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Nonlinear wave modulations in plasmas*

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Abstract. A review of the generic features as well as the exact analytical solutions of coupled scalar field equations governing nonlinear wave modulations in plasmas is presented. Coupled sets of equations like the Zakharov system, the Schrödinger–Boussinesq system and the Schrödinger–KDV system are considered. For stationary solutions, the latter two systems yield a generic system of a pair of coupled, ordinary differential equations with many free parameters. Different classes of exact analytical solutions of the generic system which are valid in different regions of the parameter space are obtained. The generic system is shown to generalize the Hénon–Heiles equations in the field of nonlinear dynamics to include a case when the kinetic energy in the corresponding Hamiltonian is not positive definite. The relevance of the generic system to other equations like the self-dual Yang–Mills equations, the complex KDV equation and the complexified classical dynamical equations is also pointed out.

Keywords. Nonlinear waves; modulational instability; solitons; NLS equation; KDV equation; Hénon–Heiles Hamiltonian; integrability; complexification.

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1. Introduction

Coupled second-order ordinary differential equations for a pair of real scalar fields occur in many branches of physics. For example, in particle physics, the relativistic quantum field theories in $1+1$ dimensions for finite-energy localized fields lead to a set of nonlinear equations for a pair of scalar fields [1]. On the other hand, in plasma physics, the nonlinear development of the instability associated with the envelope modulation of an high-frequency wave packet coupled to an appropriate low frequency wave field is governed [2] by a pair of coupled equations like Zakharov equations, or the Schrödinger–Korteweg-de Vries (KDV) (or, –Boussinesq) system. For stationary solutions, the latter two systems lead to a generic system of a pair of coupled ordinary differential equations. Depending on the problem at hand, the generic system has varying number of free parameters which can take values over a wide range. An outstanding mathematical problem associated with such nonlinear coupled equations is to obtain their exact analytical solutions valid over as much of the parameter space as possible.

The generic equations are of interest in other contexts also. Recently, it was pointed out [3] that the stationary set derived from the Schrödinger–KDV (or, –Boussinesq) system is not only very similar to but also generalizes the well-known Henon–Heiles system [4]

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which has been extensively studied over the last decade in the field of nonlinear dynamics. On the other hand, there is the interesting possibility that the self-dual Yang–Mills field equations [5] could be related to the stationary equations derived from the Schrödinger–KDV (or, –Boussinesq) system. Furthermore, when the dependent variable of an uncoupled KDV equation is made complex [6, 7], the resulting set of real equations are structurally very similar to those contained in the generic system. The latter reflects also certain interesting properties of the coupled equations obtained by the 'complexification' of the dynamical equation for a classical particle with one-degree of freedom in a general conservative potential field [6].

In this review, I will briefly discuss some of the above features. In the process, I will describe a method of solution that can be used for obtaining analytical solutions of coupled scalar field equations. In particular, I will use this method to obtain different classes of exact analytical solutions of the generic equations obtained from the Schrödinger–KDV (or, –Boussinesq) system. Wherever possible, the examples are chosen from the field of plasma physics where such equations are commonly encountered. Unless otherwise stated, all the variables are assumed to be suitably normalized so that the equations are dimensionless throughout.

2. Modulational instability

Consider the propagation along $\hat{x}$-direction of a plane wave of frequency ω_0 and wavenumber k_0 represented by

$$\mathcal{E} = E(x,t)\exp[i(k_0 x - \omega_0 t)], \tag{1}$$

where the complex amplitude $E(x,t)$ is a slowly varying function of the space and the time variables. In order to derive a governing equation for the wave propagation, we start with the nonlinear dispersion relation, namely, $\omega = \omega(k, |E|^2)$ and Taylor expand it around (ω_0, k_0) to obtain

$$\omega - \omega_0 = (k - k_0)\frac{\partial\omega}{\partial k} + \frac{1}{2}(k - k_0)^2\frac{\partial^2\omega}{\partial k^2} + |E|^2\frac{\partial\omega}{\partial(|E|^2)}, \tag{2}$$

where the partial derivatives have to be evaluated at $k = k_0$ and $|E| = 0$. Replacing the frequency shift $(\omega - \omega_0)$ by $i\partial/\partial t$ and the wavenumber shift $(k - k_0)$ by $-i\partial/\partial x$, we obtain the following evolution equation for the slowly varying complex amplitude $E(x,t)$

$$i\left(\frac{\partial E}{\partial t} + V_g\frac{\partial E}{\partial x}\right) + P\frac{\partial^2 E}{\partial x^2} + (QEE^*)E = 0, \tag{3}$$

where asterisk denotes complex conjugate, $V_g \equiv \partial\omega/\partial k$ denotes the group velocity, $P \equiv (\partial^2\omega/\partial k^2)/2$ represents the group dispersion, and $Q \equiv -\partial\omega/\partial(|E|^2)$ is the nonlinear coefficient. In view of its structural similarity, eq. (3) is called the 'nonlinear Schrödinger equation', and is known to govern the evolution of different types of high-frequency waves in plasmas. Examples are the Langmuir waves, the upper-hybrid waves and the electromagnetic waves.

Equation (3) can be analysed to determine the stability properties of an high-frequency carrier wave when its amplitude is modulated with a lower frequency. Depending on the relative sign between the dispersive and the nonlinear terms, the modulation can become unstable. Linear stability analysis shows [2] that for $PQ > 0$ the waves are modulationally unstable. Physically, the instability arises due to the self-trapping of the wave field in the 'potential', which is determined by the wave itself [cf. the last term on the left-hand side of eq. (3)].

We shall now look for solutions of eq. (3) which are stationary in a frame moving with a constant speed. For this, we note that the second term in eq. (3) can be eliminated by going into a Galilean frame defined by $\zeta = x - V_g t$ and $\tau = t$:

$$i\frac{\partial E}{\partial \tau} + P\frac{\partial^2 E}{\partial \zeta^2} + (QEE^*)E = 0. \tag{4}$$

We now look for stationary solutions in the (ζ, τ) coordinates by defining the variable, $\eta = \zeta - M\tau \equiv x - (V_g + M)t$ where M is the normalized speed, called the 'Mach number', of the stationary frame. In order to allow for any possible shift in the frequency as well as in the wave number of the carrier wave, we represent the amplitude field as

$$E = E_a(\eta)\exp[i\{X(\zeta) + T(\tau)\}], \tag{5}$$

where $E_a(\eta)$ is the real, stationary amplitude of the modulated wave. Substituting the solution (5) into eq. (4), we obtain from the imaginary part, $X(\zeta) = M\zeta/2P$ whereas the real part yields the following equation for the amplitude E_a:

$$2P\frac{\mathrm{d}^2 E_a}{\mathrm{d}\eta^2} = \lambda E_a - 2QE_a^3, \tag{6}$$

where $\lambda = 2(\mathrm{d}T/\mathrm{d}\tau) + (M^2/2P)$ is the nonlinear shift parameter. For localized boundary conditions, eq. (6) can be easily integrated to obtain the so-called 'envelope soliton', solution, namely,

$$E_a(\zeta, \tau) = \pm\left(\frac{\lambda}{Q}\right)^{1/2}\mathrm{sech}\left\{\left(\frac{\lambda}{2P}\right)^{1/2}(\zeta - M\tau)\right\}. \tag{7}$$

Clearly, the total high-frequency field $\mathcal{E}(x, t)$ has a structure wherein the amplitude of the carrier wave field is modulated, leading to a bell-shaped profile and the structure itself propagates with a constant velocity with respect to the laboratory frame.

Before considering further generalizations of the above equation, let us briefly summarize the physical mechanism that leads to such solutions. In plasma physics, the variable E can be taken to represent the electric field of a suitable high-frequency field. It can be easily shown that when the amplitude of such a wave field is slowly modulated spatially, the motion of a particle of charge q and mass m can be decomposed into two parts: the first part contains the linear motion that oscillates with the same frequency as the wave field, and is charge and mass dependent. The second part is the nonlinear motion arising because of the time-averaged force, and is independent of the sign of the charge. Such a nonlinear force is called the 'ponderomotive force' and is, in general, given by [8]

$$\vec{F}_p = -\frac{q^2}{4m\omega^2}\nabla(E_a^2).$$

Because of the inverse dependence on the mass, the ponderomotive force acts strongly on the electrons pushing them away from the regions where the field is stronger. However, the resulting ambipolar field causes the ions to follow the electrons, thereby creating a density well which further traps the high-frequency field. This process is continued till a dynamic balance between the nonlinear and the dispersive effects is achieved. The envelope soliton solution is simply a representation of such a state.

3. Coupled equations

The ponderomotive force due to a high-frequency field in a plasma drives low-frequency oscillation which may be a normal mode of the system. For example, the amplitude modulated Langmuir oscillations are coupled to the wave-excited, low-frequency acoustic fluctuations called the 'ion-acoustic waves'. The nonlinear Schrödinger equation derived above considers, however, only the static response of the latter waves. Such an approximation can be justified when the envelope packet is nearly static. However, when the envelope moves with finite, non-zero speed, dynamic response of the low-frequency wave should be taken into account.

3.1 *Zakharov system*

For the Langmuir-ion-acoustic waves, Zakharov [9] suggested the following pair of coupled equations:

$$i\epsilon\frac{\partial E}{\partial t}+\frac{3}{2}\frac{\partial^2 E}{\partial x^2}=\frac{1}{2}(\delta n_e)E, \tag{8}$$

$$\frac{\partial^2}{\partial t^2}(\delta n)-\frac{\partial^2}{\partial x^2}(\delta n)=\frac{\partial^2}{\partial x^2}\left(\frac{1}{4}EE^*\right), \tag{9}$$

where ϵ is a known small, real parameter, and δn is the perturbed ion number density. Under the approximation used to derive the above equations, δn is equal to the electron number density perturbation, namely, δn_e. Equation (8) is a Schrödinger-like equation with δn_e as the potential, which in turn is governed (since $\delta n_e = \delta n$) by the linear wave equation (9) driven by a source term involving the 'wave function'. The coupled set of equations (8) and (9) is called the 'Zakharov system', and has been very extensively studied in the literature. It may be noted that the nonlinear Schrödinger equation can be recovered from the Zakharov system if, under the assumption of static response, one neglects the time derivative term in the driven wave equation (9). However, for dynamic response, we obtain, using the stationary variable $\xi \equiv x - Mt$,

$$\delta n = -\frac{1}{4}\frac{EE^*}{(1-M^2)}, \quad \phi = \delta n + \frac{|E|^2}{4} \equiv M^2\delta n, \tag{10}$$

where ϕ is the self-consistent ambi–polar potential. Thus, the solutions of the nonlinear Schrödinger equation are valid when the Mach number M tends to zero.

Equations (8) and (9) (together with the quasi–neutrality condition, $\delta n_e = \delta n$) can be integrated for stationary solutions to yield

$$E_a(\xi) = \pm [8\lambda(1 - M^2)]^{1/2}\mathrm{sech}(\mu\xi), \tag{11}$$

$$\phi(\xi) = -2\lambda M^2 \mathrm{sech}^2(\mu\xi), \tag{12}$$

where $\mu^2 = \lambda/3$, and the nonlinear shift parameter given by $\lambda = 2\Delta + (\epsilon^2 M^2/3) \approx 2\Delta$ is required to be positive so that the solutions (11) and (12) satisfy the localized boundary conditions. Equation (11) then requires $M^2 < 1$ for real E_a; that is, the high-frequency envelope wave packet 'loaded' with the low-frequency density fluctuations can only travel at sub-sonic ($M < 1$) speeds. Both the solutions are symmetric with respect to $\xi = 0$. Note that the solutions (11) and (12) imply the following relation:

$$\frac{E_a^2}{4} = -\frac{1 - M^2}{M^2}\phi. \tag{13}$$

Clearly, for a given envelope field amplitude E_a, the variable ϕ becomes large when $M^2 \to 1$, in which case the driven linear wave equation (9) needs to be replaced by a suitable nonlinear equation. This is discussed in the next section.

3.2 *Schrödinger–Boussinesq system*

The nonlinear Schrödinger equation as well as the Zakharov equations take into account only the linear response in the low-frequency dynamics. However, as pointed out above, for the so-called 'near-sonic' ($M \sim 1$) propagation, the amplitude of the low-frequency perturbation can be quite large requiring a nonlinear dynamical equation in place of eq. (9). For the coupled Langmuir and ion-acoustic waves, Makhankov [10] suggested the following coupled Schrödinger–Boussinesq equations as the appropriate set:

$$i\epsilon\frac{\partial E}{\partial t} + \frac{3}{2}\frac{\partial^2 E}{\partial x^2} = \frac{1}{2}(\phi)E, \tag{14}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi - \frac{\partial^4\phi}{\partial x^4} - \frac{\partial^2}{\partial x^2}(\phi^2) = \frac{\partial^2}{\partial x^2}\left(\frac{EE^*}{4}\right). \tag{15}$$

The latter equation is called the (driven) 'Boussinesq equation', and generalizes the linear wave equation (9) to include the dispersive effects (third term on the left-hand side) as well as the nonlinear effects (the fourth term).

For stationary solutions of the form

$$E = E_a(\xi)\exp[i\{X(x) + T(t)\}], \quad \phi = \phi(\xi),$$

where $\xi = x - Mt$, eqs (14) and (15) yield

$$3\frac{\mathrm{d}^2 E_a}{\mathrm{d}\xi^2} = \lambda E_a + \phi E_a, \tag{16}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = -(1 - M^2)\phi - \phi^2 - \frac{E_a^2}{4}. \tag{17}$$

Equations (16) and (17) have two free parameters λ and M defined above. They describe the stationary, bi-directional propagation of coupled Langmuir and ion-acoustic waves in unmagnetized plasmas.

3.3 *Schrödinger–KDV system*

For uni-directional propagation, Nishikawa *et al* [11] suggested a simpler set which contains a driven KDV equation instead of the driven Boussinesq equation (15). Note that the Boussinesq equation has fourth-order space and second-order time derivatives whereas the KDV has third-order space and first-order time derivative terms. The latter equation can be systematically and rigorously derived by using the reductive perturbation analysis [12] on the basic set of plasma equations describing the low-frequency wave. However, it can also be directly obtained from the driven Boussinesq equation (15) under uni-directional, near-sonic approximation which allows us to use $\partial/\partial t \approx -\partial/\partial x$ for propagation along the positive x-direction. Equation (15) then reduces to the equation,

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\phi+\frac{1}{2}\frac{\partial^3\phi}{\partial x^3}+\phi\frac{\partial\phi}{\partial x}=-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{EE^*}{4}\right), \tag{18}$$

which is the driven KDV equation. This equation is coupled, as in the case of the Boussinesq equation, to the Schrödinger-like equation (14). For stationary solutions, the Schrödinger–KDV set yields the equations

$$3\frac{\mathrm{d}^2E_a}{\mathrm{d}\xi^2}=\lambda E_a+\phi E_a, \tag{19}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=-2(1-M)\phi-\phi^2-\frac{E_a^2}{4}, \tag{20}$$

which is very similar to the set of equations (16) and (17). Note that eq. (20) can be directly obtained, as expected, from eq. (17) by using $1-M^2\approx 2(1-M)$ which is valid for $M\rightarrow 1$.

3.4 *Exact analytical solutions*

The coupled equations (19) and (20) can be solved for exact analytical solutions satisfying localized boundary conditions. The equations have two free parameters, and it is desirable to obtain solutions valid in the entire allowed regions of the parameter space. However, it has been possible so far only to obtain exact solutions valid on a straight line defined by the equation, $M=1-20\Delta/3$ in the two-dimensional space spanned by the parameters M and Δ. The solutions are given [11] explicitly by

$$E_a(\xi)=\pm 16\sqrt{3}\Delta\,\mathrm{sech}(\mu\xi)\tanh(\mu\xi), \tag{21}$$

$$\phi(\xi)=-12\Delta\,\mathrm{sech}^2(\mu\xi), \tag{22}$$

where $\mu^2=2\Delta/3$, and $\lambda\approx 2\Delta$ has been used. Similar solutions also exist [10] for the coupled equations (16) and (17).

It is interesting to compare the exact solution of the Zakharov set with that of the Schrödinger–KDV (or, –Boussinesq) set obtained above. The former has two free parameters whereas the latter has only one free parameter. In the Zakharov case, the solutions for both the field variables are symmetric with respect to $\xi=0$. On the other hand, for the Schrödinger–KDV (or, –Boussinesq) set, the E-field solution is antisymmetric with respect to $\xi = 0$ whereas the ϕ-field is symmetric as earlier. Clearly, in both cases, the solutions for $E_a^2(\xi)$ have symmetric structures but with different shapes: For the Zakharov set, $E_a^2(\xi)$ is bell-shaped with only a single-hump whereas for the Schrödinger–KDV (or, –Boussinesq) set, it has a double-hump structure with the local minimum at the centre ($\xi = 0$) touching zero. Both the solutions are localized, that is, the fields as well as their derivatives tend to zero as $|\xi| \to 0$. In contrast to eq. (13), the solutions (21) and (22) yield the relation,

$$\frac{E_a^2}{4} = (-16\Delta)\phi + (-\tfrac{4}{3})\phi^2. \tag{23}$$

Thus, the relative scaling of E_a and ϕ are different for the two cases.

3.5 *Exact nonlinear equations*

The equations describing the low-frequency dynamics in the above models have been derived perturbatively using certain approximations. For the Zakharov set, it is a linear wave equation which is derived under the assumption of quasi-neutrality. On the other hand, both the Boussinesq as well as the KDV equations take into account the effects due to charge separation effects only perturbatively. Both are derived under the assumption of weak nonlinearity and, hence, are valid only for small amplitudes. For large amplitude waves, one needs to take into account full nonlinearity as well as charge separation effects by using the Poisson equation. By starting with the relevant fluid equations for the low-frequency dynamics (namely, the continuity and the momentum equations for the ions, and the Boltzmann distribution for the electrons, together with the full Poisson equation), we derived [13, 14] the following set of exact stationary nonlinear equations for the coupled Langmuir and ion-acoustic waves:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = -\frac{M}{\sqrt{M^2 - 2\phi}} + \exp\left(\phi - \frac{E_a^2}{4}\right), \tag{24}$$

$$3\frac{\mathrm{d}^2 E_a}{\mathrm{d}\xi^2} = (\lambda - 1)E_a + E_a \exp\left(\phi - \frac{E_a^2}{4}\right). \tag{25}$$

While the first equation is essentially the stationary form of the Poisson equation, the second equation arises from the Schrödinger-like equation (8). These equations incorporate exactly all the nonlinearities in the low-frequency dynamics.

It is easy to verify that eq. (24) contains, as limiting cases, the low-frequency equations used in the earlier models. For $|\phi|, E_a^2 \ll 1$, we expand the right-hand side of eq. (24) and keep only the most dominant nonlinear terms to obtain,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} \approx -\frac{(1 - M^2)}{M^2}\phi - \frac{(3 - M^4)}{2M^4}\phi^2 - \frac{E_a^2}{4}. \tag{26}$$

The Zakharov case is obtained by neglecting (because of the quasi-neutrality assumption) the left-hand side of eq. (26) and dropping the nonlinear term in ϕ. This yields

$$\frac{E_a^2}{4} = -\frac{(1 - M^2)}{M^2}\phi, \tag{27}$$

which is just eq. (13). On the other hand, the Boussinesq limit corresponds to taking the near-sonic limit $M^2 \rightarrow 1$ in eq. (26) which becomes

$$\frac{d^2\phi}{d\xi^2} = -(1 - M^2)\phi - \phi^2 - \frac{E_a^2}{4}, \tag{28}$$

which is same as eq. (17). The KDV limit is trivially obtained from eq. (28), as earlier, by writing $1 - M^2 \approx 2(1 - M)$ which is valid for $M \rightarrow 1$.

3.5.1 *A method of solution.* The coupled equations (24) and (25) are highly nonlinear and, as such, their exact analytical solutions have not been obtained so far. However, it is possible to find approximate analytical solutions by following a novel method [13, 14]. To this end, we first note that eqs (24) and (25) can be derived from a Lagrangian

$$L = \left\{\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 - \frac{3}{4}\left(\frac{dE_a}{d\xi}\right)^2\right\} - V(E_a, \phi), \tag{29}$$

where the 'potential' $V(E_a, \phi)$ is given by

$$V(E_a, \phi) = -\left[M(M^2 - 2\phi)^{1/2} + \frac{1}{4}(1 - \lambda)E_a^2 + \exp\left(\phi - \frac{E_a^2}{4}\right)\right]. \tag{30}$$

Note that the 'kinetic energy' in eq. (29) is not positive definite, but can change sign as ξ varies. The corresponding Hamiltonian is given by

$$H = \left\{\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 - \frac{3}{4}\left(\frac{dE_a}{d\xi}\right)^2\right\} + V(E_a, \phi). \tag{31}$$

Since the Lagrangian L is independent of ξ explicitly, the Hamiltonian (H) is an 'integral of motion', and has a value equal to $-(1 + M^2)$ for the localized solutions.

Using the Hamiltonian function H, it is possible to eliminate the independent variable ξ in the two equations (24) and (25) to yield the following equation for $\psi \equiv E_a^2/4$ in terms of ϕ alone:

$$\begin{aligned}
&12\psi[M(M^2 - 2\phi)^{1/2} + (1 - \lambda)\psi + \exp(\phi - \psi) - (1 + M^2)]\left(\frac{d^2\psi}{d\phi^2}\right) \\
&+ 9[M(M^2 - 2\phi)^{-1/2} - \exp(\phi - \psi)]\left(\frac{d\psi}{d\phi}\right)^3 - 6[M(M^2 - 2\phi)^{1/2} \\
&+ 2(1 - \lambda)\psi + (1 - \psi)\exp(\phi - \psi) - (1 + M^2)]\left(\frac{d\psi}{d\phi}\right)^2 \\
&- 6\psi[M(M^2 - 2\phi)^{-1/2} - \exp(\phi - \psi)]\left(\frac{d\psi}{d\phi}\right) \\
&+ 4\psi^2[(1 - \lambda) - \exp(\phi - \psi)] = 0.
\end{aligned} \tag{32}$$

Equation (32) does not *explicitly* contain the independent variable ξ, and may be considered as the 'trajectory equation' in the (ψ, ϕ) phase plane for the motion of a classical particle in a two-dimensional potential defined by $V(\psi, \phi)$. Any exact solution of the trajectory equation (32) together with the Hamiltonian (31) yields, in principle, a solution of the coupled equations (24) and (25). However, in the absence of any known exact solution of eq. (32), we look for solutions of the form

$$\psi = \sum_{n=0}^{\infty} a_n \phi^n, \tag{33}$$

where the coefficients a_n which are functions of the free parameters have to be determined suitably. In writing eq. (33), we are guided by the relations (13) and (23) which are obtained from the exact analytical solutions of the approximate equations. For convenience, we introduce the variable, $\theta \equiv \phi/M^2$ so that eq. (33) becomes

$$\psi = \sum_{n=0}^{\infty} b_n \theta^n, \tag{34}$$

where $b_n = M^{2n} a_n$.

Using eq. (34) in eq. (32), and by equating the coefficients of like powers of θ to zero, we are able to determine *uniquely* the coefficients in terms of first order algebraic equations for every n. For localized solutions the first coefficient b_0 is zero whereas the next four coefficients are explicitly given in ref. [14].

To obtain explicit solutions which are of interest for the present discussion, we consider a two-term approximation to the expansion in eq. (34) and write

$$\psi = b_1\theta + b_2\theta^2. \tag{35}$$

Using this expression in eq. (24) and expanding the right-hand side, we obtain, after retaining terms up to ϕ^3, the following equation for $\theta(\xi)$:

$$M^2 \frac{d^2\theta}{d\xi^2} = \alpha_1\theta + \alpha_2\theta^2 + \alpha_3\theta^3, \tag{36}$$

where the coefficients α_1, α_2 and α_3 are known functions of b_1 and b_2 [13, 14]. The localized solution of eq. (36) is given by

$$\theta(\xi) = \frac{\beta_1\beta_2 \operatorname{sech}^2(\mu\xi)}{\beta_1 - \beta_2 \tanh^2(\mu\xi)}, \tag{37}$$

where $\mu^2 = \lambda/3$, and β_1 and β_2 are known functions of α_1, α_2 and α_3. The solution for $E_a(\xi)$ is obtained by simply substituting the solution (37) into eq. (35), that is

$$E_a^2(\xi) = 4[b_1\theta(\xi) + b_2\theta^2(\xi)]. \tag{38}$$

This completes the approximate solution of the coupled eqs (24) and (25).

Equations (37) and (38) can be analysed [14] for the existence of localized solutions in the (M, Δ) parameter space. For a given $\Delta < 1$, the solutions $E^2(\xi)$ and $\phi(\xi)$ have, respectively, single-hump and single-dip structures for sufficiently small Mach numbers ($M \ll 1$). This corresponds to the case of the solutions obtained from the Zakharov equations. When the Mach number is increased, the solution for $E^2(\xi)$ flattens at the top till a critical Mach number M_{crit} is reached. For further increase in the Mach number, $E^2(\xi)$ develops a local dip around the center $\xi = 0$. The depth of the dip increases with increase in the Mach number, and $E^2(\xi = 0)$ becomes zero when M is equal to a cut-off Mach number M_{cut}. Beyond this value of the Mach number, $E^2(\xi)$ becomes negative around $\xi = 0$ which violates the boundary conditions, and hence the solutions are not valid for $M > M_{\text{cut}}$. The solutions on the line $M = M_{\text{cut}}$ correspond to the solutions obtained from the Schrödinger–Boussinesq (or, –KDV) equations. In fact, for $M = M_{\text{cut}}$ and for sufficiently small values of Δ, the explicit solutions (37) and (38) can be shown to exactly reduce to the solutions (21) and (22) [anti-symmetric $E(\xi)$ and symmetric $\phi(\xi)$] obtained from the Schrödinger–Boussinesq (or, –KDV) case. For all values of M, the solution for $\phi(\xi)$ has always a single-dip, symmetric structure whose amplitude increases with the increase in the Mach number reaching the maximum value at $M = M_{\text{cut}}$.

Thus, the approximate solutions (37) and (38) of the exact governing equations (24) and (25) exactly reduce, for near-sonic propagations, to the exact solutions (21) and (22) of the approximate governing equations (19) and (20). It should be noted that there is no reason why such a complete equivalence between the two should exist at all since the approximations involved in the solutions as well as the equations are entirely of different nature. On the other hand, the very existence of such an exact reduction lends support to the suitability of the above method of solution used in solving coupled equations.

The coupled Schrödinger–Boussinesq (or, –KDV) system of equations occurs in many different problems in plasma physics where a high-frequency wave is coupled to a suitable low-frequency wave via the ponderomotive force. It has been shown that the coupled electromagnetic and ion-acoustic waves [15, 16] as well as the upper-hybrid and the magnetoacoustic waves [17] are indeed governed by a Schrödinger–Boussinesq (or, –KDV) system but with different sets of free parameters. In fact, the latter system with arbitrary free parameters for each of the terms in the two equations constitutes a general set which for stationary solutions yields a generic system of two coupled ordinary differential equations admitting different classes of exact analytical solutions. This is discussed in the next section.

4. Generic system of equations

Consider a Schrödinger-like equation with arbitrary free parameters in the form

$$i\left(\frac{\partial E}{\partial t} + \lambda_1 \frac{\partial E}{\partial x}\right) + \lambda_2 \frac{\partial^2 E}{\partial x^2} = \lambda_3 E + \lambda_4 \phi E, \tag{39}$$

which is coupled to the Boussinesq equation

$$\left(\frac{\partial^2}{\partial t^2} + \mu_1 \frac{\partial^2}{\partial x^2}\right)\phi + \mu_2 \frac{\partial^4 \phi}{\partial x^4} + \mu_3 \frac{\partial^2}{\partial x^2}(\phi^2) = \mu_4 \frac{\partial^2}{\partial x^2}(EE^*), \tag{40}$$

or, to the KDV equation

$$\left(\frac{\partial}{\partial t} + \mu_1 \frac{\partial}{\partial x}\right)\phi + \mu_2 \frac{\partial^3 \phi}{\partial x^3} + \mu_3 \phi \frac{\partial \phi}{\partial x} = \mu_4 \frac{\partial}{\partial x}(EE^*), \tag{41}$$

where λ_i and μ_i, $i = 1, 2, 3, 4$ are arbitrary free parameters which can take different values depending on the problem at hand. In a stationary frame $\xi = x - Mt$, the above equations yield the following set of generic equations [18]

$$\beta \frac{d^2 E}{d\xi^2} = b_1 E + b_2 \phi E, \tag{42}$$

$$\lambda \frac{d^2 \phi}{d\xi^2} = d_1 \phi + d_2 \phi^2 + d_3 E^2, \tag{43}$$

where $\lambda, \beta, b_1, b_2, d_1, d_2$ and d_3 are the free parameters defined in terms of λ_i and μ_i, $i = 1,2,3,4$.

Equations (42) and (43) constitute a generic set of equations having seven free parameters. However, by a proper rescaling of the variables, it is possible to reduce the number of free parameters. This will be considered in a later section [cf. eqs (65) and (66)]. It is of interest to find exact analytical solutions of the generic equations valid in as much of the region in the parameter space as possible. This is done by following the method of solution described in § 3.5.1. As earlier, we try a series solution of the form given by eq. (33) which, in general, does not truncate. However, by properly choosing certain curves or surfaces in the parameter space, it is possible to make the coefficients b_n become zero for n greater than a certain value, say, m. The resulting polynomial relationship between E and ϕ thus becomes an *exact solution* of the corresponding 'trajectory equation'. This relation together with the corresponding Hamiltonian function for the generic system yield different classes of the exact analytical solutions of the eqs (42) and (43). Omitting the details which can be found in ref. [18], we summarize below the various classes of explicit solutions.

4.1 *Exact analytical solutions*

The following classes of exact analytical solutions of the generic system of equations (42) and (43) have been obtained [18] so far:

(A) For $\beta d_1 b_2 - 2b_1(3\beta d_2 - \lambda b_2) = 0$:

$$E(\xi) = \pm \frac{6b_1}{b_2}\left(\frac{\beta d_2 - \lambda b_2}{\beta d_3}\right)^{1/2} \operatorname{sech}(\mu\xi)\tanh(\mu\xi), \tag{44}$$

$$\phi(\xi) = -\frac{6b_1}{b_2}\operatorname{sech}^2(\mu\xi); \quad \mu = \left(\frac{b_1}{\beta}\right)^{1/2}. \tag{45}$$

(B) For $3\lambda b_2 - \beta d_2 = 0$:

$$E(\xi) = \pm \left[\frac{6\lambda b_1}{\beta^2 d_2 d_3}(\beta d_1 - 4\lambda b_1)\right]^{1/2} \operatorname{sech}(\mu\xi), \tag{46}$$

$$\phi(\xi) = -\frac{2b_1}{b_2}\operatorname{sech}^2(\mu\xi); \quad \mu = \left(\frac{b_1}{\beta}\right)^{1/2}. \tag{47}$$

(C) For $\beta d_1 + 2\lambda b_1 = 0$ and $d_2 = 0$: [In this case, eq. (43) is linear in ϕ.]

$$E(\xi) = \pm\left(\frac{18 b_1 d_1}{d_3 b_2}\right)^{1/2} \operatorname{sech}(\mu\xi)\tanh(\mu\xi), \tag{48}$$

$$\phi(\xi) = \frac{3\beta d_1}{\lambda b_2}\operatorname{sech}^2(\mu\xi); \quad \mu = \left(\frac{b_1}{\beta}\right)^{1/2}. \tag{49}$$

(D) For $\lambda = 0$: [In this case, eq. (43) is singular and yields just an algebraic relation between E and ϕ.]

For this value of λ, there are two cases:

(i) For $b_2d_1 - 6b_1d_2 = 0$,

$$E(\xi) = \pm\frac{d_1}{(d_2d_3)^{1/2}}\operatorname{sech}(\mu\xi)\tanh(\mu\xi), \tag{50}$$

$$\phi(\xi) = -\frac{d_1}{d_2}\operatorname{sech}^2\left[\left(\frac{b_1}{\beta}\right)^{1/2}\xi\right]; \quad \mu = \left(\frac{d_1b_2 - 2d_2b_1}{4\beta d_2}\right)^{1/2}. \tag{51}$$

(ii) For $5b_2d_1 - 6b_1d_2 = 0$,

$$E(\xi) = \pm\frac{d_1}{(d_2d_3)^{1/2}}\operatorname{sech}(\mu\xi)\tanh(\mu\xi), \tag{52}$$

$$\phi(\xi) = -\frac{d_1}{d_2}\tanh^2\left\{\left(\frac{-b_1}{5\beta}\right)^{1/2}\xi\right\}; \quad \mu = \left(\frac{d_1b_2 - 2d_2b_1}{4\beta d_2}\right)^{1/2}. \tag{53}$$

(E) Periodic solutions: In addition to the above solutions, a new class of periodic solutions were recently reported in ref. [3]. These were found by trying a solution of the form $\phi = C_0 + C_1E$ for the 'complementary trajectory equation' for ϕ in terms of E. Two sets of parameter values were found to be admissible:

(i) For $C_0 = 0$, one requires, $\lambda b_1 - \beta d_1 = 0$;
(ii) For $d_1 + d_2C_0 = 0$ one requires $\lambda b_1 + C_0(\lambda b_2 - \beta d_2) = 0$.

In both cases, C_1 is given by

$$C_1^2 = \frac{\beta d_3}{\lambda b_2 - \beta d_2}. \tag{54}$$

The corresponding explicit solutions are given by

$$E(\xi) = h_1 sn^2(\mu\xi, k) + h_2 cn^2(\mu\xi, k), \tag{55}$$

$$\phi(\xi) = C_0 + C_1E(\xi), \tag{56}$$

where

$$\mu^2 = \frac{b_2C_1}{6\beta}(h_3 - h_2), \quad k^2 = \frac{h_1 - h_2}{h_3 - h_2} \tag{57}$$

and h_1, h_2 and h_3 are the three real roots of the cubic equation

$$h^3 + \frac{3(b_1 + b_2C_0)}{2b_2C_1}h^2 + \frac{3C}{2b_2C_1} = 0. \tag{58}$$

In eq. (55), sn and cn are the Jacobian elliptic functions, and, in eq. (58), C is a constant of integration. The solutions are, in general, periodic but for $C = 0$ are localized and given by

$$E(\xi) = -\frac{3(b_1 + b_2C_0)}{2b_2C_1}\operatorname{sech}^2(\mu\xi), \tag{59}$$

$$\phi(\xi) = C_0 + C_1E(\xi); \quad \mu^2 = \frac{b_1 + b_2C_0}{4\beta}. \tag{60}$$

Note that the solution for $\phi(\xi)$ is truly localized only when $C_0 = 0$.

5. Relation to other systems

The generic equations (42) and (43) have relevance to some other equations that are commonly used in other branches of physics. We consider below some particular examples.

5.1 *Hénon–Heiles system*

The Hénon–Heiles system has been extensively studied in the field of nonlinear dynamics since it was first proposed by Hénon and Heiles [4]. The generalized form of these equations with arbitrary coefficients can be obtained from the Hamiltonian [19]

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(Ax^2 + By^2) + (Dx^2y - \tfrac{1}{3}Cy^3), \tag{61}$$

where A, B, C and D are arbitrary real parameters, x and y are the spatial coordinates, and p_x and p_y are the corresponding conjugate momenta. Clearly, H is simply the Hamiltonian for a two-dimensional harmonic oscillator with certain specific nonlinear terms given by the last two terms in eq. (61). The standard form first investigated by Hénon and Heiles [4] corresponds to the case when $A = B = C = D = 1$. The fundamental question of general interest in such Hamiltonian systems is to identify the parameters for which the system is 'completely integrable'. Since the system is of two-degrees of freedom and the Hamiltonian is an 'integral of motion', the problem is equivalent to finding another integral of motion that is in involution with the Hamiltonian.

By proper normalization of the variables, the Hamiltonian (61) can be re-written in the canonical form, namely

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(-p_1x^2 - d_1y^2) + (x^2y + \tfrac{1}{3}d_2y^3), \tag{62}$$

where p_1 takes values $+1$ or -1, and d_1 and d_2 are the two free parameters. The corresponding equations of motion are given by

$$\frac{d^2x}{dt^2} = p_1x - 2xy, \tag{63}$$

$$\frac{d^2y}{dt^2} = d_1y - d_2y^2 - x^2. \tag{64}$$

These equations are known to be integrable [20] for the following three sets of parameter values:

(A) $d_1 = p_1, d_2 = +1$,
(B) any d_1, any p_1, $d_2 = +6$,
(C) $d_1 = 16p_1, d_2 = +16$.

Note that in all the cases d_2 is always positive which indicates that for the known integrable cases, the nonlinear terms in the equations of motion, namely, eqs (63) and (64) have the same sign.

Consider now the generic equations (42) and (43) whose variables can be suitably normalized to yield the canonical set

$$\frac{d^2E}{d\xi^2} = p_1E - 2\phi E, \tag{65}$$

$$\frac{d^2\phi}{d\xi^2} = d_1\phi - d_2\phi^2 - p_2E^2, \tag{66}$$

where p_2 can take values $+1$ or -1. The Hamiltonian for the eqs (65) and (66) is given by

$$H = \frac{1}{2}\left[p_2\left(\frac{dE}{d\xi}\right)^2 + \left(\frac{d\phi}{d\xi}\right)^2\right] + \frac{1}{2}(-p_1p_2E^2 - d_1\phi^2) + \left(p_2E^2\phi + \frac{1}{3}d_2\phi^3\right). \tag{67}$$

Comparing the two Hamiltonians given by eqs (62) and (67) [or, the equations of motion, (63), (64) and (65), (66)] we note that they are structurally very similar. In fact, for the case when $p_2 = +1$, they are exactly same if we identify the set (E, ϕ) with (x, y). However, as pointed out earlier, for all modulational problems, one finds $p_2 = -1$ and hence the kinetic energy in (67) is not positive definite. Thus, the stationary equations governing the nonlinear development of the modulational instability of an high-frequency wave coupled to a suitable low-frequency wave in plasmas are complementary to the Hénon–Heiles equations, but seem to have fundamentally different qualitative features. For example, in the case of the usual Hénon–Heiles system (with positive definite kinetic energy), any minimum in the potential guarantees local oscillatory motion of the particle. This need not be true of the generic equations since the 'kinetic energy', term can change its sign during the motion of the 'particle'. The exact solutions obtained in § 4.1 do not, however, guarantee the 'complete integrability' of the equations since they satisfy specialized boundary conditions. On the other hand, even such specialized solutions valid in the entire allowed regions of the parameter space have also not been obtained so far.

The integrable parameter regimes for the generic Hamiltonian (H) given by eq. (67) for $p_2 = +1$ or -1 has recently been discussed elsewhere [21]. In particular, the generic Hamiltonian is integrable for three sets of parameter values and it is possible to explicitly obtain, in each case, the corresponding second integral of motion. Results show that the integrable parameter regimes in the two cases corresponding to $p_2 = +1$ and -1 are complementary to each other. Furthermore, there exists a direct one-to-one correspondence between the known integrable cases of H and the stationary Hamiltonian flows associated with the only integrable nonlinear evolution equations (of polynomial and autonomous type) with a scale-weight of seven. The latter result strongly suggests that there are possibly no other integrable cases of H for both $p_2 = +1$ as well as $p_2 = -1$. An extended discussion on some of these questions together with the relevant details on the known results can be found in ref. [21].

5.2 *Self-dual Yang–Mills system*

In the last couple of years, there has been much interest in the classical Yang–Mills field equations, particularly in their reduction to simpler nonlinear equations [5] by using certain symmetry properties. The classical Yang–Mills equations are written in the form

$$D_\mu G_{\mu\nu} \equiv \partial_\mu G_{\mu\nu} + [A_\mu, G_{\mu\nu}] = 0, \tag{68}$$

where

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{69}$$

where $[A_\mu, A_\nu]$ is a suitable Lie bracket defined over the Yang–Mills field variables (A_μ), and the subscripts for ∂ denote the partial derivatives. The Yang–Mills fields are said to be self-dual if the condition

$$\tilde{G}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma} = +G_{\mu\nu}, \tag{70}$$

is satisfied where $\epsilon_{\mu\nu\rho\sigma}$ is the usual anti-symmetric tensor. Solutions satisfying the above condition satisfy also the Yang–Mills field equations. Using the gauge degree of freedom implied by the self-dual condition, it has been recently shown [5] that the Yang–Mills field equations can be reduced either to the nonlinear Schrödinger or to the KDV equation. Since nonlinear Schrödinger equation can generally be thought of as a special case (namely, the static limit) of the Zakharov or the Schrödinger–Boussinesq (or, -KDV) system, one would expect to reduce the self-dual Yang–Mills system to the Schrödinger–Boussinesq (or, -KDV) system. Since the latter system is known to yield, for stationary solutions, the generalized Hénon–Heiles system, this would establish a cascading connection from the Yang–Mills to the Hénon–Heiles system via the Schrödinger–Boussinesq (or, -KDV) system. It would also provide a good model to study the 'nonlinear dynamical' behaviour of classical Yang–Mills fields. While such a possibility is quite exciting, even a formulation of the problem is yet to be carried out.

5.3 *Complex KDV equation*

The peculiar nature of the Hamiltonian (67) for the generic system of equations (65) and (66) admitting the case when the 'kinetic energy' term is indefinite is exhibited also by the usual KDV equation when the dependent variable is made complex [6, 7]. Consider the KDV equation

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{71}$$

where α and β are the free parameters, and all the variables are real quantities. For the stationary solutions depending on a single variable $\xi \equiv x - Mt$ with one free parameter M, eq. (71) yields

$$\beta\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = Mu - \tfrac{1}{2}\alpha u^2. \tag{72}$$

The Hamiltonian for the above equation is

$$H = \frac{1}{2}\beta\left(\frac{du}{d\xi}\right)^2 + V(u), \tag{73}$$

where the potential $V(u)$ is given by

$$V(u) = -\tfrac{1}{2}Mu^2 + \tfrac{1}{6}\alpha u^3. \tag{74}$$

We now 'complexify' the KDV equation by making the dependent variable u complex and write, $u = u_1 + iu_2$. Equation (72) then yields the set of equations

$$\beta\frac{d^2u_1}{d\xi^2} = Mu_1 - \tfrac{1}{2}\alpha u_1^2 + \tfrac{1}{2}\alpha u_2^2, \tag{75}$$

$$\beta\frac{d^2u_2}{d\xi^2} = Mu_2 - \alpha u_1u_2. \tag{76}$$

Accordingly, the potential becomes, $V(u) \to V(u_1, u_2) = V_1(u_1, u_2) + iV_2(u_1, u_2)$ where

$$V_1 = -\tfrac{1}{2}M(u_1^2 - u_2^2) + \tfrac{1}{6}\alpha(u_1^3 - 3u_1u_2^2), \tag{77}$$

$$V_2 = -Mu_1u_2 - \tfrac{1}{6}\alpha(u_2^3 - 3u_1^2u_2), \tag{78}$$

and the Hamiltonian $H \to H_1 + iH_2$ where

$$H_1 = \frac{1}{2}\beta\left[\left(\frac{du_1}{d\xi}\right)^2 - \left(\frac{du_2}{d\xi}\right)^2\right] + V_1(u_1, u_2), \tag{79}$$

$$H_2 = \beta\left(\frac{du_1}{d\xi}\right)\left(\frac{du_2}{d\xi}\right) + V_2(u_1, u_2). \tag{80}$$

By identifying u_1 with ϕ and u_2 with E, we note that the eqs (75) and (76) are structurally very similar to the generic equations (42) and (43).

Equations (75) and (76) can also be written in the form

$$\beta\frac{d^2u_1}{d\xi^2} = -\frac{\partial V_1}{\partial u_1}, \quad \beta\frac{d^2u_2}{d\xi^2} = +\frac{\partial V_1}{\partial u_2}, \tag{81}$$

which involve only the potential $V_1(u_1, u_2)$ given by eq. (77). Note that unlike the equations of motion for a classical particle with two degrees of freedom, the second equation has a positive sign for the derivative on the right-hand side. This is a reflection of the fact that the kinetic energy is indefinite.

Using the Cauchy–Riemann conditions, namely

$$\frac{\partial V_1}{\partial u_1} = \frac{\partial V_2}{\partial u_2}, \quad \frac{\partial V_1}{\partial u_2} = -\frac{\partial V_2}{\partial u_1}, \tag{82}$$

equations (81) can also be written in terms of the potential $V_2(u_1, u_2)$ given by (78):

$$\beta\frac{d^2u_1}{d\xi^2} = -\frac{\partial V_2}{\partial u_2}, \quad \beta\frac{d^2u_2}{d\xi^2} = -\frac{\partial V_2}{\partial u_1}. \tag{83}$$

While both the equations have now negative signs for the derivatives on the right-hand sides (like in the case of the equations of motion in classical dynamics), the equation for u_1 has derivative of V_2 with respect to u_2, and vice versa for u_2.

5.4 *'Complexified' classical dynamics*

The above features of the 'complexified' KDV equation are also exhibited by equations obtained from classical dynamical systems with one-degree of freedom by making the dependent variable complex [6]. Consider a one-degree of freedom conservative system defined by the Lagrangian

$$L(q,\dot{q}) = \tfrac{1}{2}\dot{q}^2 - V(q), \tag{84}$$

where dot denotes the time derivative, and the corresponding Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + V(q), \tag{85}$$

where the conjugate momentum (p) is defined by $p = \dot{q}$. Clearly, the equation of motion is

$$\frac{d^2q}{dt^2} = -\frac{\partial V}{\partial q}. \tag{86}$$

Let us now make the dependent variable complex, and write, $q = q_1 + iq_2$. Accordingly, the conjugate momentum becomes complex, that is $p \to p_1 + ip_2$. Then, (q_1, p_1) and (q_2, p_2) constitute canonically conjugate variables. Under complexification, the potential $V(q)$ becomes complex, that is, $V(q) \to V(q_1, q_2) = V_1(q_1, q_2) + iV_2(q_1, q_2)$ where V_1 and V_2 satisfy the Cauchy–Riemann condition

$$\frac{\partial V_1}{\partial q_1} = \frac{\partial V_2}{\partial q_2}, \quad \frac{\partial V_1}{\partial q_2} = -\frac{\partial V_2}{\partial q_1}. \tag{87}$$

Similarly, the Lagrangian and the Hamiltonian yield, respectively,

$$L(q) \to L_1 + iL_2 \equiv [\tfrac{1}{2}(\dot{q}_1^2 - \dot{q}_2^2) - V_1(q_1, q_2)] + i[\dot{q}_1\dot{q}_2 - V_2(q_1, q_2)], \tag{88}$$

$$H(q,p) \to H_1 + iH_2 \equiv [\tfrac{1}{2}(p_1^2 - p_2^2) + V_1(q_1, q_2)] + i[p_1p_2 + V_2(q_1, q_2)]. \tag{89}$$

Note that the kinetic energy term in L_1 and H_1 is not positive definite.

The Newton's equations of motion can be obtained from the usual Euler–Lagrange equation using the Lagrangian L_1. This yields,

$$\frac{d^2q_1}{dt^2} = -\frac{\partial V_1}{\partial q_1}, \quad \frac{d^2q_2}{dt^2} = +\frac{\partial V_1}{\partial q_2}. \tag{90}$$

Note that here only the potential V_1 is involved and the second equation has a positive sign before the derivative on the right-hand side. On the other hand, one can use the Lagrangian L_2 to obtain the equations in terms of V_2 only:

$$\frac{d^2q_1}{dt^2} = -\frac{\partial V_2}{\partial q_2}, \quad \frac{d^2q_2}{dt^2} = -\frac{\partial V_2}{\partial q_1}. \tag{91}$$

Here, both the equations have negative signs before the derivatives on the right-hand sides. However, the equation for q_1 is defined in terms of the derivative of V_2 with respect to q_2, and vice versa for q_2. Similar features were also noticed in the previous section for the complex KDV equation. As expected, the two sets of eqs (90) and (91) are identical in view of the Cauchy–Riemann conditions (87).

The above peculiarities are also reflected in the Hamilton's equations of motion. Using H_1 in the usual Hamilton's canonical equations of motion, we obtain

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -\frac{\partial V_1}{\partial q_1}, \tag{92}$$

$$\dot{q}_2 = -p_2, \quad \dot{p}_2 = -\frac{\partial V_1}{\partial q_2}. \tag{93}$$

Note that if we eliminate p_1 from the first set and p_2 from the second set of equations, we recover the equations of motion given by (90). On the other hand, if we use, instead, the Hamiltonian H_2, we obtain the equations

$$\dot{q}_1 = p_2, \quad \dot{p}_1 = -\frac{\partial V_2}{\partial q_1}, \tag{94}$$

$$\dot{q}_2 = p_1, \quad \dot{p}_2 = -\frac{\partial V_2}{\partial q_2}. \tag{95}$$

Unlike the previous case, here cross mixing of the equations is necessary in order to eliminate p_1 and p_2 which leads to the equations of motion given by (91).

6. Summary

To summarize, we have discussed the properties as well as some exact analytical solutions of a pair of coupled nonlinear ordinary differential equations that occur in different branches of physics. In particular, in the field of plasma physics, such equations govern the stationary solutions of time-dependent coupled nonlinear equations [like the Schrödinger–Boussinesq (or, -KDV) system] that describe the nonlinear evolution of the amplitude modulated high-frequency waves coupled to suitable low-frequency waves. The latter system with arbitrary free parameters leads to a generic system of coupled scalar field equations whose exact analytical solutions valid in different regions of the parameter space have been explicitly obtained.

The generic system is shown to contain as special cases equations that have been studied in other fields. For example, the case when the kinetic energy in the corresponding Hamiltonian is positive definite is shown to exactly correspond to the well-known 'Hénon–Heiles' system (in the field nonlinear dynamics) in a generalized form. On the other hand, the equations obtained from the stationary KDV equation by making the dependent variable complex are shown to be structurally similar to the generic system. We have also suggested the possibility of relating the classical Yang–Mills equations to the Hénon–Heiles system by reducing them to the Schrödinger–Boussinesq (or, -KDV) system using the self-duality condition. The Hamiltonian of the generic system for the nonlinear wave modulations admits kinetic energy term which is not positive definite. It is shown that this feature is exhibited also by the equations obtained from the classical equation of motion for a particle with one-degree of freedom in a conservative potential field by making the dependent variable complex.

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