+\Theta_2}{\Theta_1-\Theta_2}=0, \tag{39}
$$

$$
\frac{1}{f}(i\partial_\tau+m\mathrm{e}^{\alpha/2})g+\frac{1}{a}\partial_\chi b+\frac{i}{\xi}\frac{b}{a}\frac{\partial_\theta(\Theta_1+\Theta_2)}{\Theta_1+\Theta_2}-\frac{n}{\xi\sin\theta}\frac{b}{a}\frac{\Theta_1-\Theta_2}{\Theta_1+\Theta_2}=0, \tag{40}
$$

$$
\frac{1}{g}(i\partial_\tau-m\mathrm{e}^{\alpha/2})f-\frac{1}{b}\partial_\chi a+\frac{i}{\xi}\frac{a}{b}\frac{\partial_\theta(\Theta_1+\Theta_2)}{\Theta_1+\Theta_2}+\frac{n}{\xi\sin\theta}\frac{a}{b}\frac{\Theta_1-\Theta_2}{\Theta_1+\Theta_2}=0, \tag{41}
$$

$$
\frac{1}{g}(i\partial_\tau-m\mathrm{e}^{\alpha/2})f+\frac{1}{b}\partial_\chi a+\frac{i}{\xi}\frac{a}{b}\frac{\partial_\theta(\Theta_1-\Theta_2)}{\Theta_1-\Theta_2}-\frac{n}{\xi\sin\theta}\frac{a}{b}\frac{\Theta_1+\Theta_2}{\Theta_1-\Theta_2}=0. \tag{42}
$$

First term of equations (39), (40), (41), (42) are dependent on τ only and rest parts of each of the equations are independent of τ. Hence the first term of each of the equations should be equal to a constant λ (say) and thus we get, for the functions $f(\tau)$ and $g(\tau)$, the coupled equations:

$$
\left(\frac{\mathrm{d}}{\mathrm{d}\tau}-im\mathrm{e}^{(1/2)\alpha(\tau)}\right)g(\tau)=-i\lambda f(\tau) \tag{43}
$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + ime^{(1/2)\alpha(\tau)}\right) f(\tau) = -i\lambda g(\tau). \tag{44}$$

Premultiplying (43) by $((\mathrm{d}/\mathrm{d}\tau) + ime^{(1/2)\alpha(\tau)})$ and equation (44) by $((\mathrm{d}/\mathrm{d}\tau) - ime^{(1/2)\alpha(\tau)})$ and using eqs (43) and (44), results in a pair of uncoupled second order equations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{1}{2} im\dot{\alpha}\mathrm{e}^{(1/2)\alpha(\tau)} + \lambda^2 + m^2\mathrm{e}^{\alpha(\tau)}\right] f(\tau) = 0 \tag{45}$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \frac{1}{2} im\dot{\alpha}\mathrm{e}^{(1/2)\alpha(\tau)} + \lambda^2 + m^2\mathrm{e}^{\alpha(\tau)}\right] g(\tau) = 0. \tag{46}$$

The result obtained in this section i.e. equations (43) to (46) differs from Barut and Singh but is consistent with Sahni. We will consider (43) and (44) to understand particle production in Kasner spacetime and in some other cases.

4. Parametric resonance particle production

We write eqs (43) to (46) in the form

$$i\frac{\partial\psi}{\partial\tau} = H\psi, \tag{47}$$

and in the second order form

$$\left[\frac{\partial^2}{\partial\tau^2} + (\lambda^2 + m^2c^2(\tau) \pm im\dot{c}(\tau))\right]\psi = 0. \tag{48}$$

Here

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad H = \sigma^3 mc(\tau) + \sigma^1\lambda, \tag{49}$$

and σ^i are the usual Pauli 2×2 matrices. The complex time WKB (CWKB) approximation allows us to calculate pair production amplitude as follows [9–12]. We determine the complex turning points $\tau_{1,2}$ from

$$\omega(\tau_{1,2}) \equiv [\lambda^2 + m^2c^2(\tau) \pm im\dot{c}(\tau)]^{1/2} = 0, \tag{50}$$

and the pair production amplitude is given by

$$R = -\frac{i\exp[2i\int^{\tau_1}\omega_\lambda(\tau)\mathrm{d}\tau]}{1 + \exp[2i\int_{\tau_1}^{\tau_2}\omega_\lambda(\tau)\mathrm{d}\tau}. \tag{51}$$

In obtaining eq. (51) we assumed that particle–antiparticle states are determined through zeroeth order WKB approximation to eq. (48). From eq. (51) it follows that R has poles at

$$\int_{\tau_1}^{\tau_2}\omega_\lambda(\tau)\mathrm{d}\tau = \left(N + \frac{1}{2}\right)\pi, \quad N = 0, 1, 2, \ldots. \tag{52}$$

When the condition (52) is violated the poles become resonances and we get resonance particle production. Because of recent interest in parametric resonance particle production [23–26], we consider Mathieu-type equation which is generally used to understand catastrophic particle production

$$\frac{d^2u}{dz^2} + [h - 2\theta\cos(2z)]u = 0. \tag{53}$$

The advantage of our method lies in the WKB vacuum definition (usually taken at $|\tau| \to \infty$). In cosmology we sometime require to know the particle production at an intermediate stage to obtain the late time evolution of the universe. In such cases the parameter θ may be large and the WKB vacuum definition remains valid even at finite τ.

To demonstrate, we consider large θ approximation and replace the periodic barriers by piece-wise inverted harmonic oscillator; and take $|z| < \frac{\pi}{2}$ such that $\cos(2z) \to (1 - 2z^2)$. We write eq.(53) as

$$\frac{d^2u}{dz'^2} + [\lambda + z'^2]u = 0, \tag{54}$$

with

$$\lambda = \sqrt{\theta}\epsilon, \quad \epsilon = \frac{h}{2\theta} - 1, \quad z' = (4\theta)^{1/4}z. \tag{55}$$

The turning points are at $z' = \pm i\lambda$. We evaluate eq. (51) to get

$$|R|^2 = \frac{\exp(-\pi\lambda)}{1 + \exp(-\pi\lambda)} = \frac{\exp(-\pi\sqrt{\theta}\epsilon)}{1 + \exp(-\pi\sqrt{\theta}\epsilon)}. \tag{56}$$

The solution of eq.(54) can be written in terms of the standard parabolic cylindrical function $D_{-(i\sqrt{\theta}\epsilon+(1/2))}(e^{i\pi/4}2\theta^{1/4}z)$ which in large θ approximation will satisfy, at finite z, the conditions of WKB definition of particle–antiparticle states. Using the relation $\Gamma(z)\Gamma(1-z) = \pi/(\sin(\pi z))$, we can determine the standard Bogolubov coefficients α and β from eq.(56) with identification $R = \beta/\alpha$ as

$$\alpha = \frac{\sqrt{2\pi}\theta^{-i\sqrt{\theta}\epsilon/4}e^{-\pi\sqrt{\theta}\epsilon/4}}{\Gamma((1 + i\sqrt{\theta}\epsilon)/2)}, \quad \beta = e^{-\pi\sqrt{\theta}\epsilon/2}. \tag{57}$$

This result exactly coincides with [26]. The solution of eq.(53) is written as

$$u(z) = e^{\lambda z}P(z), \tag{58}$$

with λ real and positive, and $P(z)$, a periodic function. To obtain λ in (58) we have to consider a large number of repeated barrier crossovers to obtain $\lambda \sim (1/\pi)\ln(e^{-\pi\sqrt{\theta}\epsilon/2}) = (\sqrt{\theta}\epsilon)/2$.Thus we find that parametric resonance particle production is basically CWKB resonance particle production in parameter space (θ, h).

We have considered eq. (53) to understand the effectiveness of our approach. In the next chapter we consider the particle production in Kasner's spacetime through CWKB to understand some salient features of particle production. The above example of particle production in parameter space allows us to define particle–antiparticle states even when $|\eta| \not\to \infty$.

5. The Kasner spacetime

We now concentrate primarily on the Kasner spacetime to study the pair production. In this case we find that temporal equation shows the existence of complex turning points which are needed to evaluate the pair production.

The Kasner metric is given by

$$ds^2 = dt^2 - t^{2a_1}dx^2 - t^{2a_2}dy^2 - t^{2a_3}dz^2, \tag{59}$$

where a_1, a_2 and a_3 are real numbers. The spacetime (59) is a model of anisotropically expanding universe and is a solution of vacuum Einstein equation subject to the constraint

$$a_1 + a_2 + a_3 = a_1^2 + a_2^2 + a_3^2 = 1. \tag{60}$$

In the non-degenerate case the a_is are non-vanishing and none of the a_is are equal. On the other hand in the degenerate case one of the a_is is equal to unity, while the others are vanishing. We choose corresponding to eq. (59)

$$\gamma^0(t) = \gamma_0 = \beta, \quad \gamma^i(t) = t^{-a_i}\gamma^i, \quad i = 1, 2, 3. \tag{61}$$

Also we have

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{62}$$

and

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix}; \quad \gamma^i = \beta\alpha^i; \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{63}$$

represent flat γ-matrices, while σ^i are 2×2 Pauli matrices. The Christoffel symbols $\Gamma^\mu_{\nu\rho}$ and the spin connections Γ_μ are evaluated to get

$$\tilde{\gamma}^\mu \Gamma_\mu = -\frac{\gamma^0}{2t}. \tag{64}$$

Using eq. (64) in equation (1) we have the Dirac equation in Kasner spacetime to have the form

$$\left[i\gamma^0\partial_0 + i(t^{-a_1}\gamma^1\partial_1 + t^{-a_2}\gamma^2\partial_2 + t^{-a_3}\gamma^3\partial_3) + \frac{i\gamma^0}{2t} - m \right]\psi = 0. \tag{65}$$

Substituting

$$\psi = t^{-1/2}\bar{\psi}, \tag{66}$$

in eq. (65) and using eq. (63) we have

$$[\partial_0 + t^{-a_1}\alpha^1\partial_1 + t^{-a_2}\alpha^2\partial_2 + t^{-a_3}\alpha^3\partial_3 + i\gamma^0 m]\bar{\psi} = 0. \tag{67}$$

If we introduce

$$\bar{\psi} = \frac{1}{2\pi^{3/2}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{68}$$

and also using eq. (63) then, like eqs (43) and (44), the functions f_1 and f_2 satisfy the coupled equations

$$(\partial_0 + im) f_1 + i(t^{-a_1}\sigma^1 k_1 + t^{-a_2}\sigma^2 k_2 + t^{-a_3}\sigma^3 k_3) f_2 = 0, \tag{69}$$

$$(\partial_0 - im) f_2 + i(t^{-a_1}\sigma^1 k_1 + t^{-a_2}\sigma^2 k_2 + t^{-a_3}\sigma^3 k_3) f_1 = 0. \tag{70}$$

We note that we can go over to the flat space by identifying

$$\beta = \gamma^0; \quad \gamma^i \partial_i = \gamma \cdot \nabla = i\beta\alpha \cdot \nabla. \tag{71}$$

We seek to find the WKB solutions of (69) and (70) both in non-degenerate and in degenerate case.

5.1 *Non-degenerate case*

We use the transformations

$$A' = UAU^\dagger, \quad \partial_{t'} = U\partial_0 U^\dagger, \quad f = Uf_1, \quad g = Uf_2, \tag{72}$$

where

$$A = t^{-a_1}\sigma^1 k_1 + t^{-a_2}\sigma^2 k_2 + t^{-a_3}\sigma^3 k_3, \quad A' = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

$$\lambda = \sqrt{c_1^2 + c_2^2 + c_3^2}, \quad c_i = t^{-a_i} k_i, \quad U = \begin{pmatrix} r_1 \mathrm{e}^{i\phi/2} & r_2 \mathrm{e}^{-i\phi/2} \\ r_2 \mathrm{e}^{i\phi/2} & -r_1 \mathrm{e}^{-i\phi/2} \end{pmatrix}$$

and

$$r_1 = \sqrt{\frac{1}{2}\left(1 + \frac{c_3}{\lambda}\right)}, \quad r_2 = \sqrt{\frac{1}{2}\left(1 - \frac{c_3}{\lambda}\right)}, \quad \phi = \tan^{-1}\left(\frac{c_2}{c_1}\right).$$

As a result $\partial_{t'}$ takes the form

$$\partial_{t'} = \partial_t - \frac{ic_1 c_2 c_3 (a_1 - a_2)}{2\lambda(c_1^2 + c_2^2)} t^{-1}. \tag{73}$$

Equations (69) and (70) becomes

$$(\partial_{t'} + im) f + iA' g = 0 \tag{74}$$

and

$$(\partial_{t'} - im) g + iA' f = 0. \tag{75}$$

Premultiplying eq. (74) by iA'/λ^2 and operating by $(\partial_{t'} - im)$ from left and also using eq. (75) we get uncoupled equation for f,

$$[\partial_{t'}{}^2 - \partial_{t'}(\ln\lambda)\partial_{t'} - im\partial_{t'}(\ln\lambda) + \lambda^2 + m^2] f = 0. \tag{76}$$

Similarly for g using $f(t,-m)$,

$$g(t,m)=f(t,-m). \tag{77}$$

Substituting $f=\mathrm{e}^{(1/2)\int\partial_{t'}(\ln\lambda)\mathrm{d}t'}y$ in eq. (76) and also using eq. (73) it turns to be of the form

$$[\partial_t^2+\Omega^2(t)]z=0, \tag{78}$$

where

$$y=\mathrm{e}^{(1/2)\int 2iq(t)\mathrm{d}t}z,\quad q(t)=\frac{c_1c_2c_3(a_1-a_2)}{2\lambda(c_1^2+c_2^2)}t^{-1}.$$

5.2 *Degenerate case*

In curved spacetime the role of vacuum is not well resolved. To understand the role of vacuum, we concentrate on degenerate Kasner spacetime in the light of CWKB technique and evaluate the pair production amplitude. This is done to elucidate the vacuum with respect to strong and weak vacuum [8, 27–29]. The study of particle production in the degenerate Kasner spacetime is interesting because of its striking similarity with the Milne universe.

In the degenerate case we take $a_1=a_3=0$ and $a_2=1$, so that Kasner metric (59) reduces to

$$\mathrm{d}s^2=\mathrm{d}t^2-(\mathrm{d}x^2+t^2\mathrm{d}y^2+\mathrm{d}z^2). \tag{79}$$

Accordingly eq. (67) becomes

$$[\partial_0+\alpha^1\partial_1+t^{-1}\alpha^2\partial_2+\alpha^3\partial_3+i\gamma^0m]\bar{\psi}=0. \tag{80}$$

We insert in eq. (80) the following

$$\bar{\psi}\sim\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\chi, \tag{81}$$

where

$$\chi=\begin{pmatrix}f_1\\ f_2\end{pmatrix} \tag{82}$$

and define

$$\begin{aligned}{k_+}^2&={k_1}^2+{k_3}^2,\\ M^2&={k_+}^2+m^2,\\ \beta'M&=\alpha^1k_1+\alpha^3k_3+\beta m.\end{aligned} \tag{83}$$

Equation (80) then takes the form

$$[i\partial_0-\alpha^2k_2t^{-1}-\beta'M]\chi=0. \tag{84}$$

As eq. (84) has only two anticommuting variables we may choose

$$\beta'=\sigma^3\quad\text{and}\quad\alpha^2=\sigma^1, \tag{85}$$

as if (84) reduces to Dirac equation

$$i\partial_0\psi = H\psi, \tag{86}$$

where

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{87}$$

$$H = \sigma^1 k_2 t^{-1} + \sigma^3 M. \tag{88}$$

This equation has a remarkable similarity with Dirac equation in time dependent electromagnetic field [18]. The basic feature of reflection and particle production in the work of Cornwall and Tiktopoulos [18] is that the external field (in our case it is the gravitational field) acts between t_1 and t_2. For $t < t_1$ there is no particle but at $t > t_2$ (or $t \to +\infty$) the solutions behave such that it becomes a mixture of particle–antiparticle solutions and is written in terms of free orthonormals. Cornwall and Tiktopoulos demanded that between t_1 and t_2 the spinor suffers a rotation of odd multiple of π in order to have pair production. It may be mentioned that the particle production in an electric and gravitational field was also explicitly calculated for scalar and spinors by Sahni [30]; however, our approach is very much similar to particle production in flat spacetime using Feynman–Stuckleburg prescription.

We generalize the above approach through our *complex time multiple reflection technique* for which it is necessary to find out two solutions that will behave like particle and antiparticle at $t \to \infty$ in order to fix up the vacuum. Though not mentioned in the work of Cornwall and Tiktopoulos we observe that $t = t_1$ and $t = t_2$ must be two turning points to be determined from the second order Dirac equation corresponding to eq. (84). We show that for a system having turning points this particle–antiparticle transformation takes place very naturally [15].

(i) To generalize the work of Cornwall and Tiktopoulos [18] we used particle–antiparticle definitions in curved spacetime as follows. We adopt WKB definition of particle–antiparticle [8, 27–29] solutions such that

$$\lim_{\eta \to \mp\infty} U^{\text{in}}_{\text{out}} \simeq \exp(iS^{\text{in}}_{\text{out}}) \tag{89}$$

and

$$\lim_{\eta \to \mp\infty} V^{\text{in}}_{\text{out}} \simeq \exp(-iS^{\text{in}}_{\text{out}}), \tag{90}$$

respectively. Here

$$S^{\text{in}}_{\text{out}} = \lim_{\eta \to \mp\infty} S \tag{91}$$

and S is a solution of Hamilton–Jacobi equation.

(ii) To have reflection in time we must have turning points.

(iii) The pair production amplitude depends on two fixed points i.e. the turning points.

We use eq. (82) in eq. (84) to have coupled equations

$$(\partial_0 + iM) f_1 + ik_2 t^{-1} f_2 = 0, \tag{92}$$

$$(\partial_0 - iM) f_2 + ik_2 t^{-1} f_1 = 0. \tag{93}$$

Premultiplying eq. (92) by t and then operating by $(\partial_0 - iM)$ from left and using (93) we have, like eq. (78),

$$[\partial_\eta^2 + \Omega^2(\eta)]f_1 = 0, \tag{94}$$

where we have introduced

$$t = e^\eta \tag{95}$$

and

$$\Omega(\eta) = [M^2 e^{2\eta} + iMe^\eta + k_2^2]^{1/2}. \tag{96}$$

We now find the particle production amplitude $|R|^2$ by our non-perturbative method of WKB approximation in complex time. Equation (94) is of the same form of a Schrödinger-like equation (not in space) in conformal time coordinate viz.,

$$\frac{d^2\psi}{d\eta^2} + \Omega^2(\eta)\psi = 0, \tag{97}$$

where

$$\Omega(\eta) = (\omega^2 - V)^2, \tag{98}$$

with ω playing the role of energy and V being a potential term. We define the following

$$\begin{aligned} \tau &= Me^\eta, \\ b &= -\frac{i}{2}, \\ c &= -ik_2. \end{aligned} \tag{99}$$

The turning points τ_1 and τ_2 are now obtained from

$$\Omega(\tau_{1,2}) = 0, \tag{100}$$

to be

$$\begin{aligned} \tau_1 &= b + \sqrt{b^2 + c^2}, \\ \tau_2 &= b - \sqrt{b^2 + c^2}. \end{aligned} \tag{101}$$

Since the particle production amplitude R (see Appendix C) is given by

$$R = \frac{e^{2iS(\tau_1)}}{1 + e^{2iS(\tau_1,\tau_2)}}, \tag{102}$$

where from equation

$$S(\eta) = \int \Omega(\eta) d\eta, \tag{103}$$

we get

$$S(\tau_1) = \frac{\pi}{4} - \frac{\pi i k_2}{2} \tag{104}$$

and

$$S(\tau_1, \tau_2) = \frac{\pi}{2} - \pi i k_2. \tag{105}$$

The pair production amplitude can now be obtained employing (102) to be

$$|R|^2 = \frac{e^{2\pi k_2}}{1 + e^{4\pi k_2} - 2e^{2\pi k_2}}. \tag{106}$$

The result shows that there is particle production. If we wish to obtain the result for Milne universe herefrom, we may set $k_1 = k_3 = 0$ which means from eq. (83)

$$M = m, \tag{107}$$

and the pair production amplitude will be of the same form of eq. (106), as it is independent of $M = m$ and thus we get particle production even in Milne universe in two dimension. It is a new result and needs careful analysis. We have based our calculation on WKB approximation and adopted the WKB prescription of particle–antiparticle. So the in vacuum and out vacuum are adiabatic vacuum. It is surprising to find pair production even in adiabatic vacuum which is possible when $|0, \text{in}\rangle \neq |0, \text{out}\rangle$ though both are defined in terms of adiabatic modes. The standard result for the Milne universe in two dimension [9, 16, 31] are as follows:

(1) There is no pair production in adiabatic vacuum.
(2) There is pair production in conformal vacuum.

These two facts are based upon the solutions of the temporal equation [16]. However, our result does not rely upon a particular solution and depends only on the turning points and the extra phase obtained through reflections. It is a global feature of Robertson–Walker spacetime.

The standard arguments for the Milne or Chitre Hartle universe both in two and four dimensions reveal that under coordinate transformation the metric reduces to Minkowski spacetime and hence we do not see any pair production in Milne universe. From our approach it is clear that as soon as one does the coordinate transformation to reduce to a Minkowski spacetime-like metric, (a) one squeezes the spacetime instead of the whole as in the original universe. (b) It also gets rid of the turning points. This in turn means that an electron in a Milne universe is definitely not equivalent to some other electron in another spacetime obtained after coordinate transformation. Hence seeing the similarity of Milne universe with Minkowski spacetime it is not desirable to comment on the existence or non-existence of pair production.

6. Discussion

The method which we have developed has its basis in the Feynman–Stuckleburg prescription according to which negative energy particle solutions propagating backward in time is considered equivalent to positive energy antiparticle solutions propagating forward in time. Accordingly the e^+e^- pair creation from a potential (where we have positive energy e^+ and e^-, both propagating forward in time) may be treated as being equivalent to the reflection of a negative energy electron initially propagating backward in time, to suffer reflection in time thereby giving way to a positive energy electron moving forward in time. Alternatively, particle production can be viewed upon as a process of reflection in time due to which there is no particle present at $t \rightarrow -\infty$ but at $t \rightarrow +\infty$ there is a particle moving forward in time and an antiparticle moving backward in time. In CWKB the wave starts from a real point and suffers reflection at the complex turning points. Knoll and Schaeffer [32, 33], Schrempp and Schrempp [34] developed the method of complex semiclassical paths in space which they have utilized in calculating scattering amplitude (in case of reflection in space). We extended their method invoking complex semiclassical paths in time which is used to calculate pair production amplitude (in the case of reflection in time).

Elsewhere [10] we have established an analogy of CWKB with moving mirror example in curved spacetime. Similarity of pair production in gravitational field with that of electromagnetic field has been elucidated in our earlier works [12]. As the production of particles carried out in our approach is remarkably similar to pair production in electromagnetic field, it is thus supposed that instability of the vacuum is the root cause of pair production in such a situation as in electromagnetic case. $\mathrm{Im}\mathcal{L}_{\mathrm{eff}}$, which is needed to measure the instability of vacuum, can be calculated, as in our earlier work [10], in CWKB, without taking recourse to Bogolubov coefficient.

Let us now present a synopsis of the work done here. (i) The particle–antiparticle model in curved spacetime is fixed by zeroeth order WKB solution of the field equation. In other words we have invoked Parker's definition. (ii) The vacuum is chosen from the CWKB solutions which serve as mode solutions for proper adiabatic vacuum. (iii) Occurrence of turning points ensures particle production. This fact is hitherto unnoticed in the literature. These turning points are the characteristic of a given spacetime. (iv) Occurrence of turning points is a common feature of all standard expanding spacetimes. Hence particle production is a general characteristic of expanding spacetimes. (v) The turning points are determined utilizing eq. (98). (vi) The in vacuum in our degenerate Kasner case is Hartle Hawking vacuum as in collapsing star problem [10, 31]. The out vacuum is such that as the particle evolves from the in vacuum, it does not find the out vacuum as a physical vacuum i.e. $|0,\mathrm{in}\rangle \neq |0,\mathrm{out}\rangle$. This occurs because $\int \Omega(\tau)\mathrm{d}\tau|_{\tau=\tau_1} \neq 0$. (vii) Mixing of positive and negative frequency modes result in particle production. The situation is like the blackhole problem [10, 30] where mixing of positive and negative frequency states occur at the horizon. (viii) Vacuum acts like a blackhole. In blackhole case hole mass decreases but here vacuum energy changes. In both cases $\mathrm{Im}\mathcal{L}_{\mathrm{eff}}$ remains finite. (ix) The standard explanation of no particle production in Milne universe is that under coordinate transformation, (for two dimension)

$$\begin{aligned} y^0 &= \mathrm{e}^{\eta} \cosh x, \\ y^1 &= \mathrm{e}^{\eta} \sinh x, \end{aligned} \tag{108}$$

with $0 < y^0 < \infty$ and $-\infty < y^1 < \infty$, it reduces to Minkowski spacetime. Hence we do not expect particle production in two dimensional Milne universe. Our calculation, however, contradicts the situation both in two as well as in four dimension. The coordinate transformation washes out the turning points and hence there is no particle production. But the turning points are crucial and acts like horizon [10, 31, 35] and is responsible for mixing of positive and negative frequency states resulting in pair production.

Details in this respect are discussed in our earlier works [10, 11]. The basic calculation is that the vacuum in Milne universe is not the same as Minkowski vacuum and in a gravitational field, the instability of this vacuum results in particle production. In complex time WKB approximation (CWKB) technique, the aforementioned ideas find an elegant manifestation.

We have also studied catastrophic particle production under periodic perturbation through CWKB and find results consistent with other works. This allows us to study particle production in the light of CWKB to understand some aspects of cosmological evolution of the universe.

Appendix A

Among the three only one of the identities

$$\begin{aligned}
&S^{-1}\gamma_2(\partial_\theta + \tfrac{1}{2}\cot\theta)(S\psi)\\
&\quad = -\tfrac{1}{2}(\gamma_1\sin\theta\cos\varphi + \gamma_2\sin\theta\sin\varphi + \gamma_3\cos\theta)\psi\\
&\qquad + \tfrac{1}{2}\cot\theta(\gamma_1\cos\theta\cos\varphi + \gamma_2\cos\theta\sin\varphi - \gamma_3\sin\theta)\psi\\
&\qquad + (\gamma_1\cos\theta\cos\varphi + \gamma_2\cos\theta\sin\varphi - \gamma_3\sin\theta)\partial_\theta\psi,
\end{aligned} \tag{A1}$$

is shown in detail here and others follow the same rules.

Here we strictly follow

$$\bar{\psi} = S\psi$$

and

$$S = \tfrac{1}{2}(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 + 1)\mathrm{e}^{-(\theta/2)\gamma_3\gamma_1}\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}.$$

Then the left hand side of equation (A1) may be written as

$$S^{-1}\gamma_2(\partial_\theta + \tfrac{1}{2}\cot\theta)(S\psi) = (S^{-1}\gamma_2 S\partial_\theta + S^{-1}\gamma_2(\partial_\theta S) + \tfrac{1}{2}\cot\theta S^{-1}\gamma_2 S)\psi. \tag{A2}$$

Here

$$\begin{aligned}
S^{-1}\gamma_2 S &= \tfrac{1}{4}\mathrm{e}^{(\varphi/2)\gamma_1\gamma_2}\mathrm{e}^{(\theta/2)\gamma_3\gamma_1}(1 - \gamma_1\gamma_2 - \gamma_2\gamma_3 - \gamma_3\gamma_1)\gamma_2\\
&\quad \times (\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 + 1)\mathrm{e}^{-(\theta/2)\gamma_3\gamma_1}\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}\\
&= \tfrac{1}{4}\mathrm{e}^{(\varphi/2)\gamma_1\gamma_2}\mathrm{e}^{(\theta/2)\gamma_3\gamma_1}(1 - \gamma_1\gamma_2 - \gamma_2\gamma_3 - \gamma_3\gamma_1)\\
&\quad \times (\gamma_1 - \gamma_3 + \gamma_1\gamma_2\gamma_3 + \gamma_2)\mathrm{e}^{-(\theta/2)\gamma_3\gamma_1}\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}\\
&= \tfrac{1}{4}\mathrm{e}^{(\varphi/2)\gamma_1\gamma_2}\mathrm{e}^{(\theta/2)\gamma_3\gamma_1}(\gamma_1 - \gamma_3 + \gamma_1\gamma_2\gamma_3 + \gamma_2 - \gamma_2 + \gamma_1\gamma_2\gamma_3 + \gamma_3 + \gamma_1\\
&\quad - \gamma_1\gamma_2\gamma_3 - \gamma_2 + \gamma_1 - \gamma_3 + \gamma_3 + \gamma_1 + \gamma_2 - \gamma_1\gamma_2\gamma_3)\mathrm{e}^{-(\theta/2)\gamma_3\gamma_1}\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}\\
&= \frac{1}{4}\cdot 4\mathrm{e}^{(\varphi/2)\gamma_1\gamma_2}\mathrm{e}^{(\theta/2)\gamma_3\gamma_1}\gamma_1\left(\cos\frac{\theta}{2} - \gamma_3\gamma_1\sin\frac{\theta}{2}\right)\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}\\
&= \mathrm{e}^{(\varphi/2)\gamma_1\gamma_2}\left(\cos\frac{\theta}{2} + \gamma_3\gamma_1\sin\frac{\theta}{2}\right)\left(\gamma_1\cos\frac{\theta}{2} - \gamma_3\sin\frac{\theta}{2}\right)\mathrm{e}^{-(\varphi/2)\gamma_1\gamma_2}
\end{aligned}$$

$$= e^{(\varphi/2)\gamma_1\gamma_2}(\gamma_1\cos\theta - \gamma_3\sin\theta)\left(\cos\frac{\varphi}{2} - \gamma_1\gamma_2\sin\frac{\varphi}{2}\right)$$
$$= \left(\cos\frac{\varphi}{2} + \gamma_1\gamma_2\sin\frac{\varphi}{2}\right)\left\{(\gamma_1\cos\theta - \gamma_3\sin\theta)\cos\frac{\varphi}{2}\right.$$
$$\left. + (\gamma_2\cos\theta + \gamma_1\gamma_2\gamma_3\sin\theta)\sin\frac{\varphi}{2}\right\}$$
$$= \gamma_1\cos\theta\cos\varphi - \gamma_3\sin\theta + \gamma_2\cos\theta\sin\varphi, \tag{A3}$$

$$S^{-1}\gamma_2(\partial_\theta S) = \frac{1}{2}S^{-1}\gamma_2(\gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_3\gamma_1+1)\partial_\theta\left(\cos\frac{\theta}{2} - \gamma_3\gamma_1\sin\frac{\theta}{2}\right)e^{-(\varphi/2)\gamma_1\gamma_2}$$
$$= \frac{1}{4}e^{(\varphi/2)\gamma_1\gamma_2}e^{(\theta/2)\gamma_3\gamma_1}(1 - \gamma_1\gamma_2 - \gamma_2\gamma_3 - \gamma_3\gamma_1)\gamma_2$$
$$\times(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 + 1)\left(-\frac{1}{2}\sin\frac{\theta}{2} - \frac{1}{2}\gamma_3\gamma_1\cos\frac{\theta}{2}\right)e^{-(\varphi/2)\gamma_1\gamma_2}$$
$$= -\frac{1}{8}4e^{(\varphi/2)\gamma_1\gamma_2}\left(\cos\frac{\theta}{2} + \gamma_3\gamma_1\sin\frac{\theta}{2}\right)\left(\gamma_1\sin\frac{\theta}{2} + \gamma_3\cos\frac{\theta}{2}\right)e^{-(\varphi/2)\gamma_1\gamma_2}$$
$$= -\frac{1}{2}e^{(\varphi/2)\gamma_1\gamma_2}(\gamma_3\cos\theta + \gamma_1\sin\theta)\left(\cos\frac{\varphi}{2} - \gamma_1\gamma_2\sin\frac{\varphi}{2}\right)$$
$$= -\frac{1}{2}\left(\cos\frac{\varphi}{2} + \gamma_1\gamma_2\sin\frac{\varphi}{2}\right)\left\{(\gamma_3\cos\theta + \gamma_3\sin\theta)\cos\frac{\varphi}{2}\right.$$
$$\left. + (-\gamma_1\gamma_2\gamma_3\cos\theta + \gamma_2\sin\theta)\sin\frac{\varphi}{2}\right\}$$
$$= -\tfrac{1}{2}(\gamma_3\cos\theta + \gamma_1\sin\theta\cos\varphi + \gamma_2\sin\theta\sin\varphi), \tag{A4}$$
$$\tfrac{1}{2}\cot\theta S^{-1}\gamma_2 S = \tfrac{1}{2}\cot\theta(\gamma_1\cos\theta\cos\varphi - \gamma_3\sin\theta + \gamma_2\cos\theta\sin\varphi). \tag{A5}$$

Using equations (A3), (A4) and (A5) in equation (A2) give the required result of equation (36).

Appendix B

The unitary transformation (34) maps equation (32) into (28) which has to be proved here.

From equations (29) and (31) we get

$$\dot{a}(t) = \frac{1}{2}\dot{\alpha}(\tau), \quad \partial_\tau + \frac{3}{4}\dot{\alpha}(\tau) = e^{(1/2)\alpha(\tau)}\left(\partial_t + \frac{3}{2}\frac{\dot{a}}{a}\right) \quad \text{and} \quad \partial_\chi + \frac{\xi'}{\xi} = \rho\partial_r + \frac{\rho}{r}.$$

Using these with $\alpha_i = \gamma_0\gamma_i$ and the identities of equation (36), equation (32a) becomes

$$\left[e^{(1/2)\alpha(\tau)}\left(\partial_t + \frac{3}{2}\frac{\dot{a}}{a}\right) - \gamma_0\left\{(\gamma_1\sin\theta\cos\varphi + \gamma_2\sin\theta\sin\varphi + \gamma_3\cos\theta)\left(\rho\partial_r + \frac{\rho}{r}\right)\right.\right.$$
$$- \frac{1}{2r}(\gamma_1\sin\theta\cos\varphi + \gamma_2\sin\theta\sin\varphi + \gamma_3\cos\theta)$$
$$+ \frac{1}{2r}\left(\gamma_1\frac{\cos^2\theta}{\sin\theta}\cos\varphi + \gamma_2\frac{\cos^2\theta}{\sin\theta}\sin\varphi - \gamma_3\cos\theta\right)$$

$$+\frac{1}{r}(\gamma_1\cos\theta\cos\varphi+\gamma_2\cos\theta\sin\varphi-\gamma_3\sin\theta)\partial_\theta-\frac{1}{2r\sin\theta}(\gamma_1\cos\varphi+\gamma_2\sin\varphi)$$

$$\left.\left.+\frac{1}{r\sin\theta}(-\gamma_1\sin\varphi+\gamma_2\cos\varphi)\partial_\varphi\right\}+im\gamma_0\mathrm{e}^{(1/2)\alpha(\tau)}\right]\psi=0$$

or

$$\left[\partial_t+\frac{3}{2}\frac{\dot{a}}{a}-\frac{\gamma_0}{a}\left\{(\gamma_1\sin\theta\cos\varphi+\gamma_2\sin\theta\sin\varphi+\gamma_3\cos\theta)\left(\rho\partial_r+\frac{\rho}{r}\right)\right.\right.$$

$$-\frac{1}{r}(\gamma_1\sin\theta\cos\varphi+\gamma_2\sin\theta\sin\varphi+\gamma_3\cos\theta)$$

$$+\frac{1}{r}(\gamma_1\cos\theta\cos\varphi+\gamma_2\cos\theta\sin\varphi-\gamma_3\sin\theta)\partial_\theta$$

$$\left.\left.+\frac{1}{r\sin\theta}(-\gamma_1\sin\varphi+\gamma_2\cos\varphi)\partial_\varphi\right\}+im\gamma_0\right]\psi=0.$$

This equation becomes, after using the transformation (26),

$$\left[\partial_t+\frac{3\dot{a}}{2a}+im\gamma_0-\frac{\gamma_0}{a}\left\{\gamma_r\left(\rho\partial_r+\frac{\rho}{r}\right)-\frac{1}{r}\gamma_r+\frac{1}{r}\gamma_\theta\partial_\theta+\frac{1}{r\sin\theta}\gamma_\varphi\partial_\varphi\right\}\right]\psi=0,$$

$$\left[\partial_t+\frac{3\dot{a}}{2a}+im\gamma_0-\frac{1}{a}\left\{\alpha_r\partial_r+\alpha_\theta\frac{1}{r}\partial_\theta+\alpha_\varphi\frac{1}{r\sin\theta}\partial_\varphi+(\rho-1)\alpha_r\left(\partial_r+\frac{1}{r}\right)\right\}\right]\psi=0,$$

$$\left[\partial_t+\frac{3\dot{a}}{2a}+im\gamma_0-\frac{1}{a}\left\{\alpha\cdot\nabla+(\rho-1)\alpha_r\left(\partial_r+\frac{1}{r}\right)\right\}\right]\psi=0$$

and thus the result.

Appendix C

We determine the turning points, using eq. (98), from

$$\Omega(\eta_{1,2})=0. \tag{C1}$$

Defining

$$S(\eta_i,\eta_f)=\int_{\eta_i}^{\eta_f}\Omega(\eta)\mathrm{d}\eta \tag{C2}$$

and

$$S(\eta_i)=\left[\int\Omega(\eta)\mathrm{d}\eta\right]_{\eta=\eta_i}, \tag{C3}$$

the boundary condition takes the form

$$\psi_{\eta\,\widetilde{\rightarrow}\,-\infty}\quad\exp\left[iS(\eta,\eta_1)\right], \tag{C4}$$

$$\psi_{\eta \rightarrow \infty} \exp [iS(\eta, \eta_1)] - iR \exp [-iS(\eta, \eta_1)], \tag{C5}$$

where R is the reflection amplitude. In eqs (C4) and (C5) η is real but the turning points are complex. In CWKB the reflection amplitude is identified as the pair production amplitude. The CWKB solution of eq. (97) is

$$\psi \xrightarrow[\eta \rightarrow \infty]{} e^{iS(\eta,\eta_0)} - ie^{iS(\eta_1,\eta_0)-iS(\eta,\eta_1)} \sum_{\mu=0} [-ie^{iS(\eta_1,\eta_2)}]^{2\mu}, \tag{C6}$$

where η is real and η_0 is real, arbitrary and $\eta_0 > \eta$. In (C6), the first term represent a wave starting at η_0 and moving left reaches real $\eta < \eta_0$. The second term takes the contribution of the reflected part as follows. A wave starting at η_0 reaches the complex turning point η_1 and after bouncing back from η_1 reaches η. This is represented as

$$(-i) \exp[iS(\eta_1, \eta_0) - iS(\eta, \eta_1)]. \tag{C7}$$

This contribution (C7) to the second term of (C6) is then multiplied by the repeated reflections between η_1 and η_2 and so comes the following factor in the second term of (C6), namely

$$\sum_{\mu=0} [-i \exp\{iS(\eta_1, \eta_2)\}]^{2\mu} = \frac{1}{1 + e^{2iS(\eta_1,\eta_2)}}. \tag{C8}$$

For convenience we have neglected the WKB pre-exponential all through. Comparing (C6) with the boundary condition (C5) we have

$$R = \frac{\exp[2iS(\eta_1)]}{1 + \exp[2iS(\eta_1, \eta_2)]}. \tag{C9}$$

If we consider $\exp[iS(\eta, \eta_0)]$ as antiparticle solution then (C6) is interpreted as the reflection of antiparticle from the turning point η_1 and the reflected part is interpreted as particle moving forward in time. This is Klein paradox-like situation not in space but in time.

References

[1] V Fock, *Z. Phys.* **53**, 592 (1928)
[2] H Tetrode, *Z. Phys.* **50**, 336 (1928)
[3] E Schrödinger, *Preuss. Akad. Wiss. Berlin* **11–12**, 105 (1932)
[4] G C McVittie, *Mon. Not. R. Astron. Soc.* **92**, 868 (1932)
[5] D R Brill and J A Wheeler, *Rev. Mod. Phys.* **29**, 465 (1957)
[6] L P Chimento and M S Mollerach, *Phys. Rev.* **D34**, 3689 (1986)
[7] J Audretsch and G Schäfer, *J. Phys.* **A11**, 1583 (1978)
[8] M A Castagnino and F D Mazzitelli, *Phys. Rev.* **D31**, 742 (1985)
[9] S Biswas and J Guha, *Pramana–J. Phys.* **40(6)**, 467 (1993)
[10] S Biswas, J Guha and N G Sarkar, *Class. Quantum Gravit.* **12**, 1591 (1995)
[11] J Guha, D Biswas, N G Sarkar and S Biswas, *Class. Quantum Gravit.* **12**, 1641 (1995)
[12] S Biswas, J Guha and N G Sarkar, *Pramana–J. Phys.* **42(4)**, 319 (1994)
[13] K H Lotze, *Class. Quant. Gravit.* **3**, 81 (1986)
[14] A O Barut and I H Duru, *Phys. Rev.* **D36**, 3705 (1987)

[15] S Biswas, J Guha and P Dasgupta, *Int. J. Mod. Phys.* **A10**, (1995)
[16] V Sahni, *Class. Quant. Gravit.* **1**, 579 (1984)
[17] A O Barut and L P Singh, *Exact solution of the Dirac equation in spatially non-flat RW spacetime II*, ICTP, IC/90/205 (1990)
[18] J M Cornwall and G Tiktopoulos, *Phys. Rev.* **D39**, 334 (1989)
[19] L Parker, *Phys. Rev.* **183**, 1057 (1969)
[20] M V Fischetti, J B Hartle and B L Hu, *Phys. Rev.* **D20**, 1757 (1979)
[21] P Anderson, *Phys. Rev.* **D28**, 271 (1983)
[22] L H Ford, *Phys. Rev.* **D22**, 3003 (1980b)
[23] L Kofman, A Linde and A A Starobinsky, *Phys. Rev. Lett.* **73**, 3195 (1994)
[24] D Boyanovsky, M D'Attanasio, H J de Vega, R Holman, D S Lee and A Singh, Report No. PITT-09–95 (unpublished); *Phys. Rev.* **D51**, 4419 (1995) and references therein
[25] M Yoshimura, *Prog. Theor. Phys.* **94**, 873 (1995)
[26] H Fujisaki, K Kumekawa, M Yumaguchi and M Yoshimura, *Phys. Rev.* **D53**, 6805 (1996)
[27] M A Castagnino and L Chimento, *Phys. Rev.* **D34**, 3676 (1986)
[28] M A Castagnino, E Gunzig, P Nardone and J P Paz, *Phys. Rev.* **D34**, 3698 (1986)
[29] F D Mazettelli, J P Paz and M A Castagnino, *Phys. Rev.* **D34**, 2994 (1987)
[30] V Sahni, *Phys. Lett.* **A86**, 87 (1981)
[31] N D Birrell and P C W Davies, *Quantum fields in curved space* (Cambridge University Press, 1982)
[32] J Knoll and R Schaeffer, *Ann. Phys.* **97**, 307 (1976)
[33] J Knoll and R Schaeffer, *Phys. Rep.* **31**, 159 (1977)
[34] F Schrempp and B Schrempp, *Nucl. Phys.* **B163**, 397 (1980)
[35] T Padmanaban, *Ann. Phys. (NY)* **141**, 316 (1982)