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Higher dimensional Vaidya metric in Einstein and de Sitter background

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Abstract. Spherically symmetric non-static higher dimensional metrics are considered in connection with Einstein's field equations. Two exact solutions are derived. One of them corresponds to a mixture of perfect fluid and pure radiation field and represents higher dimensional Vaidya metric in the cosmological background of Einstein static universe. The other corresponds to a pure radiation field and represents higher dimensional Vaidya metric in the background de Sitter universe. For both of these solutions, the cosmological constant is taken to be non-zero. Many known solutions are derived as particular cases.

Keywords. Higher dimensional Vaidya metric; Einstein universe; de Sitter universe.

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1. Introduction

Multi-dimensional spacetime is now an active field of research in its attempts to unify gravitation with other forces of nature [1]. In view of recent developments in superstring theory [2, 3], higher dimensional physics has attained a new measure of importance.

The implications of theories in which the dimension of the spacetime is greater than four have been discussed by several authors. What is required from the viewpoint of physics is direct observational evidence supporting theories of higher dimensions. Many exact solutions of Einstein field equations in 4-dimensional spacetime are available in the literature [4]. But the number of articles dealing with exact solutions of Einstein's equations in higher dimensions is relatively small.

In connection with localized sources, the higher dimensional generalizations of Schwarzschild exterior and Reissner–Nordstrom solutions [5, 6] have been obtained. Higher dimensional generalization of the well-known Kerr solution is also available in the literature [7–9]. Iyer and Vishveshwara [10] have generalized the Vaidya metric [11] representing the gravitational field of a radiating star to space times of dimensions greater than four. Their solution can be expressed in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \left[1 - \frac{2m(u)}{(n-1)r^{n-1}}\right]\mathrm{d}u^2 - r^2\mathrm{d}\Omega_n^2, \tag{1}$$

where

$$\mathrm{d}\Omega_n^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + \sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{n-1}\mathrm{d}\theta_n^2 \tag{2}$$

is the metric on the n-sphere in polar co-ordinates. Here the function $m(u)$ represents the mass of the radiating star.

In recent times considerable attention has been given to the solutions of Einstein's equations that represent metrics embedded in a cosmological background. The generalizations of de Sitter's and Friedmann's cosmological solutions with a point mass are well-known in the literature [12, 13]. Patel and Akbari [14] have discussed the usual Vaidya metric in the cosmological background of Einstein static universe. Mallett [15] and Vick [16] have examined the field produced by a radiating mass in de Sitter universe. It would be interesting to obtain the metric (1) in the cosmological backgrounds of Einstein universe and de Sitter universe rather than in the standard Minkowskian background, and the purpose of the present note is to do just that.

The geometry of $(n+2)$-dimensional Einstein's universe is described by the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{\mathrm{d}\bar{r}^2}{1 - \frac{\bar{r}^2}{R^2}} - \bar{r}^2 \mathrm{d}\Omega_n^2,$$

where R is a constant.

By the coordinate transformation

$$\bar{r} = R\sin\left(\frac{r}{R}\right), \quad u = t - r,$$

the above metric reduces to

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \mathrm{d}u^2 - R^2 \sin^2\left(\frac{r}{R}\right)\mathrm{d}\Omega_n^2. \tag{3}$$

We shall take the metric of higher dimensional Einstein universe in the form (3). Similarly one can easily see that the line element of $(n+2)$-dimensional de Sitter universe can be expressed in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \left[1 - \frac{2\Lambda r^2}{n(n+1)}\right]\mathrm{d}u^2 - r^2 \mathrm{d}\Omega_n^2, \tag{4}$$

where Λ is the cosmological constant.

2. Higher dimensional Vaidya metric in the de Sitter universe

We begin with the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + 2L\mathrm{d}u^2 - r^2 \mathrm{d}\Omega_n^2, \tag{5}$$

where L is a function of r and u and u is the retarded time. We name coordinates as $x^0 = u$, $x^1 = r$, $x^{i+1} = \theta_i$ $(i = 1, 2, \ldots, n)$.

The non-vanishing components of the Einstein tensor G_{ik} for the metric (5) are found to have the following expressions:

$$G_{01} = \frac{n}{r}L_r + \frac{n(n-1)(2L-1)}{2r^2}, \tag{6a}$$

$$G_{22} = (n-1)\left[(2L-1)\left(\frac{2-n}{2}\right) - 2rL_r\right] - r^2 L_{rr}, \tag{6b}$$

$$G_{00} = -\frac{nL_u}{r} + \frac{2nLL_r}{r} + \frac{n(n-1)L(2L-1)}{r^2}, \tag{6c}$$

$$G_2^2 = G_3^3 = G_4^4 = \cdots = G_{n+1}^{n+1}. \tag{6d}$$

Here and in what follows a suffix denotes partial derivative e.g.

$$L_u = \frac{\partial L}{\partial u}, \quad L_{rr} = \frac{\partial^2 L}{\partial r^2} \cdots.$$

We wish to solve the field equations

$$G_{ik} = -8\pi\sigma w_i w_k - \Lambda g_{ik}, \tag{7}$$

where $w^i w_i = 0$ and σ is the radiation density. One can easily check that

$$w_i = (1, 0, 0, \ldots, 0) \tag{8}$$

is a null vector for the metric (5). In view of eqs (6a)–(6d) and (8), the field equation (7) led to the relation

$$G_{01} = -\Lambda, \quad G_{22} = \Lambda r^2, \quad G_{00} = -8\pi\sigma - 2L\Lambda. \tag{9}$$

The equation $G_{01} = -\Lambda$ can be easily integrated to have

$$2L = 1 - \frac{2m(u)}{(n-1)r^{n-1}} - \frac{2\Lambda r^2}{n(n+1)}, \tag{10}$$

where $m(u)$ is an arbitrary function of u. Using (10) it is easy to verify that the equation $G_{22} = \Lambda r^2$ is identically satisfied. From the last equation in (9), we can determine the value of σ. It is given by

$$8\pi\sigma = -\frac{nm_u}{(n-1)r^n}. \tag{11}$$

Thus the cosmological constant has no effect on the radiation density.

The line element of our above solution can be expressed in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \left[1 - \frac{2m(u)}{(n-1)r^{n-1}} - \frac{2\Lambda r^2}{n(n+1)}\right]\mathrm{d}u^2 - r^2\mathrm{d}\Omega_n^2. \tag{12}$$

In the absence of the source (i.e. $m = 0$), the metric (12) reduces to the metric (4) of higher dimensional de Sitter universe. When $\Lambda = 0$, the metric (12) becomes the metric (1). Thus the metric (12) describes the higher dimensional Vaidya solution in the cosmological background of the de Sitter universe. When $m =$ constant, the metric (12) represents higher dimensional Schwarzchild exterior metric in the background of the de Sitter universe. In the case $n = 2$, the metric (12) gives the usual Vaidya metric embedded in de Sitter universe. The choice $\Lambda = 0$ and $n = 2$ in (12) leads to the usual 4-dimensional Vaidya solution.

3. Higher dimensional Vaidya metric in the Einstein universe

We consider a spherically symmetric non-static line element in null coordinates in $(n+2)$ dimensions in the form

$$ds^2 = 2\mathrm{d}u\mathrm{d}r + 2L\mathrm{d}u^2 - R^2\sin^2\left(\frac{r}{R}\right)\mathrm{d}\Omega_n^2, \tag{13}$$

where L is a function of r and u and u is the retarded time. Here it should be noted that a metric more general than (13) has been considered by Chatterjee and Bhui [17] in connection with mass-energy in higher dimensions.

By a routine calculation one can obtain the expressions for Einstein tensor $G_{ik} = R_{ik} - (1/2)R\, g_{ik}$ for the metric (13). The surviving G_{ik} are listed below for ready reference.

$$G_{11} = -\frac{n}{R^2}, \tag{14a}$$

$$G_{01} = \frac{n}{R}L_r\cot\left(\frac{r}{R}\right) - \frac{2nL}{R^2} + \frac{n(n-1)}{2R^2}\left[2\cot^2\left(\frac{r}{R}\right)L - \mathrm{cosec}^2\left(\frac{r}{R}\right)\right], \tag{14b}$$

$$G_{00} = \frac{2nL}{R}L_r\cot\left(\frac{r}{R}\right) - \frac{nL_u}{R}\cot\left(\frac{r}{R}\right) - \frac{4L^2 n}{R^2}$$
$$+ \frac{2L^2 n(n-1)}{R^2}\cot^2\left(\frac{r}{R}\right) - \frac{n(n-1)}{R^2}L\,\mathrm{cosec}^2\left(\frac{r}{R}\right), \tag{14c}$$

$$G_{22} = (n-1)\left[2L - 2L_r R\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) - nL\cos^2\left(\frac{r}{R}\right) + \frac{n}{2} - 1\right]$$
$$- R^2 L_{rr}\sin^2\left(\frac{r}{R}\right), \tag{14d}$$

$$G_2^2 = G_3^3 = G_4^4 = \cdots = G_{n+1}^{n+1}. \tag{14e}$$

We take the space surrounding the radiating particle to be occupied by a matter distribution with n-spherical symmetry of density ρ and pressure p. The relevant Einstein field equations are

$$G_{ik} + \Lambda g_{ik} = -8\pi[(p+\rho)v_i v_k - p\, g_{ik} + \sigma\omega_i\omega_k] \tag{15}$$

with

$$v^i v_i = 1, \quad \omega_i\omega^i = 0, \quad v^i\omega_i = 1. \tag{16}$$

Here v^i is the flow vector of the fluid, ω^i is the null vector and $\sigma\omega^i\omega^k$ is the energy momentum tensor arising out of the flowing null radiation.

We take v_i and ω_i in the forms

$$v_i = \left(Lx + \frac{1}{2x}, x, 0, \ldots, 0\right), \quad \omega_i = \left(\frac{1}{x}, 0, 0, \ldots, 0\right), \tag{17}$$

where x is a function of coordinates to be determined from the field equations. We have checked that v_i and ω_i given by (17) satisfy the conditions (16).

Using (14a)–(14e) and (17) in the field equation (15) we obtain

$$x^2 = \frac{1}{2L} \tag{18}$$

and

$$L_{rr} + \frac{(n-2)}{R} L_r \cot\left(\frac{r}{R}\right) + \frac{(n-1)(1-2L)}{R^2} \operatorname{cosec}^2\left(\frac{r}{R}\right) = 0. \tag{19}$$

Here we have assumed that $2L$ is positive.

The differential equation (19) can be easily integrated and the solution can be expressed in the form

$$2L = 1 - \frac{2m(u)\cos\left(\frac{r}{R}\right)}{(n-1)R^{n-1}\sin^{n-1}\left(\frac{r}{R}\right)}, \tag{20}$$

where $m(u)$ is an arbitrary function of u.

Using (20) we can find explicit expressions for pressure, density and radiation density. They are given by

$$8\pi\, p = \Lambda + \frac{n(2-n)}{2R^2} - \frac{nL}{R^2}, \tag{21}$$

$$8\pi\, \rho = -\Lambda - \frac{n(2-n)}{2R^2} + \frac{3nL}{R^2}, \tag{22}$$

$$16\pi\sigma L = -\frac{nm_u}{(n-1)R^n} \cot^2\left(\frac{r}{R}\right)\sin^{2-n}\left(\frac{r}{R}\right). \tag{23}$$

From the results (21) and (22) it is clear that

$$8\pi(\rho + 3p) = -2\Lambda + \frac{n(2-n)}{R^2}$$

i.e. $\rho + 3p = \text{constant}$.

The metric of our solution can be expressed in the final form as

$$\begin{aligned} ds^2 = {} & 2\,du\,dr - R^2 \sin\left(\frac{r}{R}\right) d\Omega_n^2 \\ & + \left[1 - \frac{2m(u)}{(n-1)R^{n-1}} \cos\left(\frac{r}{R}\right)\sin^{1-n}\left(\frac{r}{R}\right)\right] du^2. \end{aligned} \tag{24}$$

In the absence of the source (i.e. $m = 0$), the metric (24) reduces to the metric (3) of higher dimensional Einstein's universe. When $R \to \infty$, the metric (24) reduces to the higher dimensional Vaidya metric (1). Thus the metric (24) describes the higher dimensional Vaidya solution in the cosmological background of Einstein's universe. When $m = \text{constant}$, the metric (24) describes the higher dimensional Schwarzschild exterior metric in the background of Einstein universe. When $n = 2$, the metric (24) reduces to the Vaidya metric in Einstein universe.

Here it should be noted that our non-radiating ($m =$ constant) metric is closely related to higher dimensional version of the solution obtained by Whittaker [18]. He assumed the equation of state $\rho + 3p =$ constant.

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