PRAMANA
—journal of
physics

© Printed in India

Vol. 49, No. 4,
October 1997
pp. 385–388

Accurate energy-levels of the sextic-double-well oscillator $V(X) = X^6 - 3X^4$ using modified Hill determinant approach

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MS received 4 November 1996; revised 5 March 1997

Abstract. Using scaled harmonic oscillator wave functions, accurate energy levels of the sextic-double-well oscillator $V(X) = X^6 - 3X^4$ are calculated through modified Hill determinant method. Present groundstate energy remains the same as reported by Tater and Turbiner, *J. Phys.* **A26**, 697 (1993).

Keywords. Sextic-double-well oscillator; scaled wave function; groundstate energy; modified Hill determinant approach.

PACS No. 03.65

The one dimensional anharmonic oscillator potential has been used extensively in nuclear physics, particle physics, solid state physics and atomic and molecular physics. This potential has the characteristics of being a rather simple model where many non-trivial features essential to understanding quite complicated system may be implemented. Their exact solutions for arbitrary couplings are hard to find. This has culminated into the development of many fascinating approximation techniques, both perturbative [1–4] and non-perturbative [5–10]. The most commonly used oscillator is

$$H = P^2 + X^2 + \lambda X^4. \tag{1}$$

Accurate energy levels of the oscillator have been calculated [1–8] using perturbation series and Hill determinant approach. However perturbation theory has not been used successfully for double well oscillator. It is worth mentioning that original Hill determinant method of Biswas *et al* [5] has undergone many modifications for sextic well oscillator. In this context it was pointed out by Tater and Turbiner [10] that Hill determinant approach fails to yield correct groundstate energy of the sextic double well oscillator characterized by the Hamiltonian [10]

$$H = P^2 + X^6 - 3X^4. \tag{2}$$

Hence the aim of this communication is to calculate groundstate as well as excited energy levels of the sextic-double-well oscillator characterized by the Hamiltonian in (2) using modified Hill determinant approach. The procedure is as follows.

The Hamiltonian in (2) is written in terms of creation and annihilation operator using the co-ordinate transformation

$$X = \frac{(a + a^+)}{\sqrt{2W}} \tag{3a}$$

and

$$P = \sqrt{\frac{W}{2}}\, i(a^+ - a). \tag{3b}$$

Further parameter W is obtained by making the coefficient of a^2 or $(a^+)^2$ to zero so that the matrix element $\langle 0 \,|\, H \,|2\rangle_W$ vanishes. This procedure has been applied for a long time in nuclear physics [11] to find out the most effective Hamiltonian. Recently this procedure was suggested for parameter calculation in anharmonic oscillators either using perturbative approach [12] or non-perturbative approach [13, 14]. One can see that W satisfies the following equation

$$4W^4 - 36W - 45 = 0, \tag{4}$$

which has only one positive definite root. Now we solve the Schrödinger equation

$$H\Psi = E\Psi \tag{5}$$

by assuming

$$\Psi = \sum_{m=0}^{\infty} A_m |m\rangle_W \tag{6}$$

whose $|m\rangle_W$ are the scaled harmonic oscillator function which satisfies the relation

$$(P^2 + X^2)|m\rangle_W = (2m + 1)|m\rangle_W. \tag{7}$$

Now substituting $\Psi = \Sigma_m A_m |m\rangle_W$ in (5) we find the $A_m^{'}$s satisfies the following seven-term recursion relation

$$P_m A_{m-6} + Q_m A_{m-4} + R_m A_{m-2} + S_m A_m + T_m A_{m+2} + U_m A_{m+4} + V_m A_{m+6} = 0, \tag{8}$$

where

$$P_m = \frac{1}{8W^3}[m(m-1)(m-1)(m-3)(m-4)(m-5)]^{1/2}, \tag{9a}$$

$$Q_m = \left[\frac{3(2m-3)}{8W^3} - \frac{3}{4W^2}\right][m(m-1)(m-2)(m-3)]^{1/2}, \tag{9b}$$

$$R_m = \left[\frac{15(m+1)}{8W^3} - \frac{3}{W^2}\right][(m-2)[m(m-1)]^{1/2}], \tag{9c}$$

$$S_m = mW - \frac{18}{4W^2}m(m+1) + \frac{1}{8W^3}(20m^3 + 30m^2 + 40m) + \frac{W}{2} - \frac{9}{4W^2} + \frac{15}{8W^3} - E, \tag{9d}$$

Table 1. Energy levels of the sextic-double-well oscillator $V(X) = X^6 - 3X^4$.

Present calculation							
Even parity ($n = 0$)		Odd parity ($n = 1$)		Exact		Previous [10] Hill approach	
−0.660	546	0.103	079	−0.660	5	−1.672	3
3.423	833	7.586	637				
12.574	054	18.409	220				
24.989	883	32.249	273				
40.138	756	48.619	117				

Table 2. First two even states of the $V(X) = X^6 + \lambda X^2$.

λ	Present			Previous [10] Exact
3	1.935	482	1	1.935 483
	11.680	970	8	
0	1.144	802	4	1.144 802
	9.073	084	5	
−3	0.000	000	0	0
	6.298	495	9	
−11	−8.000	000	0	−8
	0.000	000	0	0

$$T_m = R_{m+2}, \tag{9e}$$

$$U_m = Q_{m+4}, \tag{9f}$$

$$U_m = P_{m+6}. \tag{9g}$$

The eigenvalue condition of the modified Hill determinant in the limit of large N is

$$\operatorname{Det} D_N = 0, \tag{10}$$

where

$$D_N = \begin{bmatrix} S_n & T_n & U_n & V_n & 0 & \cdots \\ R_{n+2} & S_{n+2} & T_{n+2} & U_{n+2} & V_{n+2} & \cdots \\ Q_{n+4} & R_{n+4} & S_{n+4} & T_{n+4} & U_{n+4} & \cdots \\ P_{n+6} & Q_{n+6} & R_{n+6} & S_{n+6} & T_{n+6} & \cdots \end{bmatrix},$$

and where $n = 0$ for even parity and $n = 1$ for odd parity. The zeros of D_N as a function of the parameter E give the energy eigenvalues of the problem.

The first five convergent odd levels ($n = 1$) and even level ($n = 0$) are tabulated in table 1. It is seen that the present groundstate energy yields the correct agreement with the groundstate energy (up to four decimals) calculated earlier by Tater and Turbiner [10]. However there are no results for comparison with the present excited values. It is worth mentioning that the previous [10] convergent groundstate energy using Hill determinant approach is −1.6723.

In this approach, the method is very simple and our choice of W reduces the Hamiltonian to its effective value. The method can be extended to any parity invariant oscillator. Lastly we give some results for specific double well in table 2 and details will be reported elsewhere.

Acknowledgement

The author is grateful to the Director, Institute of Physics, Bhubaneswar for the hospitality during the summer where this work was completed.

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