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Hopping of chaotic dynamics in a doubly resonant parametric oscillator

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Abstract. We study a doubly resonant optical parametric oscillator where the pump can feed two pairs of signal-idler modes. We assume the presence of gain at the pump frequency. We investigate the various oscillation states of interest, namely, when only the first pair oscillates with the other pair having null amplitudes and vice versa. We demonstrate the exchange of dynamics between the mode pairs when the relevant parameters of the cavity, namely, the phase mismatch factors or the decay rates switch because of fluctuations. The exchange of dynamics is shown to be independent of the nature of dynamics, i.e. independent of whether the motion is n-periodic or chaotic. We also investigate the case where both the pairs can exhibit chaotic dynamics though these states are difficult to realize because of fluctuations.

Keywords. Parametric oscillation; chaos; switching.

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1. Introduction

The problem of mode hopping in optical parametric oscillators [1] has attracted a lot of attention in the past few decades [2, 3]. The doubly resonant oscillator (DRO) would be the ideal source of tunable radiation, if they were without any mode hopping and cluster jumping phenomenon [4], leading to frequency instability. Very recently a dynamical model which can account for the major observed effects in DROs was presented by Agarwal and Dutta Gupta [5]. It was demonstrated that when the relevant cavity parameters (in this case they are phase mismatch factors and cavity decay rates) are changed there can be an exchange of stability between the on-states of the down-converted modes. Since any realistic system is bound to have fluctuations of the cavity parameters the dynamics has to switch from one pair of frequencies (one mode pair) to another. In fact a change in the cavity length or the refractive indices can lead to different modes spacings at the signal and idler frequencies which may result in a new pair of modes having a larger gain, thus making the new pair the preferred one. On the other hand, phase fluctuations can lead to renormalized decay rates and alter the dynamics of evolution. Phase fluctuations of the pump can be accounted for in the theory by a redefinition of the decay rates [6]. From a different angle, the physics of the down-conversion process becomes extremely rich when there is gain at the pump frequency. It was shown by Wersinger *et al* [7] that when the pump with gain feeds only one signal-idler pair, the system can exhibit a very rich dynamics which changes from regular to chaotic (via a period-doubling bifurcation) as the decay rate of the down-converted modes increases. Besides, period-three oscillations were also observed in the system for larger values of decay rates beyond the chaotic window. Keeping the aforesaid in mind in this paper we address the question: what happens when the pump has gain and the cavity is doubly resonant? We consider the situation analogous to the case of mode hopping except that now there is an inverted medium at the pump frequency such that pump grows with a constant growth rate while propagating in the intracavity medium. To simplify the model we restrict ourselves with only two competing signal-idler pairs which are fed by the pump via a nonlinear interaction due to the presence of a $\chi^{(2)}$ material in the cavity. We show that switching of the relevant parameters (i.e., phase mismatch factor and decay rates) does lead to 'chaos hopping' for appropriate choice of other parameters. In other words, one mode pair exhibiting regular or chaotic dynamics can go to the off-state while the other starts off from the off-state to show the same response. Thus there can be an exchange of dynamics irrespective of whether it is regular or chaotic.

The organization of the paper is as follows. In § 2 we present the model and recall the results pertaining to the stability of the off-states. In § 3 we present the results of numerical integration of the dynamical system and discuss the results. Finally in § 4 we conclude the paper.

2. Mathematical formulation

We consider a pump mode with amplitude d at frequency ω_3 and two pairs of signal-idler modes, namely, a_1, a_2 and b_1, b_2, respectively. The frequencies of the modes a_1 and a_2 (b_1 and b_2) are ω_{a1} and ω_{a2} (ω_{b1} and ω_{b2}) respectively. The equations of motion for the complex amplitudes are given by

$$\dot{a}_j = -\kappa_{aj}a_j + \beta f_a \exp(-i\Delta\omega_a t) d a_{3-j}^*, \tag{1}$$

$$\dot{b}_j = -\kappa_{bj}b_j + \beta f_b \exp(-i\Delta\omega_b t) d b_{3-j}^*, \tag{2}$$

$$\dot{d} = \kappa_d d - \beta f_a^* \exp(i\Delta\omega_a t) a_1 a_2 - \beta f_b^* \exp(i\Delta\omega_b t) b_1 b_2, \tag{3}$$

where $\kappa_{aj}(\kappa_{bj})$, $j = 1, 2$, are the damping constants for the modes $a_j(b_j)$, $\Delta\omega_s = \omega_3 - \omega_{s1} - \omega_{s2}$, $(s = a, b)$ is the frequency detuning and $f_s (s = a, b)$ is the phase mismatch factor for the sth mode pair. We assumed the presence of an active medium at the pump frequency and κ_d gives the growth rate of the pump mode d. f_s, κ_{sj} and β are given by the following expressions:

$$f_s = \frac{\sin(\Delta k_s L/2)}{\Delta k_s L/2} \exp(i\Delta k_s L/2), \tag{4a}$$

$$\kappa_{sj} = \frac{\pi c}{2F_s L n_s}, \quad (s = a, b;\ j = 1, 2), \tag{4b}$$

$$\beta = \frac{2\sqrt{2} d_{\text{eff}}}{\sqrt{V}} \pi^{3/2} \sqrt{\hbar\omega_{s1}\omega_{s2}\omega_d}, \tag{4c}$$

where $\Delta k_s = k_d - k_{s1} - k_{s2}, s = a, b; F_s$ and n_s are the finesse and the refractive index at the frequency $\omega_s \cdot \kappa_{sj}(s = a, b; j = 1, 2)$ are the decay rates of the down converted modes. V and d_{eff} are the mode volume and the effective nonlinear coefficient, respectively. We assume that all the relevant frequencies are far off from any resonances of the intracavity medium. Hence we have used the same nonlinearity parameter β in eqs (1)–(3). Hereafter we will refer to $a_1, a_2(b_1, b_2)$ as the a-(b-)pair. By making use of the following transformations

$$a_j = A_j e^{-i\Omega_{aj}t}, \quad b_j = B_j e^{-i\Omega_{bj}t}, \tag{5}$$

with

$$\Omega_{a1} + \Omega_{a2} = \Delta\omega_a, \quad \Omega_{b1} + \Omega_{b2} = \Delta\omega_b, \tag{6}$$

eqs (1)–(3) can be rewritten in the following form

$$\dot{A}_j = (-\kappa_{aj} + i\Omega_{aj})A_j + \beta f_a d A^*_{3-j}, \tag{7}$$

$$\dot{B}_j = (-\kappa_{bj} + i\Omega_{bj})B_j + \beta f_b d B^*_{3-j}, \tag{8}$$

$$\dot{d} = \kappa_d d - \beta f^*_a A_1 A_2 - \beta f^*_b B_1 B_2. \tag{9}$$

The set of equations (7)–(9) is analogous to the set for a doubly resonant oscillator investigated by Agarwal *et al* [5] except for the fact that in [5] the pump was assumed to decay with the decay rate κ_d and there was a drive to compensate for the decay of the pump. However, some of the features remain the same even in presence of the active medium at the pump frequency. The set of equations (7)–(9) allow for different sets of solutions, namely when one pair oscillates with the other pair in the off-state or when both the pairs oscillate. We first look at the case when, say, the a-pair is oscillating with the b-pair having null amplitudes in the long time limit. Nontrivial solutions for the a-pair is possible if and only if the pump amplitude satisfies the following relation:

$$\bar{d}^2 = \frac{\kappa_{a1}\kappa_{a2}}{\beta^2|f_a|^2}\left[1 + \frac{\Delta\omega_a^2}{\kappa_{a1}^2 + \kappa_{a2}^2}\right] \tag{10}$$

with

$$\Omega_{aj} = \frac{\Delta\omega_a \kappa_{aj}}{\kappa_{a1} + \kappa_{a2}}. \tag{11}$$

The linear stability of the null (off) states of the b-modes can be carried out in a way analogous to that of ref. [5] leading to the following result. The inequality

$$\left|\frac{f_b}{f_a}\right| > \left(\frac{\kappa_{b1}\kappa_{b2}}{\kappa_{a1}\kappa_{a2}}\right)^{1/2} \tag{12}$$

is to be satisfied for the null state of b to be unstable. Thus for equal decay rates $(\kappa_{a1} = \kappa_{a2} = \kappa_{b1} = \kappa_{b2})$, $|f_b| > |f_a|$ implies the instability of the $B_1 = B_2 = 0$. The symmetry of eq. (12) with respect to the mode indices implies the exchange of stability (exchange of dynamics) whenever $|f_b|/|f_a|$ crosses the value unity. Similarly for a system with $|f_a| = |f_b|, \kappa_{s1} = \kappa_{s2} = \kappa_s(s = a, b)$, the inequality (12) reduces to $\kappa_b < \kappa_a$. Thus, whenever $\kappa_b < \kappa_a$ the b-mode will take off from trivial values to have a finite amplitude.

In short a mode pair with larger value of the phase mismatch factor $|f|$ or higher life time (lower decay rate) will have preference and will dominate in the long run. Note that in a realistic system fluctuations are unavoidable. Fluctuations in length or the refractive indices leads to the renormalization of the phase mismatch factor (see eq. (4a)) whereas phase or frequency fluctuations can cause a rescaling of the decay rates. Thus a random change in the physical characteristics can cause a random switching of the dynamics from one pair to the other.

It may also be noted that on-states for both the pairs are possible provided

$$\bar{d}^2 = \frac{\kappa_{a1}\kappa_{a2}}{\beta^2|f_a|^2}\left[1+\frac{\Delta\omega_a^2}{\kappa_{a1}^2+\kappa_{a2}^2}\right] = \frac{\kappa_{b1}\kappa_{b2}}{\beta^2|f_b|^2}\left[1+\frac{\Delta\omega_b^2}{\kappa_{b1}^2+\kappa_{b2}^2}\right]. \tag{13}$$

However, such states are extremely unlikely and any fluctuation leading to the violation of eq. (13) destroys the nontrivial state for one of the modes and finally the system settles down on the mode pair which has a larger $|f|$ value or a lower decay rate. Keeping in view the above analysis we now go over to the calculation of trajectories in the next section.

3. Numerical results and discussions

In this section we investigate numerically the set of equations (7)–(9). This set can be reduced to a set of real equations by introducing the real amplitudes and phases as follows

$$A_j = \bar{A}_j\, \mathrm{e}^{i\varphi_{aj}}, \quad B_j = \bar{B}_j\, \mathrm{e}^{i\varphi_{bj}}, \tag{14}$$

$$f_s = |f_s|\mathrm{e}^{i\varphi_{fs}}, \quad (s = a, b), \tag{15}$$

$$d = \bar{d}\mathrm{e}^{i\varphi_d}. \tag{16}$$

Making use of the equations (14)–(16) in (7) and (8) one can obtain the following relations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{A}_1^2 - \bar{A}_2^2) = -2(\tilde{\kappa}_{a1}\bar{A}_1^2 - \tilde{\kappa}_{a2}\bar{A}_2^2), \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{B}_1^2 - \bar{B}_2^2) = -2(\tilde{\kappa}_{b1}\bar{B}_1^2 - \tilde{\kappa}_{b2}\bar{B}_2^2), \tag{18}$$

where

$$\tilde{\kappa}_{sj} = \kappa_{sj}/\kappa_d, \quad j = 1,2; s = a,b. \tag{19}$$

For example, for $\tilde{\kappa}_{a1} = \tilde{\kappa}_{a2} = \tilde{\kappa}_a$ and $\tilde{\kappa}_{b1} = \tilde{\kappa}_{b2} = \tilde{\kappa}_b$, the quantities $(\bar{A}_1^2 - \bar{A}_2^2)$ and $(\bar{B}_1^2 - \bar{B}_2^2)$ decrease exponentially in time. Thus, if $\bar{A}_1 = \bar{A}_2(\bar{B}_1 = \bar{B}_2)$ initially, then they remain the same for all subsequent times. Let the initial conditions be such that $\bar{A}_1 = \bar{A}_2 = \bar{A}$ and $\bar{B}_1 = \bar{B}_2 = \bar{B}$. Thus the set of seven equations (7)–(9) reduces to a set of five equations as follows

$$\dot{\bar{A}} = -\tilde{\kappa}_a\bar{A} + \tilde{\beta}|f_a|\bar{d}\bar{A}\cos(\theta_a), \tag{20}$$

$$\dot{\bar{B}} = -\tilde{\kappa}_b\bar{B} + \tilde{\beta}|f_b|\bar{d}\bar{B}\cos(\theta_b), \tag{21}$$

$$\dot{\bar{d}} = \bar{d} - \tilde{\beta}|f_a|\bar{A}^2\cos(\theta_a) - \tilde{\beta}|f_b|\bar{B}^2\cos(\theta_b), \tag{22}$$

$$\dot{\theta}_a = -\Delta\tilde{\omega}_a + \tilde{\beta}|f_a|\sin(\theta_a)(\bar{A}^2 - 2\bar{d}^2)/\bar{d} + \tilde{\beta}|f_b|\bar{B}^2\sin(\theta_b)/\bar{d}, \tag{23}$$

$$\dot{\theta}_b = -\Delta\tilde{\omega}_b + \tilde{\beta}|f_b|\sin(\theta_b)(\bar{B}^2 - 2\bar{d}^2)/\bar{d} + \tilde{\beta}|f_a|\bar{A}^2\sin(\theta_a)/\bar{d}, \tag{24}$$

where

$$\theta_a = \varphi_d + \varphi_{fa} - (\varphi_{a1} + \varphi_{a2}), \tag{25}$$

$$\theta_b = \varphi_d + \varphi_{fb} - (\varphi_{b1} + \varphi_{b2}), \quad \tilde{\beta} = \beta/\kappa_d. \tag{26}$$

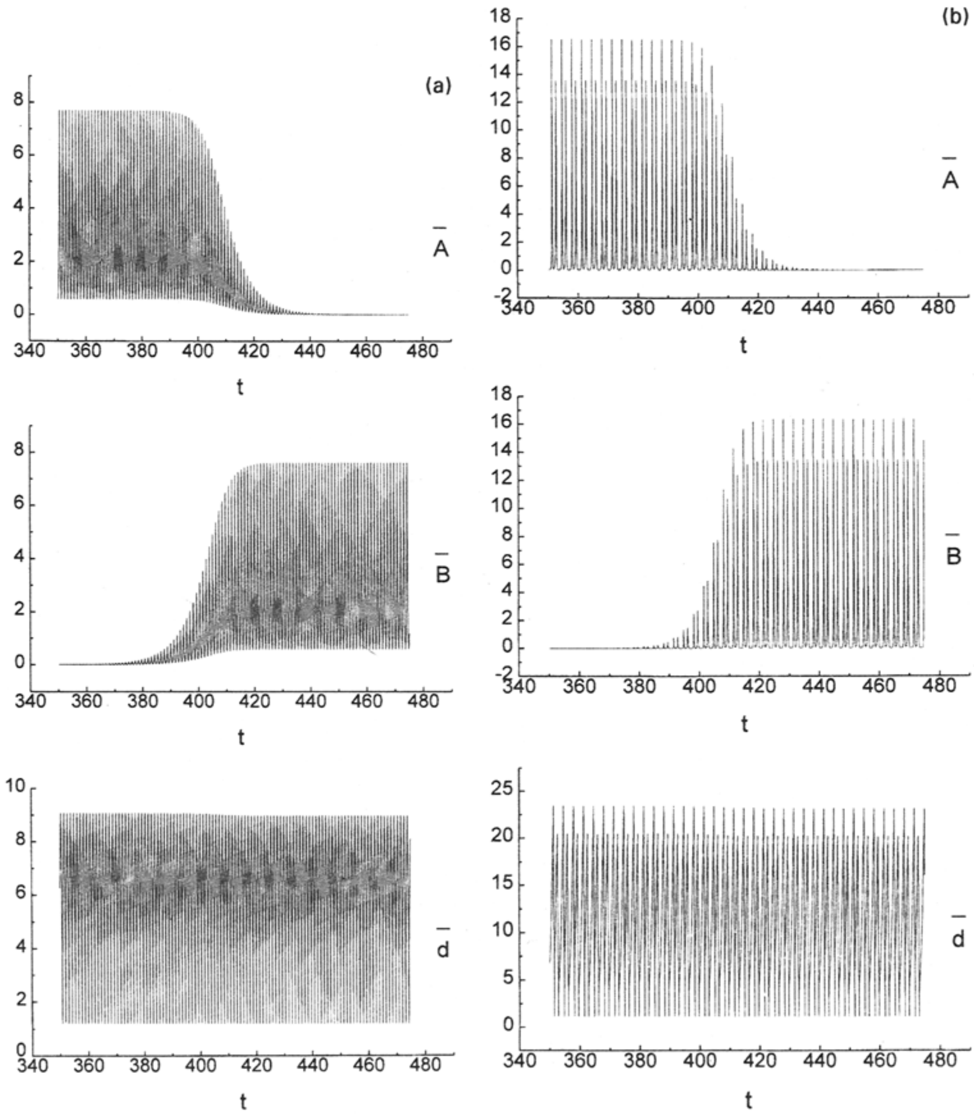

Figure 1(a) and (b).

In eqs (20)–(24) time t is scaled by κ_d. In writing eqs (20)–(24), we have used eq. (6). Note that in the absence of one of the modes a or b, the set of equations (20)–(24) reduces to eq. (1) of ref. [7], which for suitable choice of parameters exhibits regular or chaotic behavior. In the same paper a detailed analysis (including that of the Poincare section and the power spectra) for this system for $\beta = 1$ and $|f| = 1$ was carried out. It was shown that the nature of dynamics had a strong dependence on the decay rate κ of the down converted mode. For example, for $1 < \kappa \leq 3$, the motion was unbounded. In the range $3 < \kappa < 13.16$, even period bifurcations were observed. The range $13.16 \leq \kappa \leq 16.8$ was

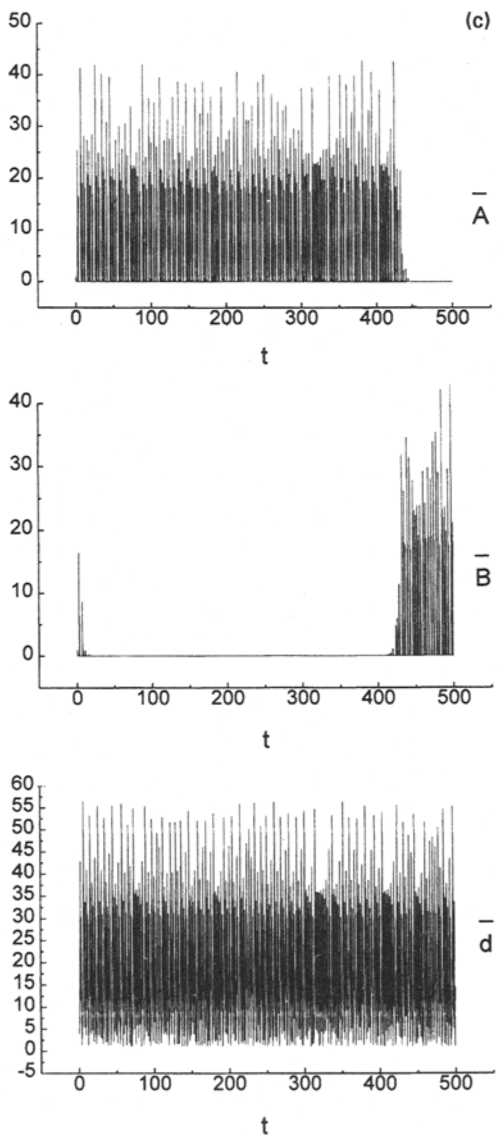

Figure 1(c).

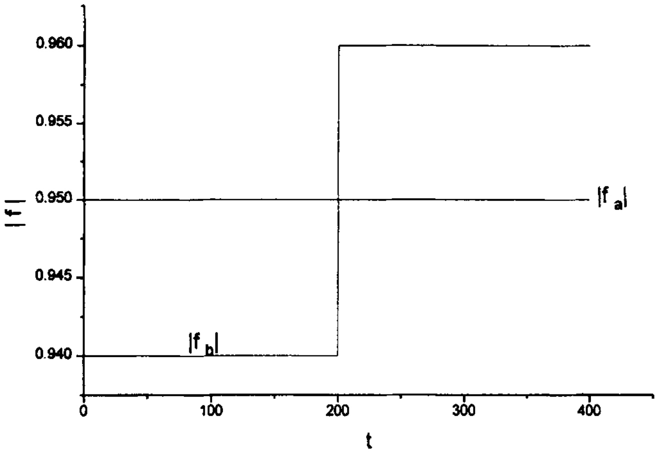

Figure 1. The temporal evolution of the mode amplitudes $\bar{A}, \bar{B}$ and $\bar{d}$ for (a) $\tilde{\kappa} = 3$, (b) $\tilde{\kappa} = 9$ and (c) $\tilde{\kappa} = 15$ when $|f_a| = 0.95$ and $|f_b|$ is switched as per eq. (27) at $t' = 200$. The other parameters are $\tilde{\beta} = 1, \delta = 0.01$ and $\Delta\tilde{\omega}_a = \Delta\tilde{\omega}_b = 2$. The inset shows $|f_a|$ and $|f_b|$ as functions of time.

characterized by lack of limit cycles and was chaotic, while larger values of κ led to period-3 bifurcation. In this paper our goal is to investigate whether the dynamics can switch from one pair to the other when the relevant parameters, namely, the phase mismatch parameter and the decay rates of the down converted modes switch at a given time, say at $t = t'$. We have carried out the numerical integration for the following two cases:

(i)
$$\tilde{\kappa}_a = \tilde{\kappa}_b = \tilde{\kappa}, |f_a| = 0.95,$$
$$|f_b| = |f_a| - \delta + \delta[1 + \text{sign}(t - t')], \quad (27)$$

(ii)
$$|f_a| = |f_b| = 1,$$
$$\tilde{\kappa}_a = \tilde{\kappa}_0\{1 + \tfrac{1}{2}[1 + \text{sign}(t - t')]\delta\}, \quad (28)$$
$$\tilde{\kappa}_b = \tilde{\kappa}_0\{1 - \delta + \tfrac{1}{2}[1 + \text{sign}(t - t')]\delta'\}, \quad \delta' = \delta^2 + 3\delta. \quad (29)$$

In eqs (28) and (29), $\tilde{\kappa}_a$ and $\tilde{\kappa}_b$ are varied such that the ratio $\tilde{\kappa}_b/\tilde{\kappa}_a$ assumes the value $1 - \delta(1 + \delta)$ before (after) switching. In all our calculations we have chosen $t' = 200$ and $\delta = 0.01, \Delta\tilde{\omega}_a = \Delta\tilde{\omega}_b = 2, \tilde{\beta} = \beta/\kappa_d = 1$. Note that such a large value of β can be achieved by a suitable choice of the pump gain, the nonlinear material or by switching to micro cavities with small quantization volume.

We now present the results for case (i). The results for the temporal evolution of the mode amplitudes $\bar{A}, \bar{B}$ and $\bar{d}$ for three values of $\tilde{\kappa}$, namely, $\tilde{\kappa} = 3, 9$ and 15 are shown in figures 1a, 1b and 1c, respectively. The inset to figure 1 shows the switching patterns for $|f_a|$ and $|f_b|$. In each of these plots we have shown the temporal evolution of the mode amplitudes. Though investigation was performed for t ranging from 0 to 600 we have shown a smaller segment in figures 1a and 1b in order to show the period one and period two oscillations. It is clear from figures 1a, 1b and 1c that as in ref. [7] the dynamics is regular for $\tilde{\kappa} = 3$ (period one), $\tilde{\kappa} = 9$ (period two) while for $\tilde{\kappa} = 15$ the motion is chaotic

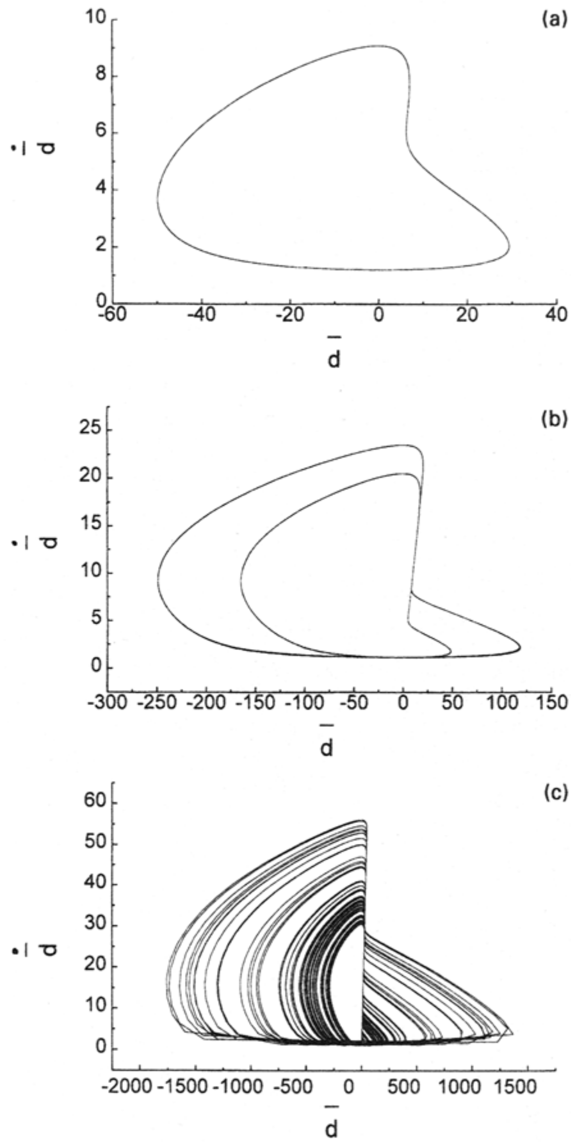

Figure 2. The phase portrait for the mode amplitudes $\bar{d}$ (i.e., $\bar{d}$ vs. $\dot{\bar{d}}$) for (**a**) $\tilde{\kappa} = 3$, (**b**) $\tilde{\kappa} = 9$ and (**c**) $\tilde{\kappa} = 15$. The other parameters are as in figure 1.

(figure 1c). This is depicted by the phase portraits of $\bar{d}$ which is shown in figures 2a, 2b and 2c. It is also clear from figure 1 that before switching, $|f_a| > |f_b|$ and the a-mode shows nonzero oscillations while the b-mode is off. The dynamics switches from a to b-mode once $|f_b|$ becomes larger than $|f_a|$. The switching of the dynamics is independent of whether the motion is with one-period, two-period or chaotic in nature. It may also be noted that the exchange of dynamics is also independent of how switching takes place.

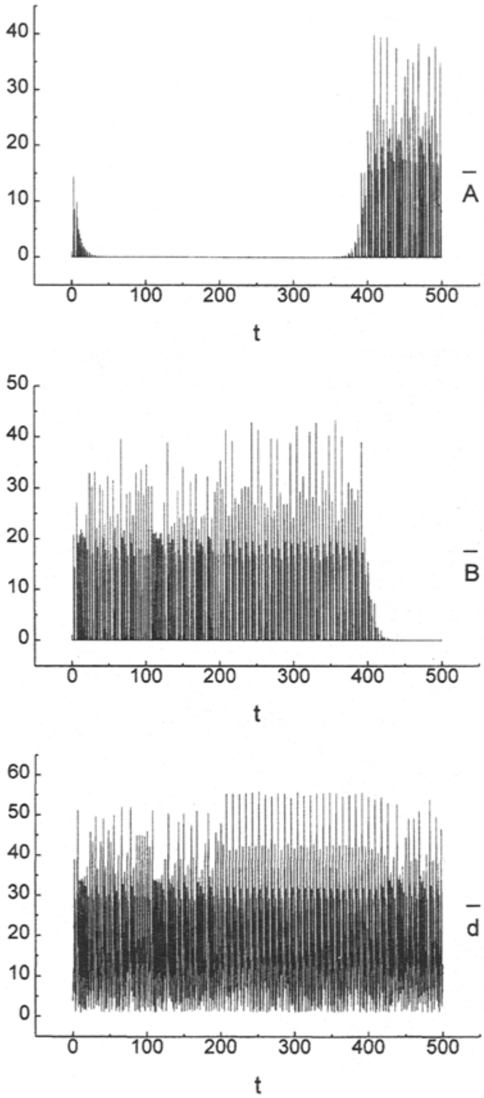

Figure 3. The temporal evolution of the mode amplitudes $\bar{A}$, $\bar{B}$ and $\bar{d}$ when $\tilde{\kappa}_a$ and $\tilde{\kappa}_b$ are switched as per (28) and (29) for $\tilde{\kappa}_0 = 15$, $|f_a| = |f_b| = 1$ and $t' = 200$. The other parameters are as in figure 1.

For example, instead of a step function (eq. (27)) one could have chosen an exponential switching.

We now look at case (ii) when $|f_a|$ and $|f_b|$ are held fixed at unity while $\tilde{\kappa}_a$ and $\tilde{\kappa}_b$ are switched at $t = t'$ such that $\tilde{\kappa}_b/\tilde{\kappa}_a$ is $1 - \delta(1 + \delta)$ before (after) switching as per eqs (28) and (29). We have performed the investigation for three values of $\tilde{\kappa}_0$, namely, $\tilde{\kappa}_0 = 3, 9$ and 15 which correspond to period one, period two and chaotic motion. We present the results only for $\tilde{\kappa}_0 = 15$ (see figure 3). It is clear from figure 3 that for the same value of

$|f|$ for both the modes, the mode with lower decay rate oscillates. Thus, before (after) switching the a-mode (b-mode) exhibits off-state.

We also investigated the case when the parameters for both the modes satisfy eq. (13) and both of them can show nontrivial oscillations (not shown). However, as mentioned earlier, on-states for both the modes are extremely unlikely. Any fluctuation leading to the violation of the second equality in eq. (13) will lead to a situation when only one mode pair survives.

4. Conclusion

We have studied a doubly resonant parametric oscillator where a pump with gain can feed two pairs of signal-idler modes. We have demonstrated both theoretically and numerically that there can be an exchange of dynamics between the down-converted mode pairs, when the relevant parameters like phase mismatch factor or cavity decay rates change due to fluctuations. The switching of dynamics from on-to-off or from off-to-on for the pairs takes place even when the dynamical system exhibits chaotic behavior. Moreover, there can be a state where both the down converted pairs exhibit chaotic motion though because of ever present fluctuations, this state is highly unlikely.

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