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Semi-empirical formula for Λ-binding energies in ground states of light hypernuclei

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Abstract. Starting with a Λ-nucleus potential, we have obtained a semi-empirical formula, which gives a fairly satisfactory account of the ground state Λ-binding energy of light hypernuclei, if the very light nuclei are ignored.

Keywords. Semi-empirical; binding energy.

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1. Introduction

Semi-microscopic calculations of the Λ-binding to nuclei (B_Λ) have been possible only for a few nuclei, ${}^3_\Lambda$H, ${}^4_\Lambda$H, ${}^4_\Lambda$He, ${}^5_\Lambda$He, and more approximately for ${}^9_\Lambda$Be, and ${}^{16}_\Lambda$O. Semi-microscopic calculations have also been carried out rather recently for a few more nuclei. On present indications, it appears unlikely, in the near future, to successfully carry out semi-microscopic calculations for many other hypernuclei. Thus, one is more or less forced to consider phenomenological calculations of B_Λ to obtain some broad insight into some of the important aspects. It is even desirable to obtain semi-empirical formulae which can give at least a rough value of B_Λ.

The Λ-nucleus potential, in the folding model with a density-dependent effective Λ-nucleon (ΛN) interaction, has been shown [1, 2] to be quite satisfactory in accounting for the ground state B_Λ data of light hypernuclei. However, all these calculations have to be carried out numerically.

By mathematical manipulations of the folded potential, several authors [3, 4] have obtained semi-empirical formulae of B_Λ for medium and heavy hypernuclei. However, none of these formulae hold for light hypernuclei for the simple reason that the core density is not of the Woods–Saxon (W–S) form as is the case for medium and heavy nuclei. No semi-empirical or even empirical formula exists for light hypernuclei because of the fact that there are different mathematical forms of the density for different light nuclei and, more importantly, simple mathematical manipulations, leading to a semi-empirical formula, do not seem to be possible for any of these forms.

So, we made a trial and error search for the Λ-nucleus potential for light hypernuclei using various analytical forms to find one that gives tolerable results by adding a plausible term that distinguishes between hypernuclei of the same core mass number but different number of protons. It was hoped that mathematical manipulation of an expression of the B_Λ may yield a semi-empirical formula. Luckily, we find that the B_Λ formula obtained using exponential form of the Λ-nucleus potential gives a semi-quantitative account of the ground state B_Λ data of light hypernuclei, ignoring only the very light hypernuclei. On general plausible grounds, we expect that our treatment would apply only to nuclei lying on the stability line and very close to it.

We obtain a semi-empirical formula for the Λ-binding energy in bound s-states of light hypernuclei using the exponential form for the Λ-nucleus potential. This formula has three adjustable parameters. After very light hypernuclei, ${}^{3}_{\Lambda}$H, ${}^{4}_{\Lambda}$H and ${}^{4}_{\Lambda}$He, are excluded from the fit, the formula gives a reasonable account of the B_Λ of nine light hypernuclei for which the experimental r.m.s. radius is available. With a little more relaxation on accuracy, a slightly modified formula accounts for seventeen nuclei. The formula is presented in the next section. Results and discussion is given in § 3 and conclusions in § 4.

2. A semi-empirical formula for the Λ-binding energies of light hypernuclei

We set the radial wave function $R_0(r) = \chi(r)/r$ and analytically solve the s-state radial Schrödinger equation with the exponential potential $V(r) = -V_0\,\mathrm{e}^{-r/a}$. The parameter a appearing in this potential is taken, in a plausible way, to be given as

$$a = r_0' A_c^{1/3},$$

where r_0' may be taken as a free parameter. We transform the Schrödinger equation by taking a new variable $z = \mathrm{e}^{-r/2a}$. Introducing the parameters $\alpha = +(8m_\Lambda V_0 a^2/\hbar^2)$, $\beta = +[8m_\Lambda |E| a^2/\hbar^2]^{1/2}$, where E is the energy eigenvalue, and, for definiteness, taking $\sqrt{\alpha}$ to be positive, we have

$$\chi(r) = AJ_\beta(\sqrt{\alpha}\,\mathrm{e}^{-r/2a}) + B\,Y_\beta(\sqrt{\alpha}\,\mathrm{e}^{-r/2a}). \tag{2.1}$$

When β is not an integer or zero, $Y_\beta(r)$ is simply equal to $J_{-\beta}(r)$. As $r \to \infty$, the argument of both J_β and Y_β vanishes and since Y_β is not well behaved at zero, the coefficient of Y_β has to be put equal to zero.

Further, since $R_0(r) = \chi(r)/r$, it follows that $\chi(0) = 0$. Therefore, we have

$$J_\beta(\sqrt{\alpha}) = 0.$$

For real β, the function $J_\beta(\sqrt{\alpha})$ has an infinite number of real zeroes. The nth positive zero of this function can be expressed as the following expansion [5]

$$\sqrt{\alpha} = \gamma - \frac{q-1}{8\gamma}\left[1 + \frac{Q_1}{3(4\gamma)^2} + \frac{2Q_2}{15(4\gamma)^4} + \cdots\right], \tag{2.2}$$

where $q = 4\beta^2$, $\gamma = (\beta - \frac{1}{2} + 2n)\frac{\pi}{2}$, $Q_1 = 7q - 31$, $Q_2 = 83q^2 - 982q + 3779, \ldots$.

In this form, it is a little complicated to get the energy for given potential parameters. Apart from other reasons, formula (2.2) is very unwieldy. One wants a simpler formula.

Here, we use (2.2) to derive the desired semi-empirical formula for B_Λ. In (2.2), we estimate the magnitudes of the 2nd and the 3rd terms within the square brackets, for a range of energy eigenvalues, say from $|E| = 0$ to roughly the expected value of B_Λ, with A_c

ranging from 2 to 25. We take $r'_0 = 0.41$ fm. This value of r'_0 is, however, not crucial for our calculations. Any other reasonable choice of r'_0 would give very similar result. Even the maximum magnitude of the 2nd and 3rd terms, in the mass number range 2 to 25, is found to be very small compared to unity and further these are of the opposite sign, making the net contribution even smaller. Their contribution for nuclei in the low mass number region is even less than it is for the relatively higher mass numbers. Thus, we neglect all the terms other than unity in the square bracket in equation (2.2). Then, (2.2) may be written as

$$\sqrt{\alpha} = \gamma - \frac{q-1}{8\gamma}. \tag{2.3}$$

Substituting for α, γ and β in the above equation and simplifying we get

$$B_\Lambda = \frac{\hbar^2 A_c^{-2/3}}{8m_\Lambda r'^2_0((\pi^2/2)-1)^2}\left\{(\xi_1+\xi_2) \pm \left[(\xi_1+\xi_2)^2 - (\pi^2-2) \times \left(\frac{\xi_1^2}{\pi^2} + \left(\frac{1}{2}-2n\right)\xi_2 + \frac{1}{8}\right)\right]^{1/2}\right\}^2, \tag{2.4}$$

where $\xi_1 = \pi^2/2((1/2)-2n)$ and $\xi_2 = \pi r'_0 A_c^{1/3}/2\ [8m_\Lambda V_0/\hbar^2]^{1/2}$. For the ground state, $n = 1$. The formula (2.4) in this form does not distinguish between hypernuclei having same A_c but different number of protons (Z). So, in order to take the Z-dependence and also the size of the core nuclei into account, we add a term $\beta' Z/r^3_{\text{rms}}$, where r_{rms} is the r.m.s. radius of the core nucleus being considered, to the B_Λ formula (2.4), i.e.

$$B_\Lambda = \frac{\hbar^2 A_c^{-2/3}}{8m_\Lambda r'^2_0(\pi^2/2-1)^2}\left\{(\xi_1+\xi_2) \pm \left[(\xi_1+\xi_2)^2 - (\pi^2-2) \times \left(\frac{\xi_1^2}{\pi^2} + \left(\frac{1}{2}-2n\right)\xi_2 + \frac{1}{8}\right)\right]^{1/2}\right\}^2 + \frac{\beta' Z}{r^3_{\text{rms}}}. \tag{2.5}$$

The last term added above is essentially same as that taken by Rahman Khan and Shoeb [6] who have given the justification for it. All we have done is not to take it as $\beta' Z/A_c$. For light nuclei it seems better to take r_{rms} directly rather than replace it with $A_c^{1/3}$.

We find that formula (2.5) with the positive sign for the 2nd term within the curly bracket gives reasonable values of B_Λ whereas when negative sign is chosen, quite unrealistic values are obtained. Thus, on this basis, we retain, in our formula (2.5) throughout, only the positive sign for the 2nd term within the curly bracket.

3. Results and discussion

Treating V_0, r'_0 and β' as adjustable parameters, χ^2-fitting is carried out for the ground state B_Λ data of the light hypernuclei with core mass number ranging from 5 to 14. Except for very light hypernuclei, ${}^3_\Lambda$H, ${}^4_\Lambda$H and ${}^4_\Lambda$He, all other light hypernuclei for which rms radii of the core nuclei, in the given mass number range, are known experimentally [7], are included in the fit. However, predictions are made regarding the B_Λ of the excluded light hypernuclei.

Table 1. Results of χ^2-fitting of B_Λ using analytical formula (2.5). The best fit parameters for $\chi^2 = 29.04$ are: $V_0 = 103.37$ MeV, $r_0' = 0.403$ fm and $\beta' = 1.65$ MeV fm^3. B_Λ^{cal}, marked by an asterisk, are the predicted values.

Hypernuclei	A_c	$B_\Lambda^{\mathrm{exp}} \pm \Delta B_\Lambda^{\mathrm{exp}}$ (MeV)	B_Λ^{cal} (MeV)
$^{3}_{\Lambda}\mathrm{H}$	2.0	0.15 ± 0.05	0.21*
$^{4}_{\Lambda}\mathrm{H}$	3.0	2.04 ± 0.04	1.32*
$^{4}_{\Lambda}\mathrm{He}$	3.0	2.39 ± 0.03	1.45*
$^{5}_{\Lambda}\mathrm{He}$	4.0	3.12 ± 0.02	3.12
$^{7}_{\Lambda}\mathrm{Li}$	6.0	5.58 ± 0.03	5.54
$^{8}_{\Lambda}\mathrm{Li}$	7.0	6.80 ± 0.03	6.85
$^{10}_{\Lambda}\mathrm{Be}$	9.0	9.11 ± 0.22	9.12
$^{11}_{\Lambda}\mathrm{B}$	10.0	10.24 ± 0.05	10.23
$^{12}_{\Lambda}\mathrm{B}$	11.0	11.37 ± 0.06	11.19
$^{13}_{\Lambda}\mathrm{C}$	12.0	11.69 ± 0.12	12.06
$^{14}_{\Lambda}\mathrm{C}$	13.0	12.17 ± 0.33	12.87
$^{15}_{\Lambda}\mathrm{N}$	14.0	13.59 ± 0.15	13.59

*Predicted values.

Table 2. Results of χ^2-fitting of B_Λ from analytical formula (2.5) in which the last term is replaced by $\beta'(Z/A_c)$. The best fit parameters for $\chi^2 = 214.15$ are: $V_0 = 98.87$ MeV, $r_0' = 0.41$ fm and $\beta' = 1.00$ MeV fm^3.

Hypernuclei	A_c	$B_\Lambda^{\mathrm{exp}} \pm \Delta B_\Lambda^{\mathrm{exp}}$ (MeV)	B_Λ^{cal} (MeV)
$^{5}_{\Lambda}\mathrm{He}$	4.0	3.12 ± 0.02	3.04
$^{6}_{\Lambda}\mathrm{He}$	5.0	4.18 ± 0.10	4.35
$^{7}_{\Lambda}\mathrm{Li}$	6.0	5.58 ± 0.03	5.77
$^{8}_{\Lambda}\mathrm{He}$	7.0	7.16 ± 0.70	6.77
$^{8}_{\Lambda}\mathrm{Li}$	7.0	6.80 ± 0.03	6.91
$^{8}_{\Lambda}\mathrm{Be}$	7.0	6.84 ± 0.05	7.06
$^{9}_{\Lambda}\mathrm{Li}$	8.0	8.50 ± 0.12	7.97
$^{9}_{\Lambda}\mathrm{B}$	8.0	8.29 ± 0.18	8.22
$^{10}_{\Lambda}\mathrm{Be}$	9.0	9.11 ± 0.22	9.06
$^{10}_{\Lambda}\mathrm{B}$	9.0	8.89 ± 0.12	9.17
$^{11}_{\Lambda}\mathrm{B}$	10.0	10.24 ± 0.05	10.05
$^{12}_{\Lambda}\mathrm{B}$	11.0	11.37 ± 0.06	10.88
$^{12}_{\Lambda}\mathrm{C}$	11.0	10.76 ± 0.19	10.97
$^{13}_{\Lambda}\mathrm{C}$	12.0	11.69 ± 0.12	11.73
$^{14}_{\Lambda}\mathrm{C}$	13.0	12.17 ± 0.33	12.44
$^{15}_{\Lambda}\mathrm{N}$	14.0	13.59 ± 0.15	13.18
$^{16}_{\Lambda}\mathrm{O}$	15.0	13.0	13.87

As usual the χ^2 is defined as

$$\sum_i \left[\frac{B_\Lambda^{\rm exp}(i) - B_\Lambda^{\rm th}(i)}{\Delta B_\Lambda^{\rm exp}(i)}\right]^2,$$

where the symbols have their usual meaning, with the summation extending to all the data-points. The experimentally quoted value is taken to be $B_\Lambda^{\rm exp}(i) \pm \Delta B_\Lambda^{\rm exp}(i)$.

The calculated B_Λ corresponding to the best fit parameters, $V_0 = 103.37\,\mathrm{MeV}$, $r_0' = 0.403\,\mathrm{fm}$ and $\beta' = 1.65\,\mathrm{MeV\,fm}^3$, are shown in table 1. Total χ^2 corresponding to the 9 data-points is 29.04. This may be taken as a fairly good agreement. Predicted B_Λ values, which are marked by an asterisk, are also shown in the table. Thus, formula (2.5) is the semi-empirical formula for B_Λ of light hypernuclei that we were looking for.

Expressing the last term of B_Λ formula (eq. (2.5)) as $\beta'(Z/A_c)$ does not give good results, presumably due to neglect of size effect which seems to have an even more important role in light hypernuclei. However, even this is qualitatively acceptable when the very light hypernuclei, ${}^3_\Lambda\mathrm{H}$, ${}^4_\Lambda\mathrm{H}$ and ${}^4_\Lambda\mathrm{He}$, and the well known 'troublesome' nuclei [8–10], ${}^7_\Lambda\mathrm{Be}$ and ${}^9_\Lambda\mathrm{Be}$, are excluded from the fit. In this way they account for 17 light hypernuclei. These are given in table 2. As error bar in the experimental B_Λ data of ${}^{16}_\Lambda\mathrm{O}$ is not available, we take a plausible value of 5% of the experimental B_Λ as the error in the datum. Besides $\beta'(Z/A_c)$, if a constant term is also added in the formula, the χ^2 reduces significantly.

4. Conclusion

The main virtue of the exponential potential, from our point of view, is that it leads to a semi-empirical formula for the light hypernuclei, which reproduces B_Λ of many light hypernuclei fairly well. We do not offer any conjectures whether any deeper significance need be ascribed to the exponential potential. From what we have said above, the answer seems to be in the negative. Still, the calculations points to an even greater importance of size effects for light hypernuclei.

The fomula with $r_{\rm rms}$ (eq. (2.5)) may be used for predicting the $r_{\rm rms}$ of the core of the hypernuclei for which it is not experimentally known.

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