Creation or destruction of sourceless abelian gauge strings in a Robertson–Walker universe

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Abstract. Following Morris’s [5] consideration of a sourceless abelian gauge string in a Robertson–Walker universe with flat space sections we have generalized the treatment to the case of arbitrary spatial curvature. We find that creation or destruction of the gauge string is not possible if the spatial curvature is nonzero.

Keywords. Cosmology; early universe; cosmic strings.

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1. Introduction

Spontaneous symmetry breaking in gauge theories in the early universe may have given rise to trapped one dimensional regions of a false vacuum called cosmic strings. These strings may have been responsible for the structure formation in the universe. A simple gauge string arising from the breaking of U(1) gauge symmetry will figure in our discussion. Various aspects of such strings have attracted attention [1, 2, 3]. The cosmic gauge string is regarded as an infinitely long very thin mass distribution centred on the symmetry axis of a cylinder. The system of equations describing a single string are so complicated that analytical solutions have not been obtained in the general case. However, Laguna-Castillo and Matzner [4] claimed to have found a regular solution by numerical methods.

Since our universe is expanding, attempts were made to find a solution corresponding to a cosmic string embedded in a Robertson–Walker universe. Morris [5] presented the model of a sourceless abelian gauge string with the complex valued Higgs field completely removed for mathematical simplicity. In this way he showed that such a string in a Robertson–Walker universe with flat space sections could be created or destroyed by a radial inflow of gauge field energy. It is interesting to see what happens if the space sections of Robertson–Walker universe are not flat.

In §2 we present the relevant equations for general curvature $k$. In §3 we give the solutions corresponding to $k = \pm 1$. Finally, in §4 we present the summary and conclusions. Like Morris, we have assumed that the cosmic scale factor has a power law behaviour. We find that in the case of nonzero spatial curvature no nonstatic exact solution of a cosmic string satisfies the required boundary conditions. This implies that creation or destruction of a straight cosmic string is possible only for a cosmological model with flat space sections. However, we assume that
the gauge field structure function can be separated into the product of a space- and
time-dependent functions.

2. The equations

We take a Robertson-Walker (RW) metric with general spatial curvature $k$ in
cylindrical polar coordinates $(\rho, z, \phi)$.

$$
\text{d}s^2 = \text{d}t^2 - a^2(t) \left( \frac{\text{d}\rho^2}{1 - k\rho^2} + (1 - k\rho^2)\text{d}z^2 + \rho^2 \text{d}\phi^2 \right).
$$

(2.1)

Here spatial curvature $k = \pm 1, 0$ and $a(t) \sim t^\alpha (2/3 \leq \alpha < 1/3)$. If we take $\rho = r \sin \theta$ and

$$
z = \tan^{-1} \left( \frac{r \cos \theta}{\sqrt{1 - r^2}} \right) \quad \text{for} \quad k = +1,
$$

(2.2a)

$$
z = (r \cos \theta) \quad \text{for} \quad k = 0,
$$

(2.2b)

$$
z = \tanh^{-1} \left( \frac{r \cos \theta}{\sqrt{1 + r^2}} \right) \quad \text{for} \quad k = -1,
$$

(2.2c)

the metric reduces to the form of RW universe in terms of spherical polar coordinates

$$
\text{d}s^2 = \text{d}t^2 - a^2(t) \left( \frac{\text{d}r^2}{1 - kr^2} + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) \right).
$$

(2.3)

The Lagrangian of the $U(1)$ gauge field is given by

$$
L = \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right).
$$

(2.4)

and the gauge field energy tensor $F_{\mu\nu}$ can be expressed as

$$
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
$$

(2.5)

where

$$
A_\mu = \frac{1}{e} [P(\rho, t) - 1]\delta_\mu^\phi
$$

(2.6)

and $P(\rho, t)$ is called the gauge field structure function. The equation of motion of the
gauge field is given by

$$
F^{\mu\nu}_{\gamma\nu} = \frac{1}{\sqrt{-g}} \left( \sqrt{-g} F^{\mu\nu}_{\gamma\nu} \right)_{,\gamma} = 0.
$$

(2.7)

The nonvanishing components of the gauge field tensor are

$$
F_{t\phi} = \frac{\dot{\rho}}{e}, \quad F^{t\phi} = -\frac{\dot{\rho}}{e a^2 \rho^2},
$$

(2.8a)

$$
F_{\rho\phi} = \frac{P'}{e}, \quad F^{\rho\phi} = \frac{P(1 - k\rho^2)}{e a^4 \rho^2}.
$$

(2.8b)
Here dot and prime represent derivatives with respect to $t$ and $\rho$ respectively. Using (2.7) we obtain
\begin{equation}
\left[ \ddot{\theta} + \frac{a}{\dot{a}} \dot{\theta} \right] - \frac{(1-k\rho^2)}{a^2} \left[ P'' - P'(1+k\rho^2) \right] = 0.
\tag{2.9}
\end{equation}

Following Morris [5] the ordinary magnetic field is
\begin{equation}
B_z = \frac{1}{\sqrt{g_{\rho\rho} g_{\phi\phi}}} F_{\phi\rho} = - \frac{P(1-k\rho^2)}{e a^2 \rho}.
\tag{2.10}
\end{equation}

The total magnetic flux can be written as
\begin{equation}
\Phi_F(\rho, t) = - \int_{A_\phi} d\phi = \frac{2\pi}{e} [1 - P(\rho, t)].
\tag{2.11}
\end{equation}

But the flux $\Phi_F$ associated with the field $F_{\phi\rho}$ in (2.8b) is
\begin{equation}
\Phi_F(\rho, t) = \int_0^{2\pi} \int_0^\rho F_{\phi\rho} \, d\rho \, d\phi = \frac{2\pi}{e} [P(0, t) - P(\rho, t)].
\tag{2.12}
\end{equation}

Here $P(0, t)$ need not be identically equal to unity. It implies
\begin{equation}
\Phi_F(\rho, t) \neq \Phi_F(\rho, t).
\tag{2.13}
\end{equation}

Therefore, aside from a magnetic field, the sourceless $U(1)$ gauge theory possesses an additional singular gauge string restricted to the $z$-axis with an associated string flux
\begin{equation}
\Phi_S(t) = \Phi_F(\rho, t) - \Phi_F(\rho, t) = \frac{2\pi}{e} [1 - P(0, t)].
\tag{2.14}
\end{equation}

3. The solutions

We assume that the gauge function remains finite throughout the universe. Hence the structure function $P(\rho, t)$ must satisfy the following asymptotic boundary conditions
\begin{equation}
P(\rho, t) \rightarrow \text{const.} \quad \text{as} \quad \rho \rightarrow \infty,
\tag{3.1a}
\end{equation}
\begin{equation}
P(\rho, t) \rightarrow \text{const.} \quad \text{as} \quad t \rightarrow \infty.
\tag{3.1b}
\end{equation}

Let us make the extra assumption that $P(\rho, t)$ can be separated as the product of two functions in the following way:
\begin{equation}
P(\rho, t) = F(\rho) G(t).
\tag{3.2}
\end{equation}

With this assumption, (2.9) separates into two ordinary equations
\begin{equation}
F'' - \frac{F'}{\rho} \left( 1 + k\rho^2 \right) - \frac{m^2 F}{(1-k\rho^2)} = 0,
\tag{3.3a}
\end{equation}
\begin{equation}
\ddot{G} + \frac{a}{\dot{a}} \dot{G} - \frac{m^2}{a^2} G = 0,
\tag{3.3b}
\end{equation}

where \( m^2 \) is the separation constant and may be positive, negative or zero.

We now consider three cases.

**Case I:** \( k = 0 \).

Equation (3.3a) reduces to the form

\[
F'' - \frac{F'}{\rho} - m^2 F = 0, \tag{3.3c}
\]

\[
\dot{G} + \frac{\dot{a}}{a} G - \frac{m^2}{a^2} G = 0. \tag{3.3d}
\]

Morris [5] found the general solution of (3.3c) and (3.3d) satisfying the boundary condition (3.1a, b) to be

\[
P(0, t) = P_0 + P_1 \exp \left[ \frac{\Gamma}{1 - \alpha} \left\{ 1 - \left( \frac{t}{t_0} \right)^{1-\alpha} \right\} \right] \tag{3.4}
\]

\[
= P_0 + P_1 G(t),
\]

where \( P_0, P_1 \) and \( \Gamma \) are constants and \( a(t) \) has been taken to be proportional to \( t^x \) with \( (2/3 \leq x < 1/3) \). From (2.14) we obtain

\[
\Phi_3(t) = \frac{2\pi}{e} \left[ 1 - [P_0 + P_1 G(t)] \right]. \tag{3.5}
\]

Depending upon the values of \( P_0 \) and \( P_1 \), \( \Phi_3 \) can be an increasing or a decreasing function of time, i.e. it may represent the creation or the destruction of strings.

**Case II:** \( k = -1 \).

Equation (3.3a) reduces to the form

\[
F'' - \frac{F'}{(1 + \rho^2)} - \frac{m^2 F}{(1 + \rho^2)} = 0. \tag{3.6}
\]

Putting \( x = -\rho^2 \) and \( u(x) = F(\rho) \) we obtain (see Murphy, p. 369, [6])

\[
x(1 - x)u'' - x u' + \frac{m^2}{4} u = 0, \tag{3.7a}
\]

\[
x(1 - x)u'' + \left[ c - (1 + a + b)x \right] u' - abu = 0. \tag{3.7b}
\]

Here a prime represents the derivative with respect to \( x \). This is Gauss’s hypergeometric equation where \( x = 0, 1 \) and \( \infty \) at the regular singular points [7, 8].

Comparing (3.7a) and (3.7b) we get \( c = 0, a = \pm m/2, b = \mp m/2 \). We must have \( r \geq 0, \rho \geq 0 \) and since \( x = -\rho^2, \rho \) is imaginary. For \( x > 0 \), the only singular point in physical space is at \( \rho = 0 \), i.e. \( x = 0 \).

**Case IIa:** \( m^2 < 0 \). In this case \( a \) and \( b \) are imaginary. So the solution is inadmissible.

**Case IIb:** \( m^2 = 0 \). Here

\[
F(\rho) = A \ln(1 + \rho^2) + F_0. \tag{3.8}
\]
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Following Morris [5]

\[ G(t) = \frac{D_t}{(1 - \alpha) a(t)} + G_0. \]  

(3.9)

On account of the boundary conditions

\[ F(\rho) = F_0, \quad G(t) = G_0. \]  

(3.10)

\[ P(\rho, t) = P_0 \] is a constant.

**Case IIc:** \( m^2 > 0 \). Since \( c = 0 \) the general solution is

\[ u(x) = (4x)_2 F_1 (1 \pm m/2, 1 \mp m/2, 2; x). \]  

(3.11)

\( _2 F_1 (1 \pm m/2, 1 \mp m/2, 2; x) \) is a Gauss’s hypergeometric function [7]. Hence

\[ F(\rho) = A(-\rho^2)_2 F_1 (1 \pm m/2, 1 \mp m/2, 2; -\rho^2), \]  

(3.12a)

\[ G(t) = \exp \left[ \frac{\Gamma}{1 - \alpha} \left\{ 1 - \left( \frac{t}{t_0} \right)^{1-\alpha} \right\} \right]. \]  

(3.12b)

Since \( F(0) = 0, P(0, t) = F(0) G(t) = 0 \), so there is no contribution to string flux in this case.

**Case IIId (General case):** The general solution consistent with the boundary conditions (3.1a,b) is

\[ P(\rho, t) = F_0 G_0 = P_0. \]  

(3.13)

The string flux

\[ \Phi_s(t) = \frac{2\pi}{e} \left[ 1 - P(0, t) \right] = \frac{2\pi}{e} \left[ 1 - P_0 \right]. \]  

(3.14)

So we have neither creation nor destruction of the string.

**Case III:** \( k = +1 \).

In this case \( \rho = r \sin \theta \) and \( z = \tan^{-1}(r \cos \theta / (\sqrt{1 - r^2})) \). The metric (2.1) takes the form

\[ ds^2 = dr^2 - a^2(t) \left( \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right). \]  

(3.15)

The universe has a finite volume. So we cannot proceed to the limit \( r \to \infty \). However, we have to examine the limit of the maximum value of \( \rho \). \( k \) is positive, \( r \) and hence \( \rho \) lies between \( 0 \) and \( 1 \).

So we have the following boundary conditions

\[ P(\rho, t) \to \text{const. as } \rho \to 0 \]  

(3.16a)

and

\[ P(\rho) \to \text{const as } t \to \infty. \]  

(3.16b)
In this case (3.3a) reduces to the form
\[ F'' - \frac{F'(1 + \rho^2)}{\rho(1 - \rho^2)} \frac{m^2 F}{1 - \rho^2} = 0. \quad (3.17) \]
Substituting \( x = \rho^2, F(\rho) \equiv u(x) \) (see Murphy [6], p. 369) we obtain
\[ x(1 - x)u'' - xu' - \frac{m^2}{4} u = 0, \quad (3.18a) \]
\[ x(1 - x)u'' + [c - (1 + a + b) x]u' - abu = 0, \quad (3.18b) \]
where prime represents a derivative with respect to \( x \).
This is Gauss's hypergeometric equation with \( x = 0, 1, \infty \) at the singular points.
Comparing (3.18a) and (3.18b) we get
\[ c = 0, \quad a = \pm \sqrt{(- m^2/4)}, \quad b = \mp \sqrt{(- m^2/4)}. \quad (3.19) \]

**Case IIIa:** \( m^2 < 0 \). In this case, we have the solution satisfying (3.1a,b).
\[ u(x) = A x \, _2F_1(1 + a, 1 + b, 2; x), \quad (3.20) \]
where the values of \( a \) and \( b \) are given above. The two regular singular points are \( x = 0, 1 \), corresponding to \( \rho = 0, 1 \). The series converges at both points.

**Case IIIb:** \( m^2 = 0 \). In this case we have the solution
\[ F(\rho) = A \ln(1 - \rho^2) + F_0. \quad (3.21) \]
This blows up at \( \rho \to 1 \). Hence we set \( A = 0 \)
\[ G(t) = \frac{D t}{(1 - x) a(t)} + G_0. \quad (3.22) \]
From the boundary conditions (since \( A = 0, D = 0 \)), we have
\[ P(\rho, t) = P_0. \quad (3.23) \]

**Case IIIc:** \( m^2 > 0 \). In this case \( a \) and \( b \) are imaginary. Hence the solution is inadmissible.

**Case IIIId (General case):**
\[ P(\rho, t) = P_0 + P_2 \rho^2 \, _2F_1(1 + a, 1 + b, 2; \rho^2) G(t), \quad (3.24) \]
where
\[ P_0 = F_0 G_0. \quad (3.25) \]
\( \, _2F_1(1 + a, 1 + b, 2; \rho^2) \to 1 \) as \( \rho \to 0 \). Here \( \, _2F_1(1 + a, 1 + b, 2; \rho^2) \) is the Gauss hypergeometric function.

The string 'magnetic' flux is
\[ \Phi_S(t) = \frac{2\pi}{e} [1 - P(0, t)] = \frac{2\pi}{e} [1 - P_0]. \quad (3.26) \]
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So we have neither creation nor destruction of the string. The radial component of the Poynting vector for the flat space sections is

$$S_\rho = ( - g_{\rho\rho})^{1/2} T^{\rho\varphi} = \frac{\dot{\rho} P'}{e^2 a^3 \dot{\rho}^2}. \quad (3.27)$$

The electromagnetic gauge field energy flows radially inwards of the string which may create a string or destroy an already preexisting string.

4. Conclusion

Following Morris's consideration of a sourceless abelian gauge string in a Robertson-Walker universe with fiat space sections, we have generalized the treatment to the case of arbitrary spatial curvature. We have found that creation or destruction of the gauge string is possible only if the spatial curvature is zero. If the spatial curvature is negative or positive, we have neither creation nor destruction of the string. We have, however, assumed like Morris that the gauge field structure function can be separated into a space-dependent and a time-dependent part.

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