PRAMANA
—journal of
physics

Vol. 49, No. 4,
October 1997
pp. 417–420

Isothermal fluid sphere: Uniqueness and conformal mapping

NARESH DADHICH
Inter-University Centre for Astronomy and Astrophysics, P.O. Box 4, Pune 411 007, India
e-mail: nkd@iucaa.ernet.in

MS received 15 April 1997

Abstract. We prove the theorem: A necessary and sufficient condition for a spacetime to represent an isothermal fluid sphere (linear equation of state with density falling off as inverse square of the curvature radius) without boundary is that it is conformal to a spacetime of zero gravitational mass ('minimally' curved).

Keywords. Fluid spheres; isothermal fluid; conformal mapping.

PACS Nos 04.20; 98.80

Except for Brinkman's theorem [1] conformally relating two Einstein spaces, only spacetimes conformal to flat spacetime have been considered. The well-known examples of such spacetimes, that are also of great astrophysical and cosmological interest, are the Friedman–Robertson–Walker (FRW) flat model of the Universe and the Schwarzschild solution describing interior of a star in hydrostatic equilibrium. These spacetimes have distinguishing physical properties, like isotropy and homogeneity for the former and uniform density for the latter. In this note we shall establish a unique association between isothermality of fluid sphere with conformal mapping of spacetime. Such a clear characterization of physical behaviour with geometric property is rather very rare.

Here the base spacetime is of course not flat but can be thought as 'minimally' curved [2]. It is a spherically symmetric spacetime from which radial acceleration as well as tidal acceleration for radial motion have been anulled out but curvature is not zero, which manifests only in tidal acceleration for transverse motion. Though it is not a solution of the Einstein vacuum equation, it presents an interesting physical situation with the property, $R_{ik}u^i u^k = 0, u^i u_i = 1$. This means spacetime is free of active gravitational mass. Further it has all but one curvature zero, which is also an invariant for spherical symmetry, and it is at any given radius proportional to curvature of sphere of that radius [3]. This is why we have termed it as 'minimally' curved spacetime (MCS) [2, 4]. We have very recently considered metric in the Kerr–Schild (KS) form with a view to find perfect fluid solutions [5]. It is however well-known that perfect fluid is not compatible with the KS form and hence Senovilla and Sopuerta [6], and Martin and Senovilla [7] have generalized it by replacing flat metric by conformally flat. This would mean that metric can be written as conformal to an original KS metric, though the base space will not in general be a solution of the Einstein equation.

A metric in KS form is given by

$$g_{ij} = \eta_{ij} + 2Hl_i l_j, \tag{1}$$

where H is a scalar field and l_i is a null vector relative to both g_{ij} and η_{ij}. For spherically symmetric vacuum solution; i.e. the Schwarzschild solution, H satisfies the Laplace equation $\nabla^2 H = 0$. It is rather interesting to note that spacetime is not flat unless $H = 0$; i.e. $H = \text{const.} \neq 0$, represents a curved spacetime and this is the MCS discussed above.

Let us consider a metric conformal to MCS by taking H constant. Recall that for perfect fluid seed metric has to be different from vacuum. So we take H constant and write

$$\bar{g}_{ij} = e^{2U}(\eta_{ij} + 2Hl_i l_j), H = \text{const.} \tag{2}$$

which for spherical symmetry can be brought to the orthogonal form to read [5]

$$\mathrm{d}s^2 = e^{2U}(-\mathrm{d}t^2 + k^2\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2). \tag{3}$$

Here $U = U(r,t)$ and $k^2 = (1+2H)^2/(1-2H)$ and the base metric is MCS.

First, the perfect fluid conditions in the comoving coordinates $(u_i = e^U\delta_i^0)$ imply $U = U(r)$, and they yield the general solution for unbounded distribution [5],

$$e^U = r^{-n} \tag{4}$$

with

$$8\pi\rho = \frac{n(n-2)}{k^2 r^{2(1-n)}}, \quad \rho\frac{n-2}{n}p, \tag{5}$$

where $k^2 = 1 + 2n(n-2)$ and removable constants have been transformed away. This is the general solution which represents isothermal fluid as it admits a linear equation of state and density falls off as inverse square of the curvature radius, $R = r^{1-n}$. It is important to note that the metric (3) admits this unique solution (4) representing isothermal fluid sphere without boundary [5].

Thus the metric ansatz (3) with the unique solution (4) is the sufficient condition for isothermal fluid sphere without boundary. We shall now turn to the necessary condition. For that we have to prove that all unbounded isothermal fluid sphere solutions can always be cast in the metric ansatz (3). Let us consider the general spherically symmetric metric

$$\mathrm{d}s^2 = -e^\nu\mathrm{d}t^2 + e^\lambda\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{6}$$

where λ and ν are functions of r only. With this we have for perfect fluid in comoving coordinates $(u_i = e^{\nu/2}\delta_i^0)$,

$$8\pi\rho = \frac{1}{r^2}[1 + e^{-\lambda}(r\lambda' - 1)], \tag{7}$$

$$8\pi p = \frac{1}{r^2}[-1 + e^{-\lambda}(r\nu' + 1)], \tag{8}$$

$$2\nu'' + \nu'^2 - \lambda'\nu' - \frac{2}{r}(\nu' + \lambda') + \frac{4}{r^2}(e^\lambda - 1) = 0, \tag{9}$$

where $\lambda' = \partial\lambda/\partial r$. Then we obtain the general solution for a regular e^{λ} at $r = 0$ [8],

$$e^{\lambda} = \text{const.} = k_1^2, \quad e^{\nu} = r^{-2m}. \tag{10}$$

This belongs to the Tolman class of solutions [9]. Thus we have obtained the general solution for unbounded isothermal fluid sphere and the metric (6) will read

$$ds^2 = r^{-2m}[-dt^2 + k_1^2 r^{2m} dr^2 + r^{2(1+m)}(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{11}$$

By redefining the radial coordinate as $\bar{r} = r^{1+m}$, the above metric on dropping overhead bar takes the form

$$ds^2 = r^{-2m/(1+m)}[-dt^2 + k_1^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{12}$$

which is exactly in the required ansatz form (3).

We have thus proved the theorem: A necessary and sufficient condition for an isothermal fluid sphere without boundary is that its spacetime metric is conformal to the minimally curved spacetime, characterized by vanishing of gravitational mass $(R_{ik}u^i u^k = 0)$,

$$ds^2 = dt^2 + k^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), k = \text{const.}. \tag{13}$$

It is quite remarkable that there exists a one-to-one association between isothermality of fluid sphere and conformally MCS. Isothermality of fluid sphere picks up uniquely the geometric property, conformally MCS. This is really very interesting because it is a rare case of a specific physical property singling out a geometric property. Like the isothermal case, stresses generated by k in MCS also fall off as $T_0^0 = T_1^1 \sim r^{-2}$. These stresses have been identified with gravitational global monopole [10]. Global monopole is an exotic object which is supposed to be created when global $O(3)$ symmetry is spontaneously broken into $U(1)$ in phase transitions in the very early Universe. The simplest way to get to MCS is that it is a spacetime corresponding to a constant potential [4]. Recall that H in (1) satisfies the Laplace equation and it corresponds to the Newtonian potential. However we confess that its interpretation as potential may not be generally agreed upon.

Viewing MCS as a small departure (as it differs from flat spacetime only in tidal acceleration for transverse motion) from flat spacetime, it is interesting to note that for its conformal spacetime it does not permit non-static perfect fluid solution. It admits the general solution having three free parameters, which can represent bounded fluid spheres as well as unbounded isothermal fluid spheres [5]. In contrast to conformally flat metric, conformally MCS is highly constrained. Even though MCS may be a small deviation from flat spacetime, their conformal spacetimes have very different physical properties.

Isothermal fluid structures have been considered in astrophysics for a long time as an equilibrium approximation to more complicated systems approaching dynamical relaxed state. Recently the spacetime (11) has been argued as an ultimate end state of the Einstein–deSitter universe [8]. It is envisioned that the Einstein–deSitter model asymptotically tends to an expansion free state, and then it condenses into a stable isothermal fluid sphere.

References

[1] H W Brinkman, *Math. Ann.* **91**, 269 (1924)
[2] N Dadhich and K Narayan, *An ansatz for spacetimes of zero gravitational mass: Global monopoles and textures* (to be submitted)
[3] N Dadhich, An analysis of some unsolved problems in general relativity, Ph.D. thesis (Pune University, 1970) (unpublished)
[4] N Dadhich, *On understanding the Schwarzschild field* (to be submitted)
[5] N Dadhich, *Gen. Relativ. Gravit.* **28**, 1455 (1996)
[6] J M M Senovilla and C F Sopuerta, *Class. Quantum Gravit.* **11**, 2073 (1994)
[7] J Martin and J M M Senovilla, *J. Math. Phys.* **27**, 265 (1986)
[8] W C Saslaw, S D Maharaj and N Dadhich, *Astrophys. J.* **471**, 571 (1996)
[9] R C Tolman, *Phys. Rev.* **55**, 365 (1934)
[10] M Barriola and A Vilenkin, *Phys. Rev. Lett.* **63**, 341 (1989)