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A new class of cosmological models in Lyra geometry*

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Abstract. FRW models have been studied in the cosmological theory based on Lyra's geometry. A new class of exact solutions has been obtained by considering a time dependent displacement field for constant deceleration parameter models of the universe.

Keywords. Cosmology; Lyra geometry; FRW models.

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1. Introduction

Einstein's idea of geometrizing gravitation in the form of general theory of relativity inspired the idea of geometrizing other physical fields. Weyl [1] suggested the first so-called unified theory which is a geometrized theory of gravitation and electromagnetism. But this theory was never taken seriously as it was based on non-integrability of length transfer. Lyra [2] proposed a new modification of Riemannian geometry by introducing a gauge function which removes the non-integrability condition of the length of a vector under parallel transport. In consecutive investigations Sen [3], Sen and Dunn [4] proposed a new scalar-tensor theory of gravitation and constructed an analog of the Einstein field equations based on Lyra's geometry which in normal gauge may be written as

$$R_{ij} - \tfrac{1}{2} g_{ij} R + \tfrac{3}{2}\phi_i \phi_j - \tfrac{3}{4} g_{ij}\phi_k \phi^k = -8\pi G T_{ij}, \tag{1}$$

where ϕ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [5] has pointed out that the constant vector displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general relativistic treatment. It is shown by Halford [6] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects, within observational limits, as the Einstein's theory. Bhamra [7], Karade and Borikar [8], Kalyanshetti and Wagmode [9], Reddy and Innaiah

*Dedicated to Professor V B Johri on his sixtieth birthday.

[10], Beesham [11], Reddy and Venkateswarlu [12], Soleng [13] have studied cosmology in Lyra's geometry with a constant displacement field. However, this restriction of the displacement field to be constant is merely one of convenience and there is no *a priori* reason for it. Beesham [14] considered FRW models with time dependent displacement field. He has shown that by assuming the energy density of the universe to be equal to its critical value, the models have the $k = -1$ geometry. Singh and Singh [15–18] have presented Bianchi-type I, III and Kantowaski–Sachs cosmological models with time dependent displacement field and have made a comparative study of Robertson–Walker models with constant deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. They [18] have given a review of cosmological models in Lyra's geometry.

Most of the well known FRW models of the universe with curvature parameter $k = 0$ are models with constant deceleration parameter. Therefore, in this paper we have studied FRW models with time dependent displacement field and have solved the field equations by taking the deceleration parameter to be constant.

2. Field equations

The time-like displacement vector ϕ_i in (1) is given by

$$\phi_i = (\beta(t), 0, 0, 0). \tag{2}$$

We assume a perfect fluid for which the energy-momentum tensor has the form

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}. \tag{3}$$

For the FRW metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Phi^2)\right],$$

where $k = 1, -1, 0$, the field equations (1) become with the eqs (2) and (3)

$$3H^2 + \frac{3k}{R^2} - \frac{3\beta^2}{4} = \chi\rho, \tag{4}$$

$$2\dot{H} + 3H^2 + \frac{k}{R^2} + \frac{3\beta^2}{4} = -\chi p, \tag{5}$$

where $\chi = 8\pi G$ and $H = \dot{R}/R$ is the Hubble's function. Equations (4) and (5) lead to the continuity equation

$$\chi\dot{\rho} + \tfrac{3}{2}\beta\dot{\beta} + 3[\chi(\rho + p) + \tfrac{3}{2}\beta^2]H = 0. \tag{6}$$

Assuming an equation of state

$$p = \gamma\rho, \qquad 0 \leq \gamma \leq 1 \tag{7}$$

and eliminating $\rho(t)$ from (4) and (5) we have

$$2\dot{H} + 3(1 + \gamma)H^2 + (1 + 3\gamma)\frac{k}{R^2} + (1 - \gamma)\frac{3\beta^2}{4} = 0. \tag{8}$$

Here β^2 plays the role of a variable cosmological term $\Lambda(t)$. We have two independent equations in three unknown viz $R(t)$, $\rho(t)$ and $\beta(t)$. Therefore, we need one more relation among the variables in order to obtain a unique solution. Hence, in what follows we have taken the deceleration parameter to be constant.

3. Solutions of the field equations

Let the deceleration parameter

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = -\frac{(\dot{H} + H^2)}{H^2} = b \text{ (constant)}. \tag{9}$$

The above equation may be rewritten as

$$\frac{\ddot{R}}{R} + b\frac{\dot{R}^2}{R^2} = 0. \tag{10}$$

On integration, equation (10) gives the exact solution

$$R(t) = \begin{matrix} (D + Ct)^{1/(1+b)} & b \neq -1, \\ R_0\, e^{H_0 t} & b = -1, \end{matrix} \tag{11}$$

where C, D, R_0 and H_0 are constants of integration. Using (9) in (8) leads to

$$\beta^2 = \frac{4}{3(\gamma - 1)}\left[(1 + 3\gamma - 2b)H^2 + (1 + 3\gamma)\frac{k}{R^2}\right]. \tag{12}$$

Now using (12) in (4) gives

$$\chi\rho = \frac{1}{(\gamma - 1)}\left[2(b - 2)H^2 - \frac{4k}{R^2}\right]. \tag{13}$$

Flat models

With $k = 0$, equations (12) and (13) reduce to

$$\beta^2 = \frac{4}{3(\gamma - 1)}(1 + 3\gamma - 2b)H^2 \tag{14}$$

and

$$\chi\rho = \frac{1}{(\gamma - 1)}2(b - 2)H^2. \tag{15}$$

From (15) we see that $\rho \geq 0$ if $b - 2 \leq 0$ i.e. $b \leq 2$ since $(\gamma - 1) < 0$. From (14) we see that since $(\gamma - 1) < 0$,

$$\beta^2 > 0 \quad \text{if} \quad b > \frac{(1 + 3\gamma)}{2} \tag{16}$$

and

$$\beta^2 < 0 \quad \text{if} \quad b < \frac{1 + 3\gamma}{2}. \tag{17}$$

Table 1. Values of β^2 and ρ for dust and radiation power-law models.

γ	β^2	ρ
0	$\frac{4}{3}\frac{(2b-1)}{(1+b)^2t^2}$	$\frac{2(2-b)}{\chi(1+b)^2t^2}$
$\frac{1}{3}$	$\frac{4(b-1)}{(1+b)^2t^2}$	$\frac{3(2-b)}{\chi(1+b)^2t^2}$

Again from equation (14) we see that when $b=(1+3\gamma)/2, \beta^2=0$ and the equations reduce to those of the standard FRW flat universe.

Case a: $b\neq -1$. For singular models since $R(0)=0$, (11) leads to

$$R=R_0t^{1/(1+b)}. \tag{18}$$

Using (18) in (14) and (15) yields

$$\beta^2=\frac{4(1+3\gamma-2b)}{3(\gamma-1)}\frac{1}{(1+b)^2t^2} \tag{19}$$

and

$$\chi\rho=\frac{2(b-2)}{(\gamma-1)(1+b)^2}\frac{1}{t^2}. \tag{20}$$

The above expressions for β^2 and energy density $\rho(t)$ are similar to those obtained by Beesham [19] for a variable cosmological term $\Lambda(t)$ and energy density $\rho(t)$. Here β^2 plays the role of a variable cosmological term Λ.

Equations (7) and (20) yield

$$\rho+3p=\frac{2(1+3\gamma)(b-2)}{(\gamma-1)(1+b)^2}\frac{1}{t^2}.$$

It can be seen from the above expression that the condition $\rho+3p\geq 0$ would hold only for $1+3\gamma\geq 0$ i.e. $\gamma\geq -1/3$. So for values of $\gamma<-1/3$ we cannot have viable models.

We observe from (19) and (20) that β^2 and ρ fall off as $1/t^2$ irrespective of the equation of state. The expressions for β^2 and ρ corresponding to $\gamma=0, 1/3$ are summarized in the table 1.

It can be readily seen from (19) and (20) that the expressions for β^2 and ρ will not be valid for the empty universe (i.e. $p=\rho=0$) and the stiff matter (i.e. $p=\rho$) models. We shall now discuss these models.

Empty universe

In the case of empty universe ($p=\rho=0$) eqs (4) and (5), with $k=0$, now become

$$3H^2-\frac{3\beta^2}{4}=0, \tag{21}$$

$$2\dot{H}+3H^2+\frac{3\beta^2}{4}=0. \tag{22}$$

Adding equations (21) and (22) yields

$$\dot{H} + 3H^2 = 0. \tag{23}$$

On comparing (9) and (23) we get $b = 2$. Therefore, (18) reduces to

$$R = R_0 t^{1/3}. \tag{24}$$

Using (24) in (21) leads to

$$\beta^2 = \frac{4}{9t^2}. \tag{25}$$

From (25) it can be seen that $\beta^2 > 0$ for all values of the deceleration parameter.

Stiff matter model

In the case of Zeldovich fluid viz., $p = \rho$ (i.e. $\gamma = 1$) equations (4), (5) and (8), with $k = 0$, reduce to

$$3H^2 - \frac{3\beta^2}{4} = \chi\rho, \tag{26}$$

$$2\dot{H} + 3H^2 + \frac{3\beta^2}{4} = -\chi\rho, \tag{27}$$

$$\dot{H} + 3H^2 = 0. \tag{28}$$

From (26) it can be seen that $\rho \geq 0$ and $\rho + 3p \geq 0$ for $\beta^2 \leq 0$ since $H^2 \geq 0$ and for $0 \leq \beta^2 \leq 4H^2$.

On comparing equations (9) and (28) we see that $b = 2$. Therefore, (18) reduces to

$$R = R_0 t^{1/3}. \tag{29}$$

Here the expressions for β^2 and ρ cannot be determined uniquely.

Case b: $b = -1$. In this case, equation (9) becomes

$$\dot{H} = 0 \quad \text{and} \quad H = H_0 = \text{constant}. \tag{30}$$

Using (30) in (14) and (15) we have

$$\beta^2 = \frac{4(1+\gamma)}{(\gamma - 1)} H_0^2 = \text{constant} \tag{31}$$

and

$$\chi\rho = \frac{-6H_0^2}{(\gamma - 1)} = \text{constant}. \tag{32}$$

From (31) it can be seen that since $(1+\gamma) > 0$ and $(\gamma - 1) < 0, \beta^2 < 0$ and from (32) we have $\rho > 0$ for all times since $(\gamma - 1) < 0$.

The expression for β^2 given by (31) is similar to the expression for the cosmological term Λ given by (32) with $k = 0$ in ref. [19]. From (7) and (32) we have

$$\rho + 3p = \frac{6(1+3\gamma)H_0^2}{(1-\gamma)}.$$

Table 2. Values of β^2 and ρ for dust and radiation exponential models.

γ	β^2	ρ
0	$-4H_0^2$	$\frac{6H_0^2}{\chi}$
$\frac{1}{3}$	$-8H_0^2$	$\frac{9H_0^2}{\chi}$

As in the previous case, we have $\rho + 3p \geq 0$ only for $1 + 3\gamma \geq 0$ i.e. $\gamma \geq -\frac{1}{3}$. The expressions for β^2 and ρ corresponding to $\gamma = 0, 1/3$ are given in table 2.

For the empty universe and stiff matter models the expressions for β^2 and ρ cannot be determined.

In the next section we shall discuss models with non-zero curvature parameter k.

Non-flat models

Case a: $b \neq -1$. Using (18) in (12) and (13) yields

$$\beta^2 = \frac{4}{3(\gamma - 1)} \left[\frac{(1 + 3\gamma - 2b)}{(1 + b)^2 t^2} + \frac{(1 + 3\gamma)k}{R_0^2 t^{2/(1+b)}} \right] \tag{33}$$

and

$$\chi\rho = \frac{1}{(\gamma - 1)} \left[\frac{2(b - 2)}{(1 + b)^2} \frac{1}{t^2} - \frac{4k}{R_0^2 t^{2/(1+b)}} \right]. \tag{34}$$

An obvious requirement that must be imposed is $\rho \geq 0$, which can be achieved only for $k \geq 0$ with $b - 2 \leq 0$ (i.e. $b \leq 2$) as $(\gamma - 1) < 0$. The case $k = 0$ has already been discussed in the previous section.

The above expressions for ρ and β^2 are similar to those of ρ and Λ given by equations (45) and (46) of ref. [19].

From (7) and (34) we have

$$\rho + 3p = \frac{(1 + 3\gamma)}{\chi(\gamma - 1)} \left[\frac{2(b - 2)}{(1 + b)^2 t^2} - \frac{4k}{R_0^2 t^{2/(1+b)}} \right].$$

From the above expression it can be seen that the condition $\rho + 3p \geq 0$ would hold only if $1 + 3\gamma \geq 0$ (i.e. $\gamma \geq -\frac{1}{3}$) as $b \leq 2$, $k \geq 0$ and $(\gamma - 1) < 0$. For values of $\gamma < -\frac{1}{3}$ we cannot have viable models.

With $k = 1$, equations (33) and (34) reduce to

$$\beta^2 = \frac{4}{3(\gamma - 1)} \left[\frac{(1 + 3\gamma - 2b)}{(1 + b)^2 t^2} + \frac{(1 + 3\gamma)}{R_0^2 t^{2/(1+b)}} \right] \tag{35}$$

and

$$\chi\rho = \frac{1}{(\gamma-1)}\left[\frac{2(b-2)}{(1+b)^2 t^2} - \frac{4}{R_0^2 t^{2/(1+b)}}\right]. \tag{36}$$

From eq. (35), we see that for $b < (1+3\gamma)/2$, $\beta^2 < 0$ for all times $t > 0$ as $(\gamma - 1) < 0$.

Again from equation (35) we observe that for $(1+3\gamma)/2 < b \leq 2$; $\beta^2 > 0$ for

$$0 < t^{2b/(1+b)} < \frac{(2b-1-3\gamma)R_0^2}{(1+3\gamma)(1+b)^2}$$

and $\beta^2 < 0$ for

$$t^{2b/(1+b)} > \frac{(2b-1-3\gamma)R_0^2}{(1+3\gamma)(1+b)^2}.$$

At

$$t^{2b/(1+b)} = \frac{(2b-1-3\gamma)R_0^2}{(1+3\gamma)(1+b)^2}, \quad \beta^2 = 0.$$

When $b = (1+3\gamma)/2$, equations (35) and (36) reduce to

$$\beta^2 = \frac{4(1+3\gamma)}{3(\gamma-1)R_0^2 t^{4/3(1+\gamma)}} \tag{37}$$

and

$$\chi\rho = \frac{1}{(1-\gamma)}\left[\frac{3(1-\gamma)}{(1+b)^2 t^2} - \frac{4}{R_0^2 t^{4/3(1+\gamma)}}\right]. \tag{38}$$

From equation (37), it is obvious that $\beta^2 < 0$ for all times as $(\gamma - 1) < 0$. The expressions for β^2 and ρ cannot be determined for the empty universe (i.e. $p = \rho = 0$) and stiff matter $(p = \rho)$ models.

Case b: $b = -1$. Using (30) in (12) and (13) leads to

$$\beta^2 = \frac{4}{3(\gamma-1)}\left[3(1+\gamma)H_0^2 + (1+3\gamma)\frac{k}{R_0^2}e^{-2H_0 t}\right] \tag{39}$$

and

$$\chi\rho = \frac{2}{(1-\gamma)}\left[3H_0^2 + \frac{2k}{R_0^2}e^{-2H_0 t}\right]. \tag{40}$$

From equation (40) we observe that $\rho > 0$ only for $k > 0$ as $(1-\gamma) > 0$. $\beta^2 < 0$ for all times as can be seen from equation (39) as $(\gamma - 1) < 0$. For large times i.e. $t \to \infty$ we see that β^2 and ρ would reach steady state i.e.

$$\beta^2 \longrightarrow \frac{4}{(\gamma-1)}(1+\gamma)H_0^2 \quad \text{and} \quad \chi\rho \longrightarrow \frac{6}{(1-\gamma)}H_0^2.$$

The expression for β^2 given by equation (39) is similar to the expression for the cosmological term given by equation (52) of ref. [19].

As in the previous case we have $\rho + 3p \geq 0$ only for $\gamma \geq -\frac{1}{3}$. For values of $\gamma < -1/3$ we cannot have viable models.

For the empty universe and stiff matter models the expressions for β^2 and ρ cannot be determined.

4. Conclusion

In this paper we have obtained exact solutions of Sen equations in Lyra geometry for constant deceleration parameter. The behaviour of the displacement field β and the energy density ρ have been examined for both the (i) power-law and (ii) exponential expansions of both the flat universe and the non-flat universe.

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