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Causal quantum mechanics

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Abstract. We discuss the split between system and measuring apparatus, i.e. non-unitary nature, of the conventional quantum mechanics to motivate a causal unitary description of nature. We then describe causal quantum mechanics of de Broglie and Bohm. We conclude by presenting a version of recently proposed causal quantum mechanics which treats position and momentum variable symmetrically.

Keywords. Causal quantum mechanics; de Broglie–Bohm theory.

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1. Is the world classical and/or quantum?

1.1 *Classical world*

The description of the physical universe given by classical physics was internally consistent and complete. It was a realistic conception of the world. The classical physics describes the world as it is. The physical world was made up of two kinds of entities viz. particles and waves. They obeyed well defined dynamical laws of change. It was thus a deterministic world. The only role of probability in classical physics, e.g. in statistical mechanics, was due to our lack either of knowledge or of interest in the initial conditions. It was not basic. Another important feature of classical physics was that it satisfied the principle of 'Einstein-locality' i.e. the factual situation of a system S_1 does not change due to whatever measurements may be made on a system S_2, which is spatially separated from S_1.

There was no split in classical physics between system which is being studied and the apparatus with which we study it. Both were described by the classical physics. The interaction which coupled the system and the apparatus was a normal physical interaction. By using the gentle, and if needed even gentler, probe it was possible to learn about the properties of the system under study to any desired degree of accuracy. What one learnt this way were the properties of the system as they were. There were thus no epistemological problems associated with a measurement.

The description of the world in classical physics is thus unitary without a split between System-Apparatus. Even though this is a very satisfying feature of classical physics description of nature, the classical physics failed in providing a useful description of atomic phenomenon and quantum mechanics had to be introduced by founding fathers viz. Planck, Einstein, Bohr, de-Broglie, Heisenberg and Schrödinger in the first quarter of this century.

Do we continue to have a unitary picture of the physical universe in quantum physics as was the case in classical physics? The answer is no, in the standard way quantum mechanics is formulated. By standard way we refer here to Copenhagen interpretation of quantum mechanics, and since this is not monolithic, we also include its variants. We now proceed to elaborate the negative answer using Bohr's and von-Newmann's views about measurement in quantum mechanics.

1.2 *Bohr*

According to Bohr [1] the system of interest, which we are studying and which is generally microscopic, must be described as a quantum object using laws of quantum physics. However the measuring apparatus, which is generally macroscopic, must be described as a classical object obeying laws of classical physics as the language of classical physics is the only language in which we can convey the results of our experiments. When we study a system S with experimental arrangement E what we observe is not just a property of S as it was but various outcomes depend on the total setup S and E. Quantum physics provides us with a calculus of probabilities for various outcomes of the joint setup S and E. Unlike classical physics there is no way in which E dependence can be wished away.

We must emphasis that in Bohr's view the classical physics has logical priority in as much as we cannot do physics without it. This logical priority is different from chronological priority, which of course is there, of classical physics over quantum physics. Thus physical universe is classical and quantum.

These views of Bohr raise a number of problems. How is one to understand the working of an apparatus? Should it follow from classical physics or should we use quantum physics when we want to do that. In that case apparatus is a quantum system when we want to understand it but is a classical system when we want to use it as a measuring device. What is the interaction between a quantum system and its classical measuring system? Well, there is no articulated theory of that. Then there is the question of how much of a given 'system + apparatus' is to be put on the quantum side and how much on the classical side. In practice this may not cause much problem as we believe that if we will put enough on the quantum side we will achieve the required degree of accuracy. This is supposed to be ensured by the folkloric 'correspondence principle' according to which quantum mechanics should, in appropriate limits, go into classical mechanics. But in principle the split remains.

1.3 *von Neumann*

In contrast to Bohr it was decided by von Neumann [2] to treat both the system S and measuring apparatus A quantum mechanically. Let the eigenfunctions of the system S corresponding to an observable Ω, be denoted by ψ_m, i.e. $\Omega\psi_m = \omega_m\psi_m$, with an eigenvalue ω_m. Further let the pointer readings a be the eigenvalues of the 'pointer position operator' M, for the system A with the eigenfunctions $f(a)$ i.e. $Mf(a) = af(a)$.

Consider first system S to be in an eigenstate ψ_n of Ω. According to von Neumann the 'measurement interaction' causes the system-apparatus state $\psi_n f(a)$ to become $\psi_n f(a_n)$.

From the pointer reading a_n we conclude thou the system was in the state ψ_n of the observable Ω with the eigenvalue ω_n. We now go over to case where the system S is not in a pure state of Ω but is in a linear superposition given by $\psi = \sum_n C_n\psi_n$. By the linearity of quantum evolution equation, the measurement interaction would cause the system-apparatus state $(\sum_n C_n\psi_n)f(a)$ to develop into $\sum_n C_n\psi_n f(a_n)$. From this we can, of course, conclude that the probability of measurement resulting in the pointer reading a_n, implying that the system was in the eigenstate ψ_m with eigenvalue ω_m is given by $| C_m |^2$. But note that this wave function still does not correspond to a definite pointer reading for the apparatus. The apparatus is still in a linear combination of states with different, even macroscopically different, pointer readings i.e. in a grotesque state.

von Neumann postulates at this stage the infamous 'collapse of the wave-function', according to which, when the measurement is completed and the pointer reading is a_n (say), the wave function $\sum C_n\psi_n f(a_n)$ collapses to $\psi_n f(a_n)$. He calls this a process of the first kind. These are acausal, discontinuous and takes place only when the measurement is completed and observation registered. In an elaboration by London and Bauer [3], and subscribed to by Wigner [4], the measurement is completed only at the level of human consciousness of the result of the measurement. In contrast to these processes of first kind there is the usual process of second kind describing the evolution of the system between measurements. These are causal, continuous and described by Schrödinger equation.

Thus irrespective of whether we follow Bohr, where quantum physics has to be supplemented by 'classical apparatus' or von Neumann, where we have to postulate a causal processes of the first kind (wave function collapse), we find that either way quantum physics is inherently incomplete. It is inexactly formulated. Quantum physics does not provide a unitary description of the nature.

1.4 *Quantum mechanics for a closed system*

This inexactness in the fundamentals of the quantum mechanics for a theory, as basic as this theory, is intolerable and there have been various attempts to formulate a quantum mechanics for a closed system. Another motivation has been the inability of any formulation of quantum mechanics, with an artificial split between system and apparatus, to deal with the quantum mechanics of the whole universe i.e. quantum cosmology. Wheeler's phrase "Include the observer in the wave function" expresses the same desire. Some of these attempts are

(i) Causal interpretation of de Broglie [5] and Bohm [6]. This was also favoured by Bell [7] and others [8].
(ii) Many world interpretation suggested by Everetts, initially championed by Wheeler, and followed later by de Witt and others [9].
(iii) 'Quantum mechanics of the history' approach and related concept of decoherence (Griffith, Gell-Mann and Hartle, Omnes, Zurck etc.) [10].

In the next § 2, we shall describe de Broglie–Bohm causal interpretation of the quantum mechanics restricting ourselves to the nonrelativistic quantum mechanics. We also point out some aspects of this theory which make us look for a version of it treating position and momentum symmetrically. This is described in § 3 mainly in the context of nonrelativistic quantum mechanics of one nonrelativistic particle moving in one space dimension but with some remarks on situations without these restrictions.

2. de Broglie–Bohm causal interpretation of quantum mechanics

2.1 *A bit of history*

Louis de Broglie had proposed a 'pilot wave theory' interpretation of quantum mechanics in 1927 which provided a realistic interpretation of quantum mechanics [5] unlike the pragmatic one associated with Copenhagen School. Under the criticism of Pauli and others, and lack of endorsement even from those physicists, such as Einstein, who did not like Copenhagen interpretation, the interpretation did not gain currency. Even de Broglie gave it up. An important role was also played in this by so called 'von Neumann' theorem [2] which was taken to mean by physics community that a completion of quantum mechanics with extra (hidden?) variables was not possible if it had to reproduce the observable predictions of quantum mechanics.

Bohm, in 1952, independently came up with a similar proposal [6]. He was able to take care of various objections which had been earlier raised by Pauli, de Broglie and others. Besides he supplemented it with his ideas on theory of measurement.

It is a historical accident that de Broglie–Bohm theory was almost totally ignored until quite recently [11]. Bell's work showing irrelevance of 'von-Neumann theorem' to hidden variable program, and his advocacy of this theory, as an explicit counter example to Copenhagen orthodoxy's claims on nonexistence of a causal description for quantum physics, has had a lot to do with the revival of interest in it.

2.2 *Mathematical formulation of de Broglie–Bohm theory*

Consider first a single particle, with mass m, moving in the potential $V(\mathbf{r})$. The time evolution of this system is described by the Schrödinger equation for the wave function $\psi = \psi(\mathbf{r}, t)$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi. \tag{2.2.1}$$

Let us use the polar decomposition

$$\psi = R\,\mathrm{e}^{iS/\hbar} \tag{2.2.2}$$

with R and S real. Substituting (2.2.2) in eq. (2.2.1) and separating real and imaginary parts of the equation we obtain [12]

$$\frac{\partial R}{\partial t} + \frac{1}{2m}[R\nabla^2 S + 2\nabla R\cdot\nabla S] = 0 \tag{2.2.3}$$

and

$$\frac{\partial S}{\partial t} + \left[\frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right] = 0. \tag{2.2.4}$$

Defining

$$P_{(\mathbf{r},t)} = |\psi|^2 = [R(\mathbf{r},t)]^2, \tag{2.2.5}$$

$$m\mathbf{v}(\mathbf{r},t) = \nabla S(\mathbf{r},t) \tag{2.2.6}$$

and

$$U(\mathbf{r},t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{2.2.7}$$

we obtain from (2.2.3) and (2.2.4) the more suggestive equations (2.2.8) and (2.2.9) given below. We have

$$\frac{\partial P}{\partial t} + \nabla(P\mathbf{v}) = 0 \tag{2.2.8}$$

which is of the form of a continuity equation for position probability density P with an associated current density $P\mathbf{v}$. We also obtain

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + U = 0, \tag{2.2.9}$$

which is of the form of the classical Hamilton–Jacobi equation for a particle of mass m moving in a potential $V + U$ i.e. sum of the given external potential V and a quantum potential U.

We have so far only manipulated the Schrödinger equation mathematically into the suggestive forms (2.2.8) and (2.2.9). We now come to de Broglie–Bohm causal version of the quantum mechanics.

We associate with this system a wave function $\psi(\mathbf{r},t)$ obeying Schrödinger equation and a trajectory function $\mathbf{q}(t)$ of the particle. The trajectory function $\mathbf{q}(t)$ is the additional hidden variable introduced so as to be able to give a causal description. The particle trajectories are given by the dynamical equation

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{v}(\mathbf{q},t) = \frac{1}{m}\nabla S(\mathbf{r},t)\,|_{\mathbf{r}=\mathbf{q}}\,. \tag{2.2.10}$$

The particle moves in a potential $V + U$. Note that this description is nonlocal since through U the particle experiences a potential which depends on the whole set up as the wave function $\psi(\mathbf{r},t)$ is determined by the conditions over the whole space through Schrödinger equation. Further even if wave function is small in some region, i.e. the probability $P = R^2$ of its being there is small, its influence in that region need not be small as the quantum potential

$$U = -\frac{\hbar^2}{2m}\cdot\frac{\nabla^2 R}{R}$$

may not be small there.

In this theory the motion of the quantum particle is analogous to the motion of boat which is set on an automatic pilot guided by radio wave signals. What matters for this purpose is the signal, i.e. the shape of the wave, and not its intensity. The boat moves under its own energy. The wave function ψ plays the same role for directing the motion of quantum particle which radio waves played for the boat. This is the 'pilot wave' picture for the de Broglie–Bohm theory.

The N-particle generalization is straightforward. One now supplements the N-particle wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N s t)$ with N trajectory functions $\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t)$ obeying the dynamical equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = \nabla_i S \,|_{\mathbf{r}_i = \mathbf{q}_i}, \tag{2.2.11}$$

where

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = \mathrm{R}\mathrm{e}^{iS/\hbar} \tag{2.2.12}$$

and the quantum potential is given by

$$U = -\frac{\hbar^2}{2m} (\nabla_1^2 R + \nabla_2^2 R + \cdots + \nabla_N^2 R)/R \,|_{\mathbf{r}_i = \mathbf{q}_i(t)} \,. \tag{2.2.13}$$

One, of course, also has N-particle Schrödinger equation.

2.3 *Probability in de Broglie–Bohm theory*

It would be remarked that all the equations of de Broglie–Bohm theory, given in § 2.2, are deterministic. Given $q(t)$ and $\psi(\mathbf{r}, t)$ at $t = 0$ we can calculate them for later times. So how do quantum probabilities arise? They arise here in the same way as they do in classical statistical mechanics i.e. through our ignorance of initial conditions.

We are not able to control precise position of the particles. All we are able to prepare are statistical ensembles at a given time t, say $t = 0$, in which the position variable q is distributed with probability function $P(\mathbf{q}, t = 0)$ given by

$$P(\mathbf{q}, t = 0) = |\psi(\mathbf{q}, t = 0)\,|^2. \tag{2.3.1}$$

Once one has done that it can be shown that de Broglie–Bohm dynamics is such that we have

$$P(\mathbf{q}, t) = |\psi(\mathbf{q}, t)\,|^2. \tag{2.3.2}$$

We thus see that, in view of this result, the probability distribution (2.3.1) is somewhat special.

2.4 *Ontology*

In the de Broglie–Bohm interpretation a quantum object, say an electron, has both a particle aspect, it has a trajectory $q(t)$ associated with it, as well as a wave aspect, as the wave function ψ is involved in determining its velocity at any time. This is the resolution of the usual wave particle duality conundrums of the standard Copenhagen interpretation in this theory. It is not a wave or a particle depending on the measuring setup but it is always a wave and a particle. For example in the usual two slit experiment the wave always goes through both the slits but the particle goes through only one of them. The trajectories of the particles, however, are not straight lines, as they normally are in classical mechanics for free particles, due to the presence of quantum potential. The wave function, and therefore, the quantum potential, does depend on the presence of both the slits even if the particle goes through only one of them.

Particle position is a hidden variable of the theory as we cannot control its value and there is inability, at least at present, to have any other distribution than $|\psi(\mathbf{q}, t)|^2$. Particle position, in this theory, however, has a special role. It 'is intrinsic and not inherently dependent.... on the overall context' [13]. Firstly, it is conceptually independent of the wave function with its own dynamical equation of motion. Secondly, it can be measured without being changed. In the terminology of Bell it is a 'beable' [7].

Particle momentum p is postulated to be given by an expression equal to mass times particle velocity i.e.

$$p_i = \nabla_i S \equiv (p_i)_{dBB} \tag{2.4.1}$$

and is also a hidden variable of the theory. This clearly depends on the phase of the wave function of the system as a whole. The momentum, unlike position, is not regarded as an intrinsic property of the particle. A measurement does not reveal a momentum value as given by (2.4.1).

The phase space density $\rho(p, q; t)$ is taken in this theory to be given by

$$\rho(p, q; t) = |\psi(q, t)|^2 \delta(p - (p)_{dBB}). \tag{2.4.2}$$

Note that, as expected

$$\int \rho(p, q; t) \mathrm{d}p = |\psi(q, t)|^2, \tag{2.4.3}$$

but

$$\int \rho(p, q, t) \mathrm{d}q = \delta(p - (p)_{dBB}) \tag{2.4.4}$$

and

$$\neq |\tilde{\psi}(p, t)|^2, \tag{2.4.5}$$

the momentum space density, where $\tilde{\psi}(p, t)$ is the momentum space wave function of the system.

3. Causal quantum mechanics with a symmetrical treatment of position and momentum

3.1 *Asymmetry in de Boglie–Bohm theory between position and momentum*

There is a clear asymmetry in the treatment of particle position and momentum variables in the de Broglie–Bohm theory as noted earlier. This aspect of the theory was considered as a defect of the theory [14].

Takabayashi pointed out while the phase space density $\rho(p, q; t)$ correctly reproduced the phase space average of a general function $f(q)$ of position q i.e.

$$\int \rho(p, q; t) \mathrm{d}p f(q) = \langle \psi | f(q) | \psi \rangle = \int \mathrm{d}q\, |\psi(q)|^2 f(q) \tag{3.1.1}$$

in view of (2.4.3), it does not do so for an arbitrary function $g(p)$ of momentum p i.e.

$$\int \rho(p,q;t)\mathrm{d}q\, g(p) \neq \langle\psi \mid g(p) \mid \psi\rangle = \int \mathrm{d}p\,|\tilde{\psi}(p)|^2\, g(p) \tag{3.1.2}$$

in view of (2.4.4) except when $g(p)$ is a linear function of momentum [15]. In view of this

$$(\delta\mathbf{q})^2 = \langle\psi|\, q^2|\psi\rangle - |\,\langle\psi|\mathbf{q}|\psi\rangle\,|^2$$

is reproduced while

$$(\delta p)^2 = \langle\psi|\mathbf{p}^2|\psi\rangle - |\,\langle\psi|\mathbf{p}|\psi\rangle\,|^2$$

is not and as such the product $\delta p\; \delta q$ involved in Heisenberg uncertainty principle is not recovered.

We may further note that, even for a 'free' particle, i.e. particle not acted on by external potentials, the momentum $(p_i)_{dBB}$ does not satisfy the 'expected' equation of motion

$$\frac{\mathrm{d}(\mathbf{p})_{dBB}}{\mathrm{d}t} = 0.$$

Why should position variable be special? One argument in favor of this position given is that ultimately all observations can be reduced to 'pointer readings'. At least it looks like that in our normal experience despite position variable having no preferred role in standard quantum mechanics as compared to, say, momentum [16]. This could, however, a quirk of nonrelativistic mechanics. In Galilean group it is possible to define a position variable which is conjugate to momentum. For Poincaré group such is not the case. In fact for massless particles such as photons there is no generally acceptable position variable. In contrast for light we have direct comprehension that the given light is say green i.e. of its wavelength which is related to its momentum. So in relativistic S-matrix theory and zero mass particle it would seem that it is momentum variable which is special and not the position variable.

3.2 *A symmetrical treatment for the case one particle moving on a line*

We now propose a reformulation of causal quantum mechanics with a symmetrical treatment of particle position and momentum [17]. For this purpose our point of departure is to replace (2.4.1) by

$$p = \hat{p}(q,t),$$

where $\hat{p}(q,t) \neq (p)_{dBB} = \nabla S$, and is to be determined by the requirement that the phase space density $\rho(p,q;t)$ reproduces correct marginal distributions. Let the phase space density ρ be given by the ansatz

$$\rho(p,q;t) = |\psi(q,t)|^2\, \delta(p - \hat{p}(q,t)). \tag{3.2.1}$$

The requirement about the marginal distributions is given by

(i) $$\int \rho(p,q;t)\mathrm{d}p = |\psi(q,t)|^2, \tag{3.2.2}$$

(ii) $$\int \rho(p,q;t)\mathrm{d}q = |\tilde{\psi}(p,t)|^2\,. \tag{3.2.3}$$

The ansatz (3.2.1) satisfies (3.2.2) automatically. The requirement (3.2.3) leads to the equation for the determination of $\hat{p}(q,t)$ given by

$$\int_{-\infty}^{\epsilon q} \mathrm{d}x' \,|\psi(\epsilon x';t)|^2 = \int_{-\infty}^{\hat{p}(q,t)} \mathrm{d}p' \,|\tilde{\psi}(p',t)|^2, \tag{3.2.4}$$

where $\varepsilon = +1$ or -1 represents a twofold ambiguity in the determination of $\hat{p}(q,t)$.

We also require that phase space density satisfies Liouville equatiuon i.e.

(iii) $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \dot{q})}{\partial q} + \frac{\partial(\rho \dot{p})}{\partial p} = 0. \tag{3.2.5}$$

This leads to the dynamical equation of motion

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \left.\frac{\mathrm{d}S(x,t)}{\mathrm{d}x}\right|_{x=q} \tag{3.2.6}$$

which is the same relation as in the de Broglie–Bohm causal quantum mechanics, by integrating (3.2.5) over the variable p and comparing with the continuity equation for $|\psi(q,t)|^2$. We further demand that the dynamics in the phase space is Hamiltonian i.e. there exists a Hamiltonian $H_c(p,q)$ such that

(iv) $$\dot{q} = \frac{\partial H_c}{\partial p}, \quad \dot{p} = -\frac{\partial H_c}{\partial q}. \tag{3.2.7}$$

The Hamiltonian H_c is not to be confused with the Hamiltonian H which is involved in the Schrödinger equation

$$i\,\frac{\partial \psi}{\partial t} = H\psi. \tag{3.2.8}$$

The requirement (3.2.5) and (3.2.7) are satisfied provided we choose

$$H_c = \frac{(p - A(x))^2}{2m} + V(x,t) \tag{3.2.9}$$

for the case when

$$H_{QM} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x). \tag{3.2.10}$$

Here A is given by

$$\hat{p}(q,t) = (p)_{dBB} + A(q,t), \tag{3.2.11}$$

and there is an appropriate equation to determine $V(x,t)$.

Illustrative examples for a quantum free particle and oscillator can be found in ref. [17].

The potential felt by the particle in de Broglie–Bohm theory was external potential plus a quantum potential. For us, in view of (3.2.11), the momentum is not just a product of mass and particle velocity but acquires, like the potential, a quantum correction $A(q,t)$. An additional contribution like this to momentum occurs even classically in the presence of electromagnetic fields.

3.3 *Some features of the symmetrical treatment*

The expression (2.4.2), while actually symmetrical between position and momentum, is not manifestly so. It can however be put in such a form given by

$$\rho(q,p;t) = |\psi(q,t)|^2 |\tilde{\psi}(p,t)|^2 \delta\left(\int_{-\infty}^{\varepsilon q} \mathrm{d}q' \,|\,\psi(\varepsilon q',t)\,|^2 - \int_{-\infty}^{\hat{p}(q,t)} \mathrm{d}p' \,|\,\tilde{\psi}(p',t)\,|^2\right). \tag{3.3.1}$$

It is obviously positive definite i.e.

$$\rho \geq 0 \tag{3.3.2}$$

unlike the Wigner phase space density. It reproduces marginal position and momentum distributions correctly.

The class of functions for which phase space density ρ reproduces the correct expectation value now is $f(q) + g(p)$, i.e.

$$\int \mathrm{d}p \, \mathrm{d}q \rho(p,q;t)[f(q) + g(p)] = \langle \psi \,|\, f(q) + g(p) | \psi \rangle \tag{3.3.3}$$

while this was restricted to only

$$f(q) + cp$$

in de Broglie–Bohm version.

We further note that the symmetrical version also leads to Newton's law

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = 0 \tag{3.3.4}$$

for a free particle (i.e. $U = 0$ in (3.2.10)).

We may also note that despite this treatment lacking manifest Galilean invariance, it is in fact Galilean invariant [18].

3.4 *Generalizations*

If the space dimension is greater than one and/or if the number of particles is more than one, then the number of marginal distributions is no longer two as was the case for one particle in one dimension.

Consider for example two particles moving in one space dimension. Let the position and momenta of the two particles be denoted by (q_1, p_1) and (q_2, p_2). We may prefer to use centre of mass and relative positions and momenta given by

$$P = p_1 + p_2, \quad p = p_1 - p_2, \quad Q = q_1 + q_2, \quad q = q_1 - q_2.$$

We now have four marginal distributions for the joint probability densities for the pairs of variables given by (P,p), (P,q), (q,Q) and (p,Q). It turns out that it is not possible to find a positive definite $\rho(P,p,Q,q;t)$ such that all the four marginal distributions are reproduced. It is however possible to reproduce marginal joint distributions in the first three pairs, such that

$$\dot{P} = 0 \text{ (Newton's law)},$$

$$\dot{q} = \frac{\partial S}{\partial q}$$

and another equation for $\dot{Q}$.

Some results are known now as to what marginal distribution cannot be reproduced [19]. Further work is however needed to elucidate the general structure underlying these considerations.

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