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On periodically kicked quantum systems

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Abstract. The time evolution of a multi-dimensional system which is kicked periodically with a potential is obtained. The most interesting aspects of the investigation are (i) if the operator corresponding to the potential has invariant subspaces (a characteristic property of multi-dimensional systems), the states belonging to these subspace in its evolution are confined to these invariant subspaces respectively and there cannot be any mixing of states between these subspaces. Further, (ii) it leads to the existence of quasi-stationary states (determined again by the potential) which evolves independent of other similar quasi-stationary states. The method followed in the paper is the direct integration of the Schrödinger equation and then to construct the wave function from the initial wave function.

Keywords. Schrödinger equation; kicked systems; Floquet's theorem; quasi-stationary states.

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1. Introduction

In recent years, attempts have been made to understand the quantum features of systems with time in the form of delta-function kicks, depending on a potential. In most of the papers, [1–17] the discussion is confined to one-dimensional system. Recently, in a paper [19iv] the problem of kicked multi-dimensional system with potential has been investigated, where it has been shown that the system in its evolution is confined to suitable subspaces related to the invariant subspaces of the operator corresponding to the potential. The investigation was for random (δ-function) kicks. Though periodic kicks are special case of random kicks, yet periodic systems have their own characteristics. This is due to Floquet's theorem on linear differential equation with periodic coefficients. This theorem prescribes the general nature of the solution, which considerably simplifies the process to obtain the solution. Further periodic systems are of interest from the point of view of applications.

Since all information pertaining to a system can be obtained from the wave function, it is sufficient to obtain the wave function with its evolution in time. Analytically this is also simpler than treating the problem with evolution operator. The latter seems to be begging the question. In general, the evolution operator is constructed from the wave function. The general nature of the wave function in these problems has also been discussed qualitatively.

The main object of the paper is to investigate the qualitative nature of a finite dimensional periodically kicked quantum system.

The problem is formulated in the following section. Section 3 is devoted to the process of solving the Schrödinger equation. The nature of the time evolution of the wave function is examined, in detail, in § 4. As a simple example, the case of constant potential is considered in § 5. The paper ends with a short discussion.

2. Formulation of the problem

2.1 *Periodic kicking*

The periodic kicking is represented by

$$D(t) = \sum_{-\infty}^{\infty} \delta(t - n\tau) \tag{1}$$

where τ is the period. It can be easily shown that [19i]

$$D(t) = \frac{1}{\tau} \sum_{-\infty}^{\infty} \exp. i2\pi mt/\tau, \tag{2}$$

(n, m are integers).

2.2 *Quantum system*

The Hamiltonian of the general quantum system consists of two terms as

$$H(q, p; t) = H_0(p, q) + \varepsilon f(q) D(t). \tag{3}$$

q stands for a (finite) set of generalized co-ordinates $\{q_i\}$ and $p \equiv \{p_i\}$ the corresponding momenta.

(a) *Discrete systems.* Let the discrete eigenvalues of $H_0(p, q)$ be $\{E_\alpha\}$ and ϕ_α the corresponding eigenfunctions, concomitant to the prescribed (steady) boundary condition. $\{\phi_\alpha\}$ belongs to a Hilbert space (L^2). They are assumed to form a complete set of orthonormal functions. The degeneracy (the order of which is finite) is taken care of in indexing. Thus

$$H_0(q, p)\phi_\beta = E_\beta \phi_\beta, (\phi_\alpha \cdot \phi_\beta) = \delta_{\alpha\beta}. \tag{4}$$

The time dependent perturbation (kicks) is given by the second term in (3).

(b) *Potential.* $f(q)$ is the steady potential and ε represents the strength of the potential. The operator $f(q)$ is real and symmetric (this is implicitly assumed in literature)

$$f(q)\phi_\beta(q) = \sum_{0}^{\infty} F_{\gamma\beta}\phi_\gamma(q), \tag{5}$$

$$F_{\gamma\beta} = \int \phi_\gamma(q) f(q) \phi_\beta(q) \mathrm{d}q. \tag{5'}$$

(c) *Wave equation and the nature of solution.* The Schrödinger equation for the system is

$$i\hbar \frac{\partial}{\partial t} \Psi(q, t) = \{H_0(q, p) + \varepsilon f(q) D(t)\} \Psi(q, t). \tag{6}$$

Since the coefficient of (6) is periodic, the solutions are of the form (Floquet's Theorem)

$$\Psi_\alpha(q,t) = \sum_{-\infty}^{\infty} \exp.i(\lambda_\alpha + 2\pi n/\tau)t\cdot\Phi_\alpha^n(q). \tag{7}$$

Since ϕ_β's form a complete set, $\Phi_\alpha^n(q)$ may be expressed as linear combination ϕ_β. Thus

$$\Phi_\alpha^n(q) = \sum_0^\infty A_{\alpha,\beta}^n \phi_\beta(q) \tag{8}$$

[subscripts in Roman represent Fourier components ($-\infty < n < \infty$) and those in Greek, represent components in Hilbert space ($0 \leqslant \beta < \infty$)]. The general solution is the sum of expressions (7) with $\alpha = 1, 2, \ldots$;

$$\Psi(q,t) = \sum_0^\infty \Psi_\alpha(q,t). \tag{9}$$

λ_α are the constant characteristic exponents to be determined in the sequel and $A_{\alpha,\beta}^n$ are to be determined from the wave equation and the initial wave function.

3. Process of solving

3.1 *Preliminary*

It is clear from eqs (7)–(9) that one has to determine λ_α and $A_{\alpha,\beta}^n$ to construct the solution. As a prelude, one notes from (2)

$$D(t)\sum_{-\infty}^{\infty} g_n \exp.i2\pi nt/\tau = \frac{1}{\tau}\sum_{-\infty}^{\infty}\left(\sum_{-\infty}^{\infty} g_m\right)\exp.i2\pi nt/\tau \tag{10}$$

for any Fourier series with coefficients g_n, [19i].

3.2 *Determination of the characteristic exponent* λ

Substituting the expression for $\Psi_\alpha(x,t)$ from (7), in (6) and equating the Fourier coefficients and those of $\phi_\beta(q)$, one obtains, (note $\varepsilon \neq 0$)

$$\left\{\hbar\left(\lambda_\alpha + 2\pi\frac{n}{\tau}\right) + E_\beta\right\} A_{\alpha,\beta}^n + \frac{\varepsilon}{\tau}\sum_0^\infty F_{\beta\gamma}\sum_{-\infty}^{\infty} A_{\alpha,\gamma}^m = 0 \tag{11}$$

$$A_{\alpha,\beta}^n + \frac{\varepsilon}{2\hbar}\left\{(\lambda_\alpha + E_\beta/\hbar)\frac{\tau}{2} + \pi n\right\}^{-1}\sum_{-\infty}^{\infty} F_{\beta\gamma} a_{\alpha,\gamma} = 0 \tag{12}$$

with

$$a_{\alpha,\beta} = \sum_{-\infty}^{\infty} A_{\alpha,\beta}^m. \tag{13}$$

(Note $-\pi/\tau \leqslant \lambda \leqslant \pi/\tau$, as if λ is a solution, then $\lambda + 2\pi n/\tau$ is also a solution $-$ n integer.) Summing over n on both sides, we get

$$a_{\alpha,\beta} + \frac{\varepsilon}{2\hbar}\cot\Lambda_{\alpha,\beta}\cdot\sum_0^\infty F_{\beta\gamma} a_{\alpha,\gamma} = 0 \tag{14}$$

$$\Lambda_{\alpha,\beta} = \frac{\tau}{2}\left(\frac{E_\beta}{\hbar} + \lambda_\alpha\right). \tag{14'}$$

Thus

$$a'_{\alpha,\beta} + \frac{\varepsilon}{2\hbar}\sum F'_{\beta,\gamma} a'_{\alpha,\gamma} = 0, \tag{15}$$

where

$$F'_{\beta\gamma} = \sqrt{\cot \Lambda_{\alpha,\beta}}\, F_{\beta\gamma}\, \sqrt{\cot \Lambda_{\alpha,\gamma}} \tag{15'}$$

and

$$a'_{\alpha,\beta} = a_{\alpha,\beta}\sqrt{\tan \Lambda_{\alpha,\beta}}. \tag{15''}$$

The matrix $F'_{\beta\gamma}$ is real and symmetric. It can be written in terms of its normalized eigenvectors $\mathbf{v}^{\theta}$ and eigenvalues η^{θ}, as

$$F'_{\beta\gamma} = \sum_{0}^{\infty} \eta^{\theta} v^{\theta}_{\beta} v^{\theta}_{\gamma}. \tag{16}$$

From eq. (15)

$$\left(1 + \frac{\varepsilon}{2\hbar}\eta^{\alpha}\right)\sum_{0}^{\infty} v^{\alpha}_{\beta} a'_{\alpha,\beta} = 0. \tag{17}$$

For non-trivial solution, noting $\varepsilon \neq 0$

$$1 + \frac{\varepsilon}{2\hbar}\eta^{\alpha} = 0. \tag{18}$$

This relation determines λ_{α} as the eigenvalues η^{α} depends on λ_{α} from eqs (14′) and (15′).

3.3 *Determination of* $a_{\alpha,\beta}$ and $A^{n}_{\alpha,\beta}$

Next, $A^{n}_{\alpha,\beta}$ is obtained directly from (12), in terms $a_{\alpha,\gamma}$ of eq. (13). On the other hand from eq. (15), it is clear that $a'_{\alpha,\gamma}$ is the eigenvector corresponding to some eigenvalue μ^{α}. Hence, one can write

$$a_{\alpha,\beta} = p^{\alpha} v^{\alpha}_{\beta} \tag{19}$$

(no sum over α), where p^{α} are constant to be determined from initial condition, ($\mathbf{v}^{\alpha}$ are normalized).

Thus the computations related to the Hill's determinant in this case is reduced simply to obtain the eigenvalues and eigenvectors of $F_{\gamma\beta}$ (eq. (5)). It needs to be mentioned that the eigenvalues and eigenvector of $F'_{\gamma\beta}$ (eq. (15′)) are easily obtained from those of $F_{\gamma\beta}$. Since they are symmetric, the eigenvectors are mutually orthogonal, i.e.,

$$\mathbf{v}^{\alpha}\cdot\mathbf{v}^{\gamma} = \delta_{\alpha\gamma}. \tag{20}$$

3.4 *Initial state and determination of* p^{α}

The most general wave function is thus a superposition of $\Psi_{\alpha}(q,t)$ (eqs (7) and (9))

$$\Psi(q,t) = \sum_{-\infty}^{\infty}\sum_{0}^{\infty}\sum_{0}^{\infty} \varepsilon \exp . i\left(\lambda_{\alpha} + 2\pi\frac{n}{\tau}\right)t\cdot p^{\alpha} v^{\alpha}_{\beta}\phi_{\beta}/2(\pi n + \Lambda_{\alpha}). \tag{21}$$

Hence

$$p^{\alpha} = \sum_{0}^{\infty} v^{\alpha}_{\beta}\int \Psi(q,0)\phi_{\beta}(q)\mathrm{d}q. \tag{22}$$

Given the initial wave function (at $t=0$), $\Psi(q,0)$ the right hand side of eq. (22) is known, which determines p^{α}. It can be easily checked that if the wave function is initially normalized it remains so for all time (averaged over a period).

4. Time evolution of the wave-function

Though apparently (21) shows that the wave function at any instant is, in general, spread over all the eigenstates of the unperturbed (unkicked) Hamiltonian, a little critical examination reveals that there are some collections of states which evolves among themselves independent of other states. Amid these states there are two important classes.

4.1 *Evolution in sub-spaces*

Let us consider the class of potentials $f(q)$, which has invariant subspaces, i.e., there exists a subset $\phi^0_\beta \equiv (\phi_{\beta_j}; j=1,2,\ldots,k)$ such that for any $\phi_{\beta_i} \in \phi^0_\beta$ and $f(q)\phi_{\beta_i} \in \Phi^0_\beta$. The corresponding matrix $F_{\alpha\beta}$ in this case (eq. (5′)) have block diagonal structure. Hence the eigenvectors of $F_{\alpha\beta}$ are confined in these respective blocks. According to eq. (21) any state belonging to this subspace during its evolution will be confined in this subspace. In fact, the initial states belonging to these subspaces will always remain in the respective subspaces. Further, since the evolution equation, i.e., the Schrödinger equation, is linear, the states belonging initially to the subspaces evolve in the respective subspaces and there is no mixing of states among different subspaces. Thus there is no possibility of dispersion of states throughout the entire space by the kicking, as it is usually supposed to be.

This property (having invariant subspaces) of the operator corresponding to the potential function manifest itself most frequently in multidimensional problems. Even in the case of two-dimensional problems in a plane, if the unperturbed (unkicked) Hamiltonian commutes with angular momentum operator and the potential $(f(q))$ depends only on the radial variable, then the kickings do not mix up the angular momentum states and the states with same angular momentum evolve among themselves.

4.2 *Quasi-stationary states and their evolution*

(a) *Quasi-stationary states.* Let us examine the state $\Psi_\alpha(x,t)$ in eq. (21), with a fixed 'α'

$$\Psi_\alpha(q,t) = \sum_0^\infty T_\alpha(t)\cdot p^\alpha \cdot v^\alpha_\beta \cdot \phi_\beta(q) \tag{23}$$

(no sum over α), where

$$T(t) = \frac{\varepsilon}{2\hbar} e^{i\lambda_\alpha t} \sum_{-\infty}^{\infty} \frac{\exp . i2\pi n t/\tau}{\pi n + \Lambda_{\alpha\beta}}. \tag{23′}$$

Hence

$$|\Psi_\alpha|^2 \propto \{|p^\alpha|^2 + O_{sc}(t,\tau)\}. \tag{24}$$

$O_{sc}(t,\tau)$ stands for oscillatory terms of period τ. The average over a period τ of this term is zero. Thus, time average of $|\Psi_\alpha|$

$$\overline{|\Psi_\alpha(q,t)|^2} = \text{constant}. \tag{25}$$

This class of states can be looked upon as quasi-stationary states. These states are superposition of stationary states $\phi_\beta(q)$ of the unperturbed system. They evolve with a common oscillatory time dependent part, which has two factors – one factor $\exp.i\lambda_\alpha t$ and the other an oscillatory factor which is periodic with period τ.

Another important property of these states are that they are mutually orthogonal. From

$$\Phi^\alpha(q) = \sum_0^\infty v_\beta^\alpha \cdot \phi_\beta(q) \tag{26}$$

$$\int \Phi^\alpha(q) \cdot \Phi^{\alpha'}(q) \mathrm{d}q = \sum_0^\infty v_\beta^\alpha v_\gamma^{\alpha'} \int \phi_\beta(q)\phi_\gamma(q)\mathrm{d}q$$

$$= \sum_0^\infty v_\beta^\alpha v_\beta^{\alpha'} = \delta_{\alpha\alpha'}. \tag{26$'$}$$

Since $\mathbf{v}^\alpha$ forms a complete set of vector, $\Phi^\alpha(q)$ are also a complete set of orthonormal functions.

(b) *Time-evolution of quasi-stationary states.* (1) If all the p's excluding p^α $(\neq 0)$ are zero, (from eq. (23) the initial state is $\Phi^\alpha(q)$), then

$$\Psi(q,t) = \Psi_\alpha(q,t) = T_\alpha(t)\Psi_\alpha(q,0) = T_\alpha \Phi^\alpha(q). \tag{27}$$

Thus the evolution manifest itself by the oscillatory time-dependent factor and there is no dispersion among these states.

(2) Next, since the evolution equation (eq. (6)) is linear, for any initial states which are some linear superposition of $\Psi_\alpha(q,0)$

$$\Psi(q,0) = \sum_\alpha p^\alpha \Psi_\alpha(q,0), \tag{28}$$

the time evolved state is simply a superposition of the individual time evolved states, i.e.,

$$\Psi(q,t) = \sum_\alpha T_\alpha(t) p^\alpha \Psi_\alpha(q,0). \tag{29}$$

Thus, these quasi-stationary states evolve independent of each other and there is no mixing among them due to the kicking. Next, since any arbitrary initial state,

$$\Psi(q,0) = \sum_0^\infty B_\gamma \phi_\gamma(q)$$

$$= \sum_0^\infty (B_\gamma v_\gamma^\alpha)(v_\beta^\alpha \phi_\beta) = \sum_0^\infty (B_\gamma v_\gamma^\alpha)\Psi_\alpha(q,0) \tag{30}$$

can be expressed as linear superposition of $\Psi_\alpha(q,0)$, (as $\mathbf{v}^\alpha$'s forms a complete set of vectors). Hence, one can assert that, in general, the system evolves as superposition of these quasi-stationary states, each with its characteristic time-dependent oscillatory term. The latter consists of two factors, one $\exp.i\lambda_\alpha t$ (with characteristic exponent λ_α) and the other factor is periodic of period τ.

5. A special case

In order to show explicitly the effect of kicking on the system, one can consider the simple case of constant potential, $f(q) = 1$, i.e., $F_{\alpha\beta} \equiv \delta_{\alpha\beta}$ (say) eq. (14), now reduces to

$$\left\{1 + \frac{\varepsilon}{2\hbar}\cot\left(\lambda_\beta + \frac{E_\beta}{\hbar}\right)\frac{\tau}{2}\right\} a_\beta = 0. \tag{31}$$

Consequently, for nontrivial solution

$$\frac{\tau}{2}\left(\lambda_\beta + \frac{E_\beta}{\hbar}\right) = -\cot^{-1}\frac{2\hbar}{\varepsilon} \equiv K\,(\text{say}) \tag{31'}$$

and

$$\lambda_\beta = -\frac{E_\beta}{\hbar} + \frac{2K}{\tau}, \tag{31''}$$

where K is a number. Since, in this special case, E_β and ϕ_β corresponds to a distinct λ, one can omit double indices. Finally, from eq. (23)

$$\Psi_\beta(q,t) = \frac{\varepsilon}{2\hbar}\exp. -i\left(\frac{E_\beta}{\hbar} - \frac{2K}{\tau}\right)t. \sum_{-\infty}^{\infty} \frac{\exp. i2\pi nt/\tau}{\pi n + K}\phi_\beta. \tag{32}$$

It is clear that each individual term with ϕ_β is a solution of eq. (6). Thus, if initially the system is at any one of the unperturbed state ϕ_β, it remains in that state even with the kicks. They only introduce the oscillatory factor of period τ. In general

$$\Psi(q,t) = \sum_{0}^{\infty} B_\beta \Psi_\beta(q,t) \tag{33}$$

when the initial state is

$$\Psi(q,0) = \sum_{0}^{\infty} B_\beta \phi_\beta(q). \tag{33'}$$

This is also expected as q-independent potential do not produce any force. The only change is the phase factor at each kick. Consequently, each unperturbed initial stationary state evolves independently to the corresponding quasi-stationary state and the time average over a period τ remains the same.

6. Discussion

The essential point, which needs to be emphasized is that there are states which evolve independently without mixing with other states, by the kicking. These quasi-stationary states are specified by the potential. Hence any initial state, is not necessarily dispersed over all the states by kicking. Thus, there is no loss of its memory as it is often mentioned.

Next, it was shown in [19iv], that the effect of kicking (be it periodic or random) is to impart discontinuous phase factor to the stationary wave function of the unkicked system. The general wave function obtained here are (21), (27), (29) and (33) which appears to be continuous (with respect to time). (These are the wave functions as envisaged in [18] for a system with periodic coefficient.) But, this is apparent as each of them contains a series as a factor of the form

$$\sum_{-\infty}^{\infty} \exp. i\pi(p+k)\frac{t}{\tau}\bigg/(p+k).$$

This diverges at $t = n\tau$, as it should be due to δ-function in the coefficient of eq. (6). In fact the derivative of the above series is a δ-function, which leads to discontinuous change at $t = n\tau$.

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