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Darboux transformation and elementary exact solutions of the Schrödinger equation

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Abstract. Darboux transformation is applied to three classical potentials, namely the harmonic oscillator, effective Coulomb and Morse potentials to generate exactly solvable potentials of elementary form. For every potential, the isospectral families of potentials are constructed. For almost all potentials, a set of normalized discrete spectrum wave functions is given.

Keywords. Schrödinger equation; Hamiltonian; wavefunction; Darboux transformation.

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1. Introduction

Since the first reported work [1], the Darboux transformation has been an area of great interest. Perhaps this interest was first reflected in [2], then this method was rediscovered as the factorization method [3, 4]. The next stimulus to its development was given by the supersymmetric formulation of the Schrödinger equation [5]. (For a recent review in this domain see [6]). The soliton theory must be mentioned especially because the Darboux transformation played a considerable role in the development of this theory (see [7] and references therein). This transformation was investigated recently in connection with the spectral theory of the Schrödinger operator by the techniques of dressing chain [8] and as a source of new exactly solvable potentials [9–11]. Despite the fact that this type of transformation has been studied for a long period, the number of quantum mechanical problems exactly solved in elementary functions is not too large, especially if the question consists in finding the normalized discrete spectrum wave functions. In this article we make an attempt to fill this gap. We give and investigate new exactly solvable potentials for the one-dimensional Schrödinger equation as applied to the harmonic oscillator, effective Coulomb and Morse potentials. On the base of each potential we construct a family of isospectral potentials. For almost all potentials we give a set of normalized discrete spectrum wave functions. We concentrate our attention on the solutions of elementary form.

2. Generating properties of the Darboux transformation

In this section we give a brief overview of the main properties of the Darboux transformation discussed in a number of earlier publications.

Let us consider the one-dimensional Schrödinger equation (setting $\hbar = 2m = 1$)

$$H_0\psi_E(x) = E\psi_E(x), \quad H_0 = -\mathrm{d}^2/\mathrm{d}x^2 + V_0(x), \quad x \in [a,b]. \tag{1}$$

The potential $V_0(x)$ is supposed to be a real and sufficiently smooth function in the interval $R = [a,b]$ which can be infinite. It is also supposed that we know the general solution of this equation for all values of the parameter E (complex in general). Let functions $u_i(x)$ called transformation functions be particular solutions of (1) with α_i as eigenvalues. Using these functions we can introduce the following operator called the N-order Darboux transformation operator.

$$L_N = W^{-1}(u_1, u_2, \ldots, u_N) \begin{vmatrix} u_1 & u_2 & \cdots & 1 \\ u_1' & u_2' & \cdots & \mathrm{d}/\mathrm{d}x \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(N)} & u_2^{(N)} & \cdots & \mathrm{d}^N/\mathrm{d}x^N \end{vmatrix}, \tag{2}$$

where $W(u_1, u_2, \ldots, u_N)$ is the Wronskian of functions $u_1, u_2, \ldots, u_N$, $(W(u_1) = u_1)$, the prime denotes a derivative with respect to x, and the determinant is a differential operator obtained by the development of the determinant in the last column with the functional coefficients placed before the derivative operators.

Operator L_N has the property [11]

$$L_N H_0 = H_N L_N, \tag{3}$$

where

$$H_N = -\mathrm{d}^2/\mathrm{d}x^2 + V_N(x) \tag{4}$$

is a new Hamiltonian and the potential difference $A_N(x) = V_N(x) - V_0(x)$ is given by the formula [12, 13]:

$$A_N(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log W(u_1, \ldots, u_N). \tag{5}$$

The operator L_N when applied to any solution of (1) gives the known Crum–Krein formula [12, 13] for the solutions of a new Schrödinger equation with the Hamiltonian H_N

$$\varphi_E(x) = L_N\psi_E(x). \tag{6}$$

It can be shown [8, 11] that any N-order differential transformation operator satisfying (3) can always be presented in the form (2) by properly choosing its arbitrary constants.

If H_0 and H_N are formally self-adjoint operators (in the sense of some scalar product) then we have from formula (3)

$$H_0 L_N^+ = L_N^+ H_N. \tag{7}$$

It follows from this relation that the operator $L_N^+ \equiv L_N^\dagger$ [being Hermitian conjugated to L_N with respect to an inner product for which $(\mathrm{d}/\mathrm{d}x)^+ = -(\mathrm{d}/\mathrm{d}x)$] realizes the transformation in the inverse direction i.e. from the solutions of the new Schrödinger equation to the solutions of the initial one, and consequently the product $L_N^+ L_N$ is a symmetry operator for (1). In full analogy, the operator $L_N L_N^+$ is a symmetry operator for the new Schrödinger equation. Since we deal with a one dimensional Schrödinger equation the above-mentioned symmetry operators are the Hamiltonian and its polynomial functions. As a result, we have the following remarkable factorization properties [14, 11]:

$$L_N^+ L_N = (H_0 - \alpha_1)(H_0 - \alpha_2)\cdots(H_0 - \alpha_N), \tag{8}$$

$$L_N L_N^+ = (H_N - \alpha_1)(H_N - \alpha_2)\cdots(H_N - \alpha_N). \tag{9}$$

In the case $N = 1$ we have the conventional Darboux transformation [1]

$$L \equiv L_1 = -u'(x)/u(x) + \mathrm{d}/\mathrm{d}x, \quad H_0 u = \alpha u$$

which is the first order transformation. The relations (8) and (9) take for this case the form $L^+L = H_0 - \alpha$, $LL^+ = H_1 - \alpha$. These are the properties that are used in the supersymmetric quantum mechanics when one defines the superhamiltonian

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}$$

and supercharge operators

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}$$

which form superalgebra $sl(1/1)$ [5] with commutation relations $[H, Q] = [H, Q^+] = 0$ and anticommutation ones $\{Q, Q\} = \{Q^+, Q^+\} = 0, \{Q, Q^+\} = H - \alpha I$, where I is the unit 2×2 matrix. Note that the use of the N-order transformation operator L_N and its conjugate L_N^+ in supercharges Q and Q^+ gives N-order superalgebra [11, 14]

$$\{Q, Q^+\} = (H - I\alpha_1)(H - I\alpha_2)\cdots(H - I\alpha_N),$$

where superhamiltonian H has to be constructed with the help of the Hamiltonians H_0 and H_N and I is $(N+1) \times (N+1)$-dimensional unit matrix.

It follows from the definition of the operator L that Ker $L = \mathrm{span}\{u\}$ and Ker $L^+ = \mathrm{span}\{v\}$ where $v = u^{-1}$ and span denotes the linear hull over the complex number field. Note that the function v is an eigenfunction of the Hamiltonian H_1 and it corresponds to the eigenvalue α. Let $\tilde{u}$ and $\tilde{v}$ be another linearly independent solutions of the Schrödinger equations with the Hamiltonians H_0 and H_1 respectively corresponding to the same eigenvalue α. If they are chosen such that $W(u_1, u_2) = 1$ and $W(v_1, v_2) = 1$ then

$$\tilde{u} = u \int u^{-2}\mathrm{d}x, \quad \tilde{v} = v \int v^{-2}\mathrm{d}x$$

and $L\tilde{u} = v, L^+\tilde{v} = -u$. This signifies that with the help of the operators L and L^+ we can establish the one-to-one correspondence between the solutions of the initial Schrödinger equation with the Hamiltonian H_0 and the transformed one with the Hamiltonian H_1. Every two dimensional space T_E^0 of the solutions of the input equation with the eigenvalue $E \neq \alpha$ is transformed in the space T_E^1 of the solutions of the final equation with the same eigenvalue E. If $E = \alpha$ we have $\tilde{u} \to v, \tilde{v} \to -u$. This means that the Darboux transformation operator L gives a complete information on the transformed Schrödinger equation. In particular we can obtain not only the discrete spectrum eigenfunctions but the scattering data for the Hamiltonian H_1 as well.

The scattering data for the Schrödinger equation (1) considered on full axis $R=(-\infty,+\infty)$ can be obtained with the help of its Jost solutions $\psi^+(x,k)$ and $\psi^-(x,k)$ (see for example [15]) with $k^2=-E$ and k is a complex number. These solutions have the following asymptotic behavior $\psi^\pm(x,k)\to\exp(\mp kx)$ as $x\to\pm\infty$ at $\mathrm{Re}(k)>0$. We suppose that the discrete spectrum of (1) (if exists) is negative ($E_i<0$) and the continuous one is positive. The four Jost solutions $\psi^\pm(x,k)$ and $\psi^\pm(x,-k)$ are linearly dependent from each other, namely any of them can be expressed as a linear combination of any two others. One can use $\psi^+(x,k)$ and $\psi^+(x,-k)$ as the basis in the two-dimensional space of the solutions of (1) at $E=-k^2$ since their Wronskian equals $2k\neq 0$. Then $\psi^-(x,k)$ can be expanded in terms of this basis as follows [9]:

$$\psi^-(x,k)=a_0(k)\psi^+(x,-k)+b_0(k)\psi^+(x,k),\quad E=-k^2,\quad \mathrm{Re}\,k>0.$$

Coefficients $a_0(k)$ and $b_0(k)$ define the scattering data [15] for the potential $V_0(x)$. For the real valued potentials these coefficients satisfy the following conditions

$$a_0(-k)=a_0^*(k),\quad b_0(-k)=b_0^*(k),\quad |a_0(k)|^2-|b_0(k)|^2=1,$$

where the asterisk implies the complex conjugation.

Consider first the case when the transformation function u is the ground state function of the Hamiltonian H_0: $H_0u=\alpha u, E_0=\alpha=-k_0^2<0$. In this case the energy E_0 does not belong to the discrete spectrum of the transformed Hamiltonian. Using the known asymptotic behavior of the function u we find the scattering data for the Hamiltonian H_1. (Note that differentiating of the asymptotic of functions can be completely justified in this case). Since $L^+L=H_0-\alpha$ and $LL^+=H_1-\alpha$ the Jost solutions for the two Schrödinger equations with the Hamiltonian H_0 and H_1 are interrelated by the operators L and L^+ [9]

$$\varphi^\pm(x,k)=\pm(k_0-k)^{-1}L\psi^\pm(x,k),\quad \psi^\pm(x,k)=\pm(k_0+k)^{-1}L^+\varphi^\pm(x,k),$$
$$k_0^2=-\alpha>0.$$

It follows from these relations that the coefficients $a_1(k)$ and $b_1(k)$ defining the scattering data for the Hamiltonian H_1 are expressed in terms of the coefficients $a_0(k)$ and $b_0(k)$ as follows [9]:

$$a_1(k)=\frac{k+k_0}{k-k_0}a_0(k),\quad b_1(k)=-b_0(k).$$

If we insert a new eigenvalue in the spectrum of the initial Hamiltonian we should replace $k_0\to -k_0$ since this procedure is equivalent to deletion of the ground state level from the Hamiltonian H_1.

It can be shown [11] that the N-order operator L_N can always be presented as a product of N first order Darboux transformation operators: $L_N=L^{(N-1,N)}L^{(N-2,N-1)}\ldots L^{(0,1)}$ and consequently the formula (2) represents the final result of the iteration of N first order transformations. Every operator $L^{(p-1,p)}$ intertwines the Hamiltonians H_{p-1} and H_p: $L^{(p-1,p)}H_{p-1}=H_pL^{(p-1,p)}, p=1,2,\ldots,N$ and indeed we have a chain of Hamiltonians $H_0\to H_1\to\cdots\to H_N$ obtained one from another by the chain of sequential transformations $L^{(0,1)}$, $L^{(1,2)}$, $L^{(N-1,N)}$ (so called dressing chain, see for example [8]). Some properties of such a chain was recently studied in connection with the construction of reflectionless potentials with infinite discrete spectrum [9].

In the two cases the chain so obtained can have ill-defined elements [16] although the final Hamiltonian remains well-defined. The first case which corresponds to complex-valued intermediate potentials has been described in [14]. We can illustrate this case by the following simple example [17]. It can be shown that for $N = 2$ the formula (5) gives the real-valued potential if the (essentially complex) transformation functions u_1 and u_2 correspond to mutually conjugated eigenvalues: $H_0u_1 = \alpha u_1, H_0u_2 = \alpha^* u_2$. The intermediate transformations give in this case the complex-valued potential differences and the intermediate Hamiltonian H_1 is not self-adjoint.

In the second case we can obtain using (5) a regular potential $V_2(x) = V_0(x) + A_2(x)$ using two juxtaposed discrete energy spectrum eigenfunctions $u_1(x) = \psi_j(x)$ and $u_2(x) = \psi_{j+1}(x)$ of the Hamiltonian H_0 as transformation functions. This fact was proved by Krein [13] and recently rediscovered by Adler [18]. The intermediate potential $V_1(x)$ is in this case a singular function in the interval R and the spectral problem for it strongly differs from the one for the Hamiltonians H_0 and H_2. Using transformations of these kind we can construct a supersymmetric quantum mechanical model with unusual properties [19], namely, the states which are annihilated by supercharges are located in the middle of the discrete spectrum of the superhamiltonian and the ground state of the superhamiltonian is degenerate.

In the general case of N transformation functions $u_{m_1}, u_{m_2}, \ldots, u_{m_N}$ the potential difference [see the formula (5)] is a regular function if the Wronskian $W(u_{m_1}, u_{m_2}, \ldots, u_{m_N})$ conserves its sign on the interval R. For the case when $u_{m_i}, i = 1, 2, \ldots, N$, are the discrete spectrum eigenfunctions of the Hamiltonian H_0 the sign conservation condition of this Wronskian is known [13]. The Wronskian $W(u_{m_1}, u_{m_2}, \ldots, u_{m_N})$ conserves its sign for $u_{m_i} \in L^2$ (the space of square integrable functions on the interval R) if the integers $(0) \leq m_1 < m_2 < \cdots < m_N$, being equal to a number of zeros of the function u_{m_i}, satisfy the condition $(m - m_1)(m - m_2) \cdots (m - m_N) \geq 0$ for all $m = 0, 1, 2, \ldots$. This condition is satisfied, in particular, if α_{m_i} are two by two, the juxtaposed points of the discrete spectrum. Using this property we can construct exactly solvable potentials in terms of elementary functions. The levels with $E = \alpha_{m_i}$ will be absent in the discrete spectrum of the transformed Hamiltonians. Nevertheless the functions $\varphi_i = L_N\psi_i$ form the discrete basis of the space L^2 [13, 18].

It follows from the formulae (6) and (2) that $L_Nu_i = 0, i = 1, 2, \ldots, N$ i.e. $\mathrm{Ker}\, L_N = \mathrm{span}\{u_i, i = 1, 2, \ldots, N\}$. Nevertheless, with every solution u_i of the initial Schrödinger equation we can associate the function

$$\varphi_m = W^{(m)}(u_1, u_2, \ldots, u_N)W^{-1}(u_1, u_2, \ldots, u_N) \tag{10}$$

which is the solution of the Schrödinger equation with the potential $V_N(x) = V_0(x) + A_N(x), A_N(x)$ being calculated using the formula (5). $W^{(m)}(u_1, u_2, \ldots, u_N)$ is the Wronskian of the functions $u_1, u_2, \ldots, u_N$ except the function u_m. Since the operator L_N^+ assures the transformation in the inverse direction we have $L_N^+\varphi_m = 0$, $m = 1, 2, \ldots, N$, i.e. $\mathrm{Ker}\, L_N^+ = \mathrm{span}\{\varphi_m, m = 1, 2, \ldots, N\}$. If the functions φ_m satisfy the boundary condition of the discrete spectrum eigenfunctions of the Hamiltonian H_N operator L_N creates N additional discrete spectrum levels for this Hamiltonian with respect to the Hamiltonian H_0.

Operator L_N completely defines the properties of the Hamiltonian H_N. For example, using the factorization property (8) we can easily obtain the normalization constants for the transformed Hamiltonian discrete spectrum eigenfunctions

$$\langle\varphi_i|\varphi_i\rangle = \langle L_N\psi_i|L_N\psi_i\rangle = \langle\psi_i|L_N^+L_N\psi_i\rangle = \prod_{m=1}^{N}(E_i - \alpha_m). \tag{11}$$

The same formula permits one to find the coefficients $a_N(k)$ and $b_N(k)$ for the Hamiltonian H_N. For example, if all the transformation functions appertain to the discrete spectrum of the initial Hamiltonian we obtain

$$a_N(k) = \prod_{m=1}^{N}\frac{k+k_m}{k-k_m}\,a_0(k), \quad b_N(k) = (-1)^N b_0(k), \quad k_m^2 = -\alpha_m > 0.$$

Let us now consider the scattering data for the radial Schrödinger equation for the Coulomb potential. One dimensional spectral problem for this case should be considered on the half-line $R = [0,\infty)$ and the potential has an additional term $l(l+1)x^{-2}$. The notation for k used in the papers [20, 21] is different from the one introduced here. To facilitate the comparison of our results with that of [21], we change now the notation for $k : k^2 = E > 0$ for the continuous spectrum of the radial Schrödinger equation. The presence of the centripetal member $l(l+1)x^{-2}$ makes the analysis slightly different from the one on full-line. Two linearly independent Jost solutions depending now on the angular quantum number l, $f_l(k,r)$ and $f_l(-k,r)$ are defined by the following asymptotic behavior at $r \to \infty$ [20]

$$f_l(k,r) \to e^{(il\pi)/2}e^{ikr}.$$

The regular at $r = 0$ solution of the Schrödinger equation defined by the boundary condition

$$\lim_{r\to\infty}(2l+1)!!\,r^{-l-1}\psi_l(k,r) = 1$$

is decomposed in terms of the basis functions $f_l(k,r)$ and $f_l(-k,r)$ with the help of the Jost function $F_l(k)$ [20]

$$\psi_l(k,r) = \tfrac{1}{2}ik^{-l-1}[F_l(k)f_l(-k,r) - (-1)^l F_l(-k)f_l(k,r)]. \tag{12}$$

Let us choose the ground state function of the initial Schrödinger equation as the transformation function: $u = \psi_0(k,r), H_0u = \alpha u,\ k_0^2 = -\alpha = -E_0 > 0$. The transformed equation differs in this case from the initial one only by the value of l which has to be changed as follows $l \to l+1$. Using the known asymptotic behavior of the function u and the factorization property $L^+L = H_0 - \alpha$ we find

$$Lf_l^0(k,r) = (k - ik_0)f_{l+1}^1(k,r), \quad L^+f_{l+1}^1(k,r) = (k+ik_0)f_l^0(k,r), \tag{13}$$

where the value of superscript is related with the number of Hamiltonian. Moreover, for the case under consideration the Jost solutions for two Hamiltonians are related as follows: $f_{l+1}^1 = f_{l+1}^0 = if_l^0$. Using the asymptotic behavior of the regular solution $\psi_l^0(k,r)$ at $k \to \infty$ [20]:

$$\psi_l^0(k,r) \to k^{-l-1}\sin(kr - \tfrac{1}{2}l\pi)$$

it is not difficult to obtain the following relations:

$$L\psi_l^0(k,r) = (k^2 + k_0^2)\psi_{l+1}^1(k,r), \quad L^+\psi_{l+1}^1(k,r) = \psi_l^0(k,r). \tag{14}$$

It follows from the formulae (12), (13) and (14) that

$$F_{l+1}^1(k) = -\frac{k}{k - ik_0} F_l^0(k). \tag{15}$$

This formula differs from that given in [21] and obtained with the help of the integral representation of the Jost function only by the sign. This difference is due to the different definition of the tranformation operator: $L = -A^+$. The extension of the formula (15) to the iteractive procedure is obvious. Using the asymptotic behavior of other transformation functions we can find the Jost function for other transformations.

We will now make some comments on the possibility of getting multiparameter families of isospectral potentials. Such a possibility has earlier been discussed in a number of publications, for example [22–27] (see as well [6]). In [28], the author described the possibility of using the differential transformation operators to obtain a one parameter family of potentials which are identical to the ones obtained using the Gelfand–Levitan–Marchenko integral operator. The paper [16] contains the generalization of this approach making possible to get N-parameter family of potentials. As an illustration of the method a new representation of the known N-soliton potential (e.g. N-parametric potential with N arbitrary disposed discrete energy spectrum levels) has been obtained [16]. In particular case $N = 2$ this potential completely coincides with the one reported in [26].

The possibility of getting the family of isospectral potentials resides on the use of the general solution of the initial Schrödinger equation which is given $E = \alpha < E_0$ a linear combination of its two linearly independent solutions as the transformation function u. The position of the discrete energy levels does not depend on the value of the coefficients of this linear combination. Hence, we obtain strictly isospectral potentials having the same discrete spectrum levels and reflection coefficients. If the function $v = u^{-1}$ is square integrable (this is the ground state function for the whole family in this case) the potentials differ only by the value of the normalization coefficient of the function v. Its calculation method is described in [16].

3. Elementary exactly solvable potentials

If the initial Schrödinger equation has elementary solutions, the double Darboux transformation [see formulae (2), (5) and (6) at $N = 2$] with the adjacent points of the discrete spectrum functions as transformation functions gives a new elementary exactly solvable potential. Almost all elementary solutions in one-dimensional quantum mechanics are related with the hypergeometric function or its confluent form. The general potential exactly solvable in hypergeometric functions is cited in [29]. The application of the double Darboux transformation to this potential leads to very complicated formulae. We shall consider now some special simpler cases corresponding to the harmonic oscillator, effective Coulomb and Morse potentials.

3.1 *Harmonic oscillator*

Let H_0 be given by $H_0 = -\mathrm{d}^2/\mathrm{d}x^2 + x^2/4 - 1/2$. Consider first a case where the transformation function does not belong to the space L^2. There exist a family of elementary solutions of the Schrödinger equation for the harmonic oscillator. For example, for even $m = 2p$, we can choose

$$u_m(x) = \psi_{-m}(x) = \exp(x^2/4)H_m(ix/\sqrt{2}), \quad H_0 u_m(x) = -(m+1)u_m(x)$$
$$m = 0, 1, 2, \ldots, \qquad (16)$$

where $H_m(x)$ are the Hermite polynomials [30]. The Darboux transformation with these functions produces the family of potentials

$$V_1^{(2p)}(x) = x^2/4 - 3/2 + 8p^2[q_{2p-1}(x)/q_{2p}(x)]^2 - 4p(2p-1)q_{2p-2}(x)/q_{2p}(x),$$
$$q_p(x) = (-i)^p He_p(ix), \quad He_p(x) = 2^{-p/2}H_p(x/\sqrt{2}), \qquad (17)$$
$$q_{p+1}(x) = xq_p(x) + pq_{p-1}(x), \quad q_0(x) = 1, \quad q_1(x) = x.$$

Potential (17) has been studied in detail for $m = 2$ by Dubov *et al* [31]. These potentials have been previously obtained by us [11] but their eigenfunctions were not cited. All these potentials have a sharp minimum in the region $x = 0$ and with growing x they tend to the parabola $x^2/4$. The discrete spectrum of the new Hamiltonians $H_1^{(m)} = -\mathrm{d}^2/\mathrm{d}x^2 + V_1^{(m)}(x)$ has one additional level $E_0 = -(m+1)$ with respect to the Hamiltonian H_0. The discrete spectrum wave functions have the form

$$\varphi_0(x) = \sqrt{(2p)!}(2\pi)^{-1/4}\exp(-x^2/4)/q_{2p}(x),$$
$$\varphi_{n+1}(x) = (2\pi)^{-1/4}(n!)^{-1/2}(n+2p+1)^{-1/2}\exp(-x^2/4) \qquad (18)$$
$$\times [He_n(x)q_{2p+1}(x)/q_{2p}(x) - nHe_{n-1}(x)], \quad n = 0, 1, 2, \ldots.$$

With odd $m = 2p+1$, functions (16) become zero only for $x = 0$, and one can use them as transformation functions in the region $x \geq 0$. This yields the family of the new potentials:

$$V_1^{(2p+1)}(x) = x^2/4 - 3/2 + 2(2p+1)^2[q_{2p}(x)/q_{2p+1}(x)]^2$$
$$- 4p(2p+1)q_{2p-1}(x)/q_{2p+1}(x).$$

As a solution of the new Schrödinger equation for $E = n\,(n = 0, 1, 2, \ldots)$, we obtain the functions

$$\varphi_n(x) = \sqrt{2}(2\pi)^{-1/4}(n!)^{-1/2}(n+2p+2)^{-1/2}$$
$$\times [He_n(x)q_{2p+2}(x)/q_{2p+1}(x) - nHe_{n-1}(x)]\exp(-x^2/4). \qquad (19)$$

When n is odd, these functions are the solutions of the new Schrödinger equation becoming zero for $x = 0, \infty$, and when n is even, they are singular for $x = 0$. Due to this fact, the discrete spectrum has only odd n values. Functions (19) are normalized-at-unity on the interval $(0, \infty)$. When $n = 1$, we have the ground state function.

The second order Darboux transformation (see the formulae (2), (5) and (6) at $N = 2$) with the discrete spectrum functions $\psi_p(x) = \exp(-x^2/4)He_p(x)$ and $\psi_{p+1}(x)$ produces

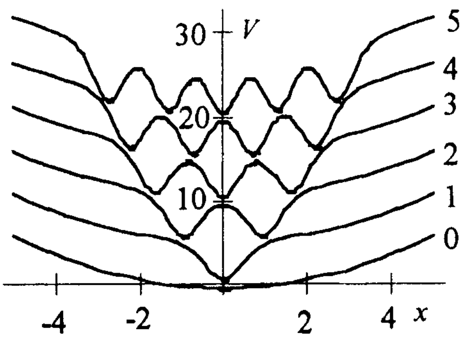

Figure 1. The k-well oscillator-like potentials.

another family of potentials:

$$V_2^{(p,p+1)}(x) = \frac{x^2}{4} + \frac{3}{2} - 2\frac{J_p''(x)}{J_p(x)} + 2\left(\frac{J_p'(x)}{J_p(x)}\right)^2, \quad J_p(x) = \sum_{i=0}^{p} \frac{\Gamma(p+1)}{\Gamma(i+1)} He_i^2(x).$$

These potentials are even functions of x and resemble the parabola $x^2/4$ with N shallow minima of equal height at the bottom. We plot in figure 1 the functions $V = V_2^{(p,p+1)} + 5p$ where p is equal to the curve number. Their discrete spectrum differs from that of the harmonic oscillator potential by the absence of the levels $E = p$ and $E = p + 1$.

The set of normalized-at-unity wave functions for these potentials can easily be obtained from formulae (2) and (6):

$$\varphi_n(x) = (2\pi)^{-1/4}(n!)^{-1/2}[(n-p)(n-p-1)]^{-1/2}\exp(-x^2/4) \times [(n-p)He_n(x) + (He_p(x)He_{n+1}(x) - He_n(x)He_{p+1}(x))He_{p+1}(x)/J_p(x)],$$
$$n \neq p, (p+1).$$

It is not difficult to study the properties of the second-order Wronskian constructed from functions (16):

$$W_{m,l}(x) \equiv W(u_m, u_l) = f_{m,l}(x)\exp(x^2/2),$$
$$f_{m,l}(x) = q_m(x)q_{l+1}(x) - q_l(x)q_{m+1}(x).$$

For the derivative of this function we obtain $W'_{m,l}(x) = (l-m)q_m(x)q_l(x)\exp(x^2/2)$. It follows from the definition of the polynomials $q_m(x)$ (17) that their parity coincides with the parity of the polynomial number m, all their coefficients are integer, and $q_m(x) > 0$ for even m values, and $q_m(x) = 0$ only for odd m value and $x = 0$, and the zero is simple. It is easy to establish that $W_{m,l}(0) = l!!(m-1)!! > 0$. Hence, it follows that if m and l, both positive, have the different parity then $W_{m,l}(x)$ with $l > m$ decreases for $x < 0$, increases for $x > 0$ and has only one minimum at $x = 0$, and $W_{m,l}(x) > 0$ is valid for all x values. If $m > l$ then $W_{m,l}(x)$ increases for $x < 0$, decreases for $x > 0$, and has two symmetrically disposed real simple zeros. If m and l have the same parity and we put for definiteness $l > m$ then for odd m, l we have $W'_{m,l}(x) \geq 0$ and the equality holds only for $x = 0$ and the zero is of a second degree, for even m, l we have $W'_{m,l}(x) > 0$ and in both cases $W'_{m,l}(x) = W'_{m,l}(-x)$. The function $W_{m,l}(x)$ in this case is an odd monotonically increasing function having at $x = 0$ the only simple zero for even m and l and triple zero for odd m and l. By this means, for $m = 0, 2, 4, \ldots$ and $l = m+1, m+3, m+5, \ldots$ the functions (16) are suitable for the second order Darboux transformation.

For the new potentials, formula (5) yields the following expression:

$$V_2^{(m,l)}(x) = x^2/4 - 5/2 - 2f''_{m,l}(x)/f_{m,l}(x) + 2[f'_{m,l}(x)/f_{m,l}(x)]^2. \tag{20}$$

Potentials (20) are regular functions on all real axes. The Hamiltonians $H_2^{(m,l)}$ have two additional discrete energy levels, $E = -l - 1$ and $E = -m - 1$, with respect to H_0. The first level corresponds to the ground state and the second one to the excited state. The normalized-at-unity wave functions for these states are

$$\varphi_0^{m,l}(x) = (2\pi)^{-1/4}\sqrt{l!(l-m)}\exp(-x^2/4)q_m(x)/f_{m,l}(x),$$
$$\varphi_1^{m,l}(x) = (2\pi)^{-1/4}\sqrt{m!(l-m)}\exp(-x^2/4)q_l(x)/f_{m,l}(x).$$

Another discrete spectrum wave functions have the form

$$\begin{aligned}\varphi_{n+2}(x) = &(2\pi)^{-1/4}(n!)^{-1/2}[(n+l+1)(n+m+1)]^{-1/2}\exp(-x^2/4)\\ &\times[(n+1)He_n(x) + ((m-l)q_m(x)q_l(x)He_{n+1}(x)\\ &- mlHe_n(x)f_{m-1,l-1}(x))/f_{m,l}(x)], \quad n = 0,1,2\ldots.\end{aligned}$$

In full analogy, one can consider higher-order transformations. For example, the fourfold Darboux transformation with two discrete spectrum wave functions with $n = 2$ and $n = 3$ and with two functions (12) with $m = 2$ and $m = 3$ generates the following potential:

$$\begin{aligned}V_4(x) = &x^2/4 - 1/2 - 24(1467x^2 + 6x^6 - x^{10})Q^{-1}(x) + 82944\\ &\times(105x^2 + 140x^6 + 3x^{10})Q^{-2}(x), \quad Q(x) = 315 + 315x^4 + 9x^8 + x^{12}.\end{aligned}$$

Consider now the generation of simpler potentials of non-elementary form. Choose as a transformation function the following solution of the initial Schrödinger equation:

$$u(x) = e^{x^2/4}(C + \mathrm{erf}(x/\sqrt{2})), \quad H_0u(x) = -u(x).$$

This transformation function generates the well-known family of isospectral potentials [32, 33]

$$V_1(x) = x^2/4 - 3/2 + 2xQ_1^{-1}(x)e^{-x^2/2} + 2Q_1^{-2}(x)e^{-x^2}$$
$$Q_1(x) = \sqrt{\frac{\pi}{2}}(C + \mathrm{erf}(x/\sqrt{2}))$$

(plotted recently in ref. [34]) but the normalizing constant of the ground state function has never been met by the authors in the available literature. For $|C| > 1$ we obtain the following system of normalized-at-unity wave functions

$$\varphi_0(x) = (2\pi)^{-1/4}\sqrt{C^2-1}\cdot u^{-1}(x),$$
$$\varphi_{n+1}(x) = (2\pi)^{-1/4}[(n+1)!]^{-1/2}(He_{n+1}(x)e^{-x^2/4} + He_n(x)Q_1^{-1}(x)e^{-3x^2/4}),$$
$$n = 0, 1, 2, \ldots$$

The general solution of the initial Schrödinger equation for $E = -2$,

$$u_1(x) = \mathrm{e}^{-x^2/4} + x\mathrm{e}^{x^2/4}\left(-C + \sqrt{\frac{\pi}{2}}\mathrm{erf}(x/\sqrt{2})\right),$$

generates another family of the isospectral potentials for $|C| < \sqrt{\pi/2}$:

$$V(x) = -\frac{x^2}{4} - \frac{7}{2} + \frac{1}{2}\left(\frac{x + (2 + x^2)(\sqrt{(\pi/2)}\,\mathrm{erf}(x/\sqrt{2}) - C)\mathrm{e}^{x^2/2}}{1 - x(C - \sqrt{(\pi/2)}\,\mathrm{erf}(x/\sqrt{2}))\mathrm{e}^{x^2/2}}\right)^2.$$

Another interesting potential occurs for the single Darboux transformation with the function $\psi_{-1/2}(x) = {}_0F_1(3/4, x^4/64)$ (${}_0F_1$ is a conventional symbol for the hypergeometric function ${}_pF_q$), which is a solution of equation (1) for the harmonic oscillator potential with $E = -1/2$. This potential is a double well with $|x_{\mathrm{min}}| \approx 1.68$, $U(x_{\mathrm{min}}) \approx -0.94$, and with a maximum at $x = 0$ and $U(0) = -1/2$, i.e., the energetic level of the ground state touches the potential curve. It occurs in the discrete spectrum of this potential an additional level $E = -1/2$ with respect to the harmonic oscillator spectrum with the ground state eigenfunction (up to a normalization factor) $\varphi_0(x) = \psi_{-1/2}^{-1}(x)$.

Every potential can be studied in more detail. Consider, for example, potential (17) for $p = 2$. The solutions of the new Schrödinger equation are intimately connected with Hermite polynomials. To avoid the square roots in their arguments, we make a change of the variable: $x^2 \to 2x^2$. Solutions (18) can be expressed through new nonclassical polynomials $P_n(x)$ orthogonal on all real axes:

$$\varphi_n(x) = N_n P_n(x) R(x) \exp(-x^2/2), R(x) = 3 + 12x^2 + 4x^4, \quad n = 0, 1, 2, \ldots.$$

The weight function of these polynomials $f(x) = R^{-2}(x)\exp(-x^2)$, in contrast with the classical ones, does not satisfy the Pearson equation. If one normalizes the polynomials $P_n(x)$ with weight $f(x)$ as

$$\int_{-\infty}^{\infty} f(x) P_n^2(x)\mathrm{d}x = 1,$$

the normalizing constants are: $N_0^2 = 24/\sqrt{\pi}$, and $N_{n+1}^2 = [2^{n+1}\sqrt{\pi}(n+5)n!]^{-1}$. The connection of the new polynomials with the Hermite ones

$$P_{n+1}(x) = H_n(x)R'(x) + H_{n+1}(x)R(x), \quad P'_{n+1}(x) = 2(n+5)R(x)H_n(x),$$

permits one to obtain their generating function

$$F(x,z) = 2[4x^5 + 20x^3 + 15x - zR(x)]\exp(2zx - z^2) = \sum_{k=0}^{\infty} \frac{P_{k+1}(x)}{k!} z^k,$$

which obeys the differential equation

$$\frac{\partial^2 F(x,z)}{\partial z^2} - 4(x-z)\frac{\partial F(x,z)}{\partial z} + 2[1 + 2(x-z)^2]F(x,z) = 0.$$

In spite of the simple connection of the new polynomials with the Hermite ones, they satisfy the five-term recursion relations

$$4n(n-1)P_{n-1}(x) - 8xnP_n(x) + 2(1+2n+2x^2)P_{n+1}(x) - 4xP_{n+2}(x) + P_{n+3}(x) = 0.$$

The differential equation for these polynomials follows from those of the Hermite one:

$$P_n''(x) - 2x[1 + 8(3+2x^2)R^{-1}(x)]P_n'(x) + 2(n+5)P_n(x) = 0.$$

The existence of the ladder operators $a = x + \mathrm{d}/\mathrm{d}x$ and $a^+ = x - \mathrm{d}/\mathrm{d}x$ for the harmonic oscillator potential permits one to construct similar operators for the new one. For example,

$$a_L = LaL^+ = x^3 + 7x + 13824x(1+2x^2)R^{-3}(x) - 384x(6-5x^2)R^{-2}(x) - 8x(15-2x^2)R^{-1}(x) + [5 + x^2 + 1152x^2R^{-2}(x) - 24(3-2x^2)R^{-1}(x)] \times \mathrm{d}/\mathrm{d}x - x\mathrm{d}^2/\mathrm{d}x^2 - \mathrm{d}^3/\mathrm{d}x^3.$$

These operators have the properties

$$a_L\varphi_{n+1}(x) = 4n(5+n)\varphi_n(x), \quad a_L^+\varphi_n(x) = 2(n+4)\varphi_{n+1}(x),$$
$$a_L\varphi_0(x) = a_L^+\varphi_0(x) = 0, \quad a_L\varphi_1(x) = 0.$$

It is remarkable that the first polynomial is equal to the unity but the nth one is a polynomial of degree $n+4$. Nevertheless, in agreement with the oscillator theorem for the transformed Hamiltonian the polynomial of order n has n zeros in full real axis. Finally, we cite some of the first polynomials

$$P_0(x) = 1, P_1(x) = 2x(4x^4 + 20x^2 + 15), \quad P_2(x) = 16x^6 + 72x^4 + 36x^2 - 6,$$
$$P_3(x) = 4x(8x^6 + 28x^4 - 14x^2 - 21),$$
$$P_4(x) = 64x^8 + 128x^6 - 480x^4 - 288x^2 + 36,$$
$$P_5(x) = 8x(16x^8 - 216x^4 + 81).$$

3.2 *Effective Coulomb potential*

The next simplest potential, after the harmonic oscillator potential, is the effective Coulomb one:

$$V_0(x) = -2z/x + l(l+1)/x^2, \quad E_n = -z^2/n^2, \quad n = 1,2,3,\ldots,$$
$$\psi_{nl}(x) = N_{nl}^0 x^{l+1} \exp(-zx/n) L_{n-l-1}^{2l+1}(2zx/n),$$
$$N_{nl}^0 = 2/n^2(2/n)^l z^{l+3/2}\sqrt{(n-l-1)!/(n+l)!}.$$

Here, $L_n^\alpha(x)$ are the generalized Laguerre polynomials [30] and functions $\psi_{nl}(x)$ are normalized by the condition $\int_0^\infty \psi_{nl}^2(x)\mathrm{d}x = 1$.

The second order Darboux transformation with these functions with $n = p$ and $n = p+1$ produces the following exactly solvable potentials:

$$V_2^{(l,p,p+1)}(x) = -2z/x + (l+1)(l+4)/x^2 - 2w_0''(x)/w_0(x) + 2(w_0'(x)/w_0(x))^2,$$

where

$$w_0(x) = 2(p+1)L^{2l+2}_{p-l-2}(2zx/p)L^{2l+1}_{p-l}(2zx/(p+1)) + L^{2l+1}_{p-l-1}(2zx/p)[L^{2l+1}_{p-l}(2zx/(p+1)) - 2pL^{2l+2}_{p-l-1}(2zx/(p+1))].$$

The expression for the normalized-at-unity wave functions follows from formulae (6) and (11)

$$\varphi_{nl}(x) = N_{np}\left[\left(\frac{1}{n^2} + \frac{w_1(x)}{np(p+1)w_0(x)}\right)\psi_{nl}(x) - N^0_{nl}\frac{2(2p+1)}{np(p+1)w_0(x)} \times L^{2l+1}_{p-l-1}(2xz/p)L^{2l+1}_{p-l}(2xz/(p+1))x^{l+1}L^{2l+2}_{n-l-2}(2xz/n)\exp(-zx/n)\right],$$

where

$$w_1(x) = 2n(p+1)L^{2l+1}_{p-l-1}(2xz/p)L^{2l+2}_{p-l-1}(2xz/(p+1)) + L^{2l+1}_{p-l}(2xz/(p+1)) \times [(n-1)L^{2l+1}_{p-l-1}(2xz/p) - 2p(nL^{2l+2}_{p-l-2}(2xz/p)) + L^{2l+1}_{p-l-1}(2xz/p)],$$

and

$$N_{np} = n^2p(p+1)[(p^2-n^2)((p+1)^2-n^2)]^{-1/2}.$$

The simplest nontrivial potentials correspond to the case $l=0, p=2$:

$$V_2^{(0,2,3)} = -2z/x + 10/x^2 + 40Q_0^{-1}(xz)(2-xz)/x^2 - 100Q_0^{-2}(xz)(2xz-3)/x^2,$$
$$Q_0(x) = -15 + 10x - 2x^2.$$

The levels with $n=2$ and $n=3$ in its discrete spectrum are absent. The normalized-at-unity discrete spectrum eigenfunctions are expressed via a system of new polynomials $P_n(x)$ orthogonal on the interval $(0,\infty)$:

$$\varphi_{nl}(x) = \tfrac{1}{6}N^0_{nl}N_{n2}\,n^{-2}z^2x^3P_n(xz)Q_0^{-1}(xz)\exp(-xz/n).$$

These polynomials are related with the Laguerre ones as follows:

$$x^3P_n(x) = 10n(-54+63x-22x^2+2x^3)L^2_{n-2}(2x/n) + L^1_{n-1}(2x/n)[270n(n-1) - 45(2-7n+5n^2)x + 10(6-11n+5n^2)x^2 - 2(6-5n+n^2)x^3].$$

We cite the first five polynomials,

$$P_1(x) = -4, \quad P_2(x) = P_3(x) = 0, \quad P_4(x) = \tfrac{1}{24}(-84+63x-18x^2+2x^3)$$
$$P_5(x) = \tfrac{8}{625}(-875+700x-220x^2+30x^3-x^4).$$

The properties of $P_n(x)$ strongly differ from the properties of the Laguerre polynomials, though they are completely defined by the latter. For example, for $n>3$ the polynomial $P_n(x)$ is a polynomial of degree n but it has $n-3$ zeros in the $(0,\infty)$ interval, and this fact corresponds well to the oscillator theorem for the Hamiltonian H_1.

The fourfold Darboux transformation gives more complicated results. For example, for $l=0,\ n_1=3,\ n_2=4,\ n_3=5$ and $n_4=6$, the new potential is

$$V_4^{(0,3,4,5,6)}(x)=-2z/x+38/x^2-144x^{-2}Q_2^{-1}(xz)(492-228xz+27x^2z^2-x^3z^3)$$
$$-31104x^{-2}Q_2^{-2}(xz)(9240-2640xz+255x^2z^2-8x^3z^3),$$
$$Q_2(x)=11880-3960x+540x^2-36x^3+x^4.$$

The levels with $n=n_1,n_2,n_3$, and n_4 are absent in its discrete spectrum.

In contrast with the harmonic oscillator potential, we can obtain here isospectral family of potentials of elementary form. For this purpose, the solutions of the initial Schrödinger equation singular at $x=0,\infty$ can be used as transformation functions. We can construct elementary solutions having such properties with the use of the following functions:

$$u_{nl}(x)=x^{-l}L_n^{-2l-1}(2zx/(n-l))\exp(-zx/(n-l)),$$
$$H_0u_{nl}(x)=-z^2/(n-l)^2u_{nl}(x).$$

For $l=1$ and $n=2$, the general solution of the Schrodinger equation has the form

$$u(x)=x^{-1}e^{-zx}Q_3(xz), Q_3(x)=1+2x+2x^2+Ce^{2x},\quad H_0u(x)=-z^2u(x).$$

This transformation function leads to the family of isospectral potentials with a hydrogen-like discrete spectrum:

$$V_1(x)=-2z/x-16z^3x(xz-1)Q_3^{-1}(xz)+32z^6x^4Q_3^{-2}(xz),$$
$$C\in(-\infty,-1)\cup(0,\infty).$$

These potentials with $z=1$ are plotted in figure 2. The normalized-at-unity eigenfunctions are

$$\varphi_1(x)=2\sqrt{z^3C(C+1)}u^{-1}(x),$$
$$\varphi_n(x)=N_{n1}^0nz^{-1}(n^2-1)^{-1/2}[x(2-xz/n)L_{n-2}^3(2xz/n)-2x^2z/nL_{n-3}^4(2xz/n)$$
$$+x(1+xz+Ce^{2xz}(1-xz)+2x^3z^3)L_{n-2}^3(2xz/n)Q_3^{-1}(xz)]\exp(-xz/n),$$
$$n=2,3,\ldots.$$

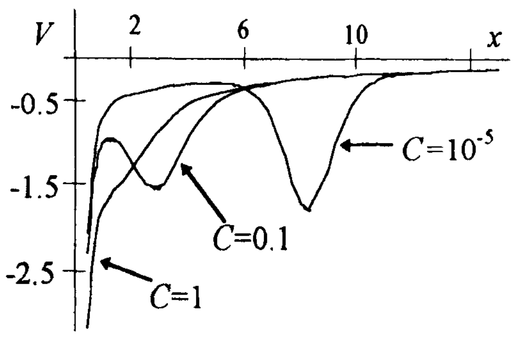

Figure 2. The isospectral Coulomb-like potentials.

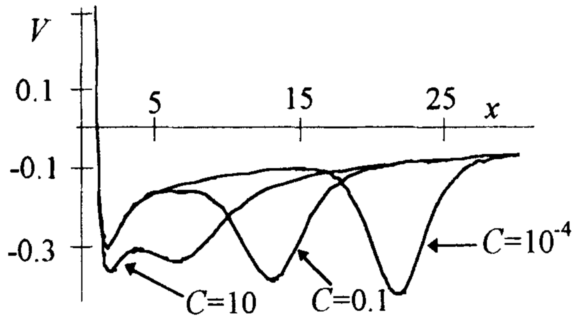

Figure 3. The isospectral Coulomb-like potentials with a centripetal member.

For $n=4,\ l=2$ the function

$$u(x)=x^{-2}\exp(-xz/2)Q_4(xz),\quad Q_4(x)=24+24x+12x^2+4x^3+x^4+Ce^x$$

produces a family of isospectral potentials with a centripetal member

$$V_1(x)=-2z/x+2/x^2+2z^5x^3(4-xz)Q_4^{-1}(xz)+2z^{10}x^8Q_4^{-2}(xz)$$

plotted for $z=1$ in figure 3. The normalized-at-unity wave functions for these potentials are given as follows:

$$\varphi_2(x)=z^{5/2}\sqrt{C(C/4!+1)}\cdot u^{-1}(x), C\in(-\infty,-4!)\cup(0,\infty)$$

$$\varphi_n(x)=2N_{n2}^0 n/z(n^2-4)^{-1/2}\exp(-xz/n)[x^2(3-xz/n)L_{n-3}^5(2xz/n)-2zx^3/n$$
$$\times L_{n-4}^6(2xz/n)+\tfrac{1}{2}x^2Q_4^{-1}(xz)(96+72xz+24x^2z^2+4x^3z^3+x^5z^5$$
$$+(4-xz)Ce^{xz})L_{n-3}^5(2xz/n)], n=3,4,5,\ldots.$$

It is interesting to note that, as can be seen from figures 2 and 3, the plots of the Coulomb-like potentials resemble the diagram of a moving soliton, especially if one changes the sign: $V\to -V$.

Note as well that the Jost functions for every potential can be found with the help of the asymptotic behavior of the transformation functions. We do not dwell on these calculations.

3.3 *Morse potential*

One more example is a rather complicated potential known as the Morse potential:

$$V_0(x)=\exp(-2\alpha x)-A\exp(-\alpha x),$$

$$E_n=-\tfrac{1}{4}[A-\alpha(2n+1)]^2, \psi_n(x)=z^\mu\exp(-z/2)\cdot L_n^{2\mu}(z),$$

$$z=\frac{2}{\alpha}\exp(-\alpha x), \mu=\sqrt{|E_n|}/\alpha,\quad n=0,1,2,\ldots.$$

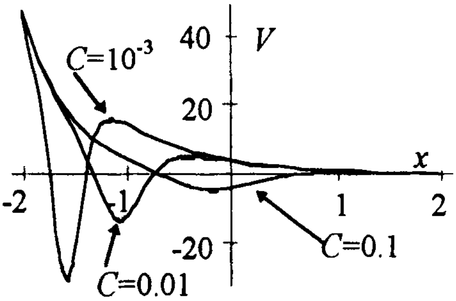

Figure 4. The isospectral one-level Morse potentials with $k = 2$ and $\alpha = 1$.

We consider the variable x to span all real axis. The second order Darboux transformation with the function $\psi_1(x)$ and $\psi_2(x)$ gives a new potential of the form

$$V_2^{(1,2)}(x) = \exp(-2\alpha x) - \exp(-\alpha x)(A - 4\alpha) + 128(2\alpha - A)^{-1}\alpha^3 Q_4^{-2}(x)$$
$$\times [(A - 3\alpha)\exp(\alpha x) - 1] + 8(2\alpha - A)^{-1}\alpha^2 Q_4^{-1}(x)$$
$$\times [4\alpha - (A^2 - 5A\alpha + 6\alpha^2)\exp(\alpha x),$$
$$Q_4(x) = 4 - 4(A - 3\alpha)\exp(\alpha x) + (A^2 - 5A\alpha + 6\alpha^2)\exp(2\alpha x), \quad A \neq 2\alpha.$$

In its discrete spectrum, the states with $n = 1$ and $n = 2$ are missing.

On the base of this potential, we can construct a family of isospectral potentials of elementary form with a single discrete spectrum energy level. This possibility is due to the fact that when μ is entire or half-integer, the general solution of the Schrödinger equation has an elementary form. Let μ take a fixed value as $\mu = -(p + 1)/2$ and $A = -p\alpha, p = 0, 1, 2, \ldots$. In this case, the Morse potential

$$V_0(x) = \tfrac{1}{4}\alpha^2 z^2 + \tfrac{1}{2}\alpha^2 pz$$

has no discrete spectrum at all. However, for $E_0 = -\frac{1}{4}(p + 1)^2\alpha^2$ we have the following general solution of the initial Schrödinger equation:

$$u(x) = \mathrm{e}^{-z/2} z^{-(p+1)/2} Q_5(z), \quad Q_5(z) = 1 + C\mathrm{e}^z \left(z^p + \sum_{i=1}^{p} (-1)^i p!/(p - i)! z^{p-i}\right).$$

This function when used as a transformation function, generates a one-level potential

$$V_1(x) = \tfrac{1}{4}\alpha^2 z^2 + \alpha^2(p/2 + 1)z - 2\alpha^2 z \mathrm{d}\log Q_5(z)/\mathrm{d}z - 2\alpha^2 z^2 \mathrm{d}^2 \log Q_5(z)/\mathrm{d}z^2,$$

with the energy level $E = E_0$ and the normalized-at-unity wave function of the form:

$$\psi_0(x) = [\alpha C((-1)^p p! C + 1)]^{1/2} u^{-1}(x).$$

It is clear from this relation that for even p values for the constant C we have $C \in (-\infty, -1/p!) \cup (0, \infty)$ and for odd p values it must fall in the interval $C \in (0, 1/p!)$.

The simplest of these potentials corresponds to $p = 0$:

Darboux transformation

$$V_1(x) = \exp(-2\alpha x) + 2\alpha\left[1 - C\exp\left(\frac{2}{\alpha}\,\mathrm{e}^{-\alpha x}\right)\right]$$
$$\times\, Q_6^{-1}(x) - 8C\exp\left(\frac{2}{\alpha}\mathrm{e}^{-\alpha x}\right)Q_6^{-2}(x),$$

$$Q_6(x) = \mathrm{e}^{\alpha x}\left(1 + C\exp\left(\frac{2}{\alpha}\mathrm{e}^{-\alpha x}\right)\right).$$

These potentials are plotted for $p = 2$ in figure 4.

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