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Classical limit of scattering in quantum mechanics–A general approach

D SEN*, A N BASU and S SENGUPTA
Condensed Matter Physics Research Centre, Department of Physics, Jadavpur University, Jadavpur, Calcutta 700 032, India
*Present address: Barasat Govt. College, Barasat, West Bengal, India

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Abstract. The classical and quantum physics seem to divide nature into two domains macroscopic and microscopic. It is also certain that they accurately predict experimental results in their respective regions. However, the reduction theory, namely, the general derivation of classical results from the quantum mechanics is still a far cry. The outcome of some recent investigations suggests that there possibly does not exist any universal method for obtaining classical results from quantum mechanics. In the present work we intend to investigate the problem phenomenonwise and address specifically the phenomenon of scattering. We suggest a general approach to obtain the classical limit formula from the phase shift δ_l, in the limiting value of a suitable parameter on which δ_l depends. The classical result has been derived for three different potential fields in which the phase shifts are exactly known. Unlike the current wisdom that the classical limit can be reached only in the high energy regime it is found that the classical limit parameter in addition to other factors depends on the details of the potential fields. In the last section we have discussed the implications of the results obtained.

Keywords. Scattering theory; quantum mechanics; classical limit.

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1. Introduction

When the conceptual structures of two theories in physics differ significantly it becomes a difficult proposition to derive the approximate theory from the more general one which replaces it. The best approach is to investigate the situations in which the observable result from the two theories would agree. This is well illustrated by the problem of getting the Newton's law of gravitation from the relativistic theory.

The problem of classical limit of quantum mechanics is, however, much more difficult because of a radical difference in the conceptual structures of classical and quantum physics. Even the basic concept, the dynamical state of a system is defined differently in the two theories. The limit $h \to 0$ is often regarded as a convenient way to obtain the classical approximation. It has become increasingly clear that an indiscriminate use of the procedure leads to absurd results. For example if we put $h \to 0$ in the Schrödinger equation nothing meaningful is obtained.

Another difficulty is that the problem of classical limit is not well defined. How much of classical physics we expect to recover from quantum mechanics is not precisely known. The domain of validity of classical physics is quite large and one has to recover not only the laws but also the description of the objective reality which forms the basis of these laws. It seems that the nature of objective reality undergoes a qualitative change with the scale of observation. The picture which is so clear at the classical level turns hazy and then completely disappears as we approach the atomic domain. In nature this change must occur smoothly. But such a smooth change seems to be ruled out by the formalism of quantum mechanics.

It is well known that even at very large energy regions, there are quantum states [1] which violate classical laws. An Arnold cat when treated classically leads to a chaotic behaviour. But a quantum mechanical Arnold cat remains perfectly reversible [2]. There is a belief that if classical particles are described by narrow wave packets, the spreading effect will be negligible (as the mass is large compared to those of atoms), and the motion of the packet will satisfy classical laws. But we can prepare a classical particle in a state so that its position lies definitely within, say, $x_0 \pm \Delta x$ and momentum within $p_0 \pm \Delta p$ (Δx, Δp of course satisfy uncertainty relation). But a wave packet description of such a state require that both $\psi(x,t)$ and its Fourier transform $\phi(k,t)$ must give compact density functions, i.e., $|\psi|^2$ (or $|\phi|^2$) must be zero for $x(/p)$ outside the region $x_0 \pm \Delta x(/p_0 \pm \Delta p)$. Quantum mechanics, however, overrules such a packet. It can be shown that if $\rho(x,t)$ is compact, $\rho(k,t)$ cannot be so. Facts like the above indicate complexities in the nature of relation between classical and quantum mechanics.

However, most investigations in the classical limit problem avoid the bigger question of retrieving from quantum mechanics, a whole domain of classical physics. Instead they confine themselves with calculations in specific areas. The aim is to find in what limiting conditions quantum mechanical results will agree with those obtained from classical mechanics. A few examples may be cited. Rowe [3] showed that in attractive Coulomb scattering ($V(r) = -\gamma/r$), the outgoing density functions at arbitrary distances from the scatterer, are quite different in classical and quantum mechanics, though scattering cross-sections agree. In the limit, the dimensionless parameter $\gamma/\hbar v_0$ (v_0 is the incoming velocity) goes to infinity, the quantum mechanical density function agrees with the classical value. Bhaumik *et al* [4] showed that with coherent packets and large average energy one can get the classical Kepler orbit for the hydrogen atom. Born and Ludwig [5] showed that at large energies, a wave packet for a particle confined in a box with rigid walls can be constructed whose motion resembles the classical motion of a particle repeatedly reflected from the rigid walls.

In the present paper we shall work out a general approach to recover the classical limit for the differential scattering cross-section ($\mathrm{d}\sigma/\mathrm{d}\Omega$), starting from the quantum mechanical expression for the phase shift δ_l. As have been pointed out by Sen *et al* [6] the classical definition for $\mathrm{d}\sigma/\mathrm{d}\Omega$ for particles is not exactly the same as the definition used in quantum mechanics. But in the classical limit, the difference in the definition affects the expression for differential cross-section around a small angle near zero degree. At all other angles, the differential cross-sections in the two definitions are identical.

Usually the exact expression for δ_l depends on a dimensionless parameter $\alpha = a_0/\hbar$ (a_0 does not involve $\hbar$). The scattering amplitude is given by

$$f(\theta) = \frac{1}{2ik}\sum_0^\infty (2l+1)P_l(\cos\theta)(e^{i2\delta_l}-1)$$
$$= \frac{1}{2ik}\sum_0^\infty (2l+1)P_l(\cos\theta)e^{i2\delta_l}, \quad \text{if } \theta \neq 0, \tag{1}$$

where k is the incoming particle wave vector and the phase shift δ_l depends, apart from the scattering angle, on the parameter α defined above. We show that in general, if we calculate $f(\theta)$ in the limit $\alpha \to \infty$ then $|f(\theta)|^2$ gives the classical scattering cross-section. In evaluating the sum(1) we use the method of stationary phase [7], when such a stationary phase point exists. In other cases one may use numerical methods.

In the following section we have applied the above method to obtain the classical limit results for scattering by two potential fields, namely the $1/r^2$ and $1/r$ and scattering by the rigid sphere. The last section contains a discussion of the implications of the results obtained and also a comparison between the Landau–Lifshitz method [7] and the present approach.

2. Classical limit calculation for different potentials

a) *Scattering by* $V(r) = \gamma/r^2$ *potential*

The expression for the phase shift δ_l, has been evaluated and is given by [8]:

$$\delta_l = \frac{\pi}{2}(l+\tfrac{1}{2}) - \frac{\pi}{2}(l+\tfrac{1}{2})\left[1 + \frac{\alpha^2}{(l+(1/2))^2}\right]^{1/2}, \tag{2}$$

where

$$\alpha = (2\gamma m)^{1/2}/\hbar. \tag{2.1}$$

We shall determine $f(\theta)$ in the limit $\alpha \to \infty$. We take α to be so large that $\sqrt{\alpha} \gg 1$. We divide the entire domain of l from 0 to ∞ into three regions: region I from $l=0$ to $l=l_1$, region II from $l=l_1$ to $l=l_2$, and region III from $l=l_2$ to infinity such that $l_1 < \alpha \leqslant l_2$ and $l_2 - l_1$ is small and of the order $\sqrt{\alpha}$. In region I, we can write δ_l in (2) as

$$\delta_l \approx \frac{\pi}{2}(l-\alpha) + \frac{\pi}{4}, \quad l < \alpha \tag{2.2}$$

and

$$\cos 2\delta_l = (-1)^l \sin\pi\alpha, \qquad \sin 2\delta_l = (-1)^l \cos\pi\alpha. \tag{2.3}$$

In region III, we get

$$\delta_l \approx -\frac{\pi\alpha^2}{2(2l+1)}, \quad l > \alpha. \tag{2.4}$$

Using the above values of δ_l it can be shown that

$$\sum_0^{l_1}(2l+1)P_l(\cos\theta)e^{i2\delta_l} \simeq 0$$

and

$$\sum_{l_2}^{\infty}(2l+1)P_l(\cos\theta)e^{i2\delta_l}\simeq 0. \tag{2.5}$$

The details of the arguments leading to the above results are given in Appendix A. The results are true for $\theta \neq 0$.

For evaluating $f(\theta)$ given by (1) we are now left with the following sum for region II:

$$f(\theta)=\frac{1}{2ik}\sum_{l_1}^{l_2}(2l+1)P_l(\cos\theta)e^{i2\delta_l}. \tag{3}$$

Since in this region $l > \alpha$ and $\alpha \gg 1$, we can use the method of stationary phase due to Landau and Lifshitz. Assuming that for a given θ the phase δ_l has a stationary value at some $l = l(\theta)$, we get from (2)

$$2\frac{d}{dl}\delta_l=\pi-\pi\Big/\left[1+\frac{\alpha^2}{(l+(1/2))^2}\right]^{1/2} \tag{3.1}$$

which on substitution in the following equation (see eq. (111.3) of ref. [7]):

$$2\frac{d}{dl}\delta_l \pm \theta = 0 \tag{3.2}$$

gives the desired l as a function of θ. The value corresponding to the stationary phase is thus given by

$$l_\theta^2 \simeq \frac{\alpha^2(\pi-\theta)^2}{(2\pi-\theta)\theta}. \tag{3.3}$$

Then for the differential cross-section we get using equations (1), (2), (3) and (3.3) the following relation (see appendix B4)

$$|f(\theta)|^2 \simeq \frac{\hbar^2}{2m^2v^2\sin\theta}\frac{d}{d\theta}l_\theta^2, \tag{3.4}$$

$$\frac{d\sigma}{d\Omega}=|f(\theta)|^2=\frac{\gamma\pi^2(\pi-\theta)}{E(2\pi-\theta)^2\theta^2\sin\theta} \tag{3.5}$$

which agrees with the classical result given by equation (4) in ref. [8]. It should, however, be emphasized that in this derivation we have not retrieved the classical phenomenon of scattering. In particular we have not recovered the tracks of the individual particles, which play an important role in the classical theory. We shall further discuss this point in the last section.

b) *Scattering by the Coulomb potential* $V(r)=\gamma_0/r$

It may appear trivial to discuss the Coulomb potential as it is well known that the classical and quantum mechanical expressions agree in this case. But it is instructive to follow the general approach outlined above and to examine whether $f(\theta)$ evaluated in the classical limit as defined here, leads to the correct expression for the differential cross-section.

The phase shift for the Coulomb potential is given by [9]

$$\exp(2i\delta_l) = \frac{\Gamma(l+1+i\gamma_0/\hbar v_0)}{\Gamma(l+1-i\gamma_0/\hbar v_0)} = \frac{Ae^{iB}}{Ae^{-iB}} = \exp(2iB), \tag{4}$$

where B is the argument of $\Gamma(l+1+i\gamma_0/\hbar v_0)$ and

$$2\delta_l = 2B. \tag{4.1}$$

In the present case the appropriate parameter is $\alpha_0 = \gamma_0/\hbar v_0$, where v_0 is the velocity of the incident particle. Using Stirling's approximation we get

$$2\delta_l = \alpha_0 \ln\{\alpha_0^2 + (l+1)^2\} + (2l+1)\tan^{-1}\{\alpha_0/(l+1)\}. \tag{4.2}$$

Now let us examine the limiting situation $\alpha_0 \to \infty$. As in the previous case we divide the sum in $f(\theta)$ into three regions $l=0$ to $l=l_1$, $l=l_1$ to $l=l_2$, and $l=l_2$ to infinity, with $l_1 < \alpha_0 < l_2$. In region I, we can approximate δ_l by

$$2\delta_l \simeq 2\alpha_0 \ln \alpha_0$$

which is a constant independent of l. It then follows that the contribution to $f(\theta)$ from region I will be zero as seen from the results of the previous example. Similarly for region III, we can write

$$2\delta_l \simeq 2\alpha_0 \ln l$$

and the contributions to $f(\theta)$ from all these terms vanish due to rapid oscillation of $P_l(\cos\theta)$. Thus the significant range is from l_1 to l_2. Using the method of stationary phase we get from (4.2)

$$2\frac{\mathrm{d}}{\mathrm{d}l}\delta_l = 2\tan^{-1}\{\alpha_0/(l+1)\} = \mp\theta, \quad \text{or}$$

$$l_\theta \simeq \alpha_0 \cot\theta/2 \tag{4.3}$$

and from (3.4) the differential cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 = \frac{\hbar^2\alpha_0^2}{4v_0^2\sin^4\theta/2} \tag{4.4}$$

which is the classical result. It is found that all the assumptions (see appendix B) made in deriving (3.4) are valid in the limit $\alpha_0 \gg 1$.

Again we have recovered the classical formula and not the entire classical phenomenon. This leads to the important question that in a specific problem how much of the classical phenomenon is recovered and why the remaining part is not obtained. In the case of Coulomb scattering for example, Rowe [3] has demonstrated that in the attractive Coulomb potential ($V = -\gamma_0/r$), where the classical limit parameter is $(\gamma_0/v\hbar)$ in the limit $\hbar \to 0$ or $\gamma_0/v\hbar \to \infty$, the quantum mechanical incoming and outgoing density functions go over to the corresponding classical quantities (defined from the classical trajectories) at all values of $\underset{\sim}{r}$. This, however, neither lead to trajectories from quantum mechanical calculation nor imply in any way the existence of the same. Furthermore the above procedure is not applicable for repulsive potential and to others in general.

c) *Scattering by a rigid sphere of radius a*

The expression for the phase shift in this case is given by [9].

$$\delta_l = \tan^{-1}\{j_l(\rho)/n_l(\rho)\}, \quad \text{where} \quad \rho = ka = \frac{mv_0 a}{\hbar} \tag{5}$$

and

$$f(\theta) = \frac{1}{2ik}\left\{\sum_0^\infty (2l+1)P_l(\cos\theta)\left(\frac{-2j_l^2}{j_l^2+n_l^2}\right) + i\sum_0^\infty (2l+1)P_l(\cos\theta)\left(\frac{2j_l n_l}{j_l^2+n_l^2}\right)\right\}. \tag{5.1}$$

In the present case the relevant parameter is ρ which goes to infinity as $\hbar \to 0$. We take $l_1 = \rho$ and in region I for $\rho \geqslant l$ to a good approximation we use the asymptotic expansion of $j_l(\rho)$ and $n_l(\rho)$, given by

$$j_l(\rho) = \frac{1}{\rho}\sin\left(\rho - l\frac{\pi}{2}\right)$$

and

$$n_l(\rho) = -\frac{1}{\rho}\cos\left(\rho - l\frac{\pi}{2}\right)$$

and get the value of δ_l for l in region I as

$$\delta_l = l\frac{\pi}{2} - \rho. \tag{5.2}$$

As in the previous two examples, we can show using the above values of δ_l that the contribution from region I to $f(\theta)$ vanishes. Also, we take $l_2 \gg \rho$ and for region III we can use the following asymptotic expansion (ρ fixed and $l \gg \rho$)

$$j_l(\rho) \sim (e\rho/2l)^l$$

and

$$n_l(\rho) \sim -(e\rho/2l)^{-l}$$

which gives

$$\frac{j_l(\rho)}{n_l(\rho)} \simeq -(e\rho/2l)^{2l} \tag{5.3}$$

and from (5) we get

$$|\delta_l| \simeq 0. \tag{5.4}$$

So the sum in $f(\theta)$ from region III vanishes according to eq. (1). Thus the significant contribution comes from region II, i.e. from the terms between $l_1 \sim \rho$ to $l_2 \sim \rho^{3/2}$. But in this case the method of stationary phase fails. From figure 1 we see that $(\mathrm{d}/\mathrm{d}l)\delta_l$ is a very small quantity close to zero for $l \geqslant \rho$. Hence the solution for (3.2) will exist only for those values of θ which are close to zero. Thus it is clear that for finite θ no stationary phase point exists. So, we have to evaluate the value of $f(\theta)$ given by (5.1) between $l_1 = \rho$ to $l_2 = \rho^{3/2}$ by numerical computation.

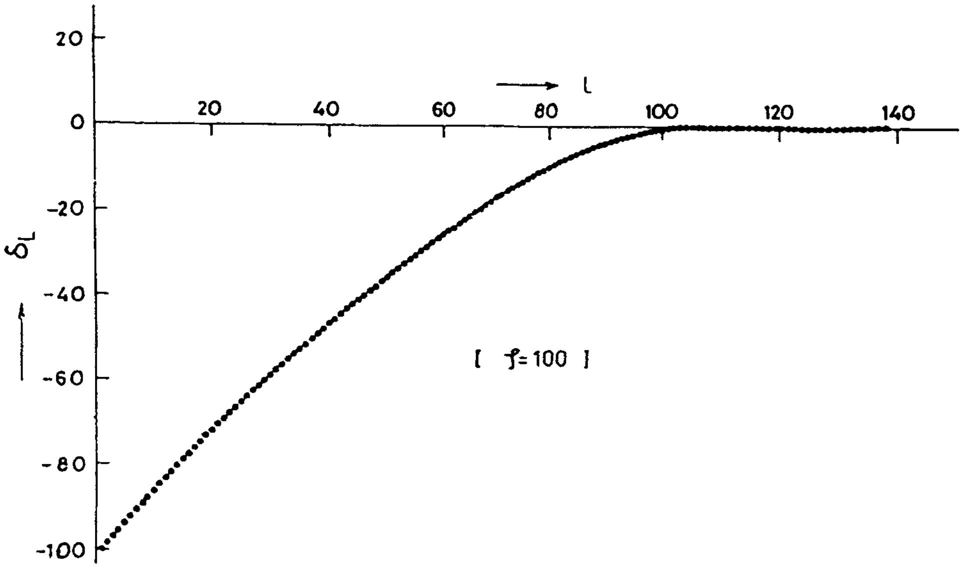

Figure 1. Scattering phase shift δ_l vs. angular momentum quantum number l for $\rho = 100$.

The differential cross-section computed numerically for several values of the scattering angle $\theta > 0$ and $\rho \gg 1$ is shown in table 1. We give results for different values of θ and ρ and notice that as ρ increases the classical values are obtained as a limit.

There is, however, a subtle difference in the definition [6, 10] in classical and quantum mechanics which has caused much confusion in the literature. According to the quantum mechanical calculation, scattering along the forward direction is non-vanishing and the total cross-section in the classical limit situation becomes $2\pi a^2$. This difference is essentially due to difference in the definition of scattering cross-section in quantum and classical mechanics. In ref. [6] we have obtained in the classical limit a general relation between the quantum mechanical and the classical values of total cross-section, $\sigma_{\rm qm}$ and $\sigma_{\rm cl}$ for different cases of scattering including that by rigid sphere and have shown that the total cross-section have to satisfy the relation, $\sigma_{\rm qm} = 2\sigma_{\rm cl}$. This difference is not only a difference in the facts produced by two theories, but also a difference in the definitions of scattering cross-section in two theories. Both, however, correctly reproduce experimental observation (in approximate limits) with different interpretations.

This analysis brings out an important aspect of the classical limit problem so far overlooked. In addition to demonstrating that the differential cross-section agree at large angles it is essential to prove the above relation for the total cross-section for obtaining correct classical limit. In conclusion we have remarked in ref. [6] that it is ironical that the well known result $\sigma_{\rm qm} = 2\sigma_{\rm cl}$, for rigid sphere scattering in the classical limit has been regarded as anomalous in most of the literature on scattering. From the present analyses we conclude that this is the correct classical limit relation. A real anomaly would be produced if the relation came out $\sigma_{\rm qm} = \sigma_{\rm cl}$.

Table 1. Some typical computed values for the differential scattering cross-section at selected angles of scattering by a rigid sphere.

	$(4\|f(\theta)\|^2/a^2)^*$		
$\rho = ka$	$\theta = 45°$	$\theta = 60°$	$\theta = 90°$
20	1·25	1·10	—
50	1·12	1·04	—
100	1·04	1·01	1·001

$^*(4|f(\theta)|^2/a^2)_{\text{classical}} = 1$.

3. Discussion

The examples given in the previous section demonstrate how the general approach suggested may be applied to different scattering cases to calculate the classical limit. In this connection it may be mentioned that the approach suggested by Landau and Lifshitz [7] and later discussed by Newton [11] has certain limitations from which the present method is free. Landau and Lifshitz have enumerated the following set of conditions:
(a) for every scattering angle θ in the classical limit the dominant contribution to the scattering amplitude must come from large l values centred around $l_0(\theta)$ determined by the condition of stationary phase, (b) the total phase factor ($\simeq 2\delta_l \pm \theta_l$) for each $f_l(\theta)$ must be large enough so that $f_l(\theta)$ is a rapidly varying function of l near l_0; for this it is necessary that both δ_{l_0} and δ''_{l_0} should be much greater than 1, and (c) there is only one value of θ for every classical impact parameter. The above conditions are sufficient but the main weakness of the method lies in the fact that it cannot state for which limiting value of a dimensionless parameter all these conditions are satisfied. Secondly, the method of stationary phase may not be applicable in all cases. For example, in the case of rigid sphere scattering the method does not work. Thirdly even if a stationary phase exists the parabolic formula for the differential cross-section (see appendix B) may become inadequate if the stationary phase point is a broad maxima or minima. On the other hand in the present procedure we have to know the phase shift δ_l as a function of l and the quantum parameter α and then evaluate $f(\theta)$ in the limit $\alpha \to \infty$. No other assumptions are required. It would indeed be surprising if the classical limit is not recovered in this limit.

A counter intuitive and interesting feature that emerges from the present analysis is that the parameter α which determines the classical limit depends on the scattering potential. If we write α in the form $a_0/\hbar$, then a_0 is the action parameter for the problem and the classical limit is reached when $a_0 \gg \hbar$. For $V = \gamma/r^2$, $a_0 = (2\gamma m)^{1/2}$; for $V = \gamma_0/r$, $a_0 = \gamma_0/v_0$; and for a rigid sphere $a_0 = mv_0 a$. These expressions clearly indicate that no universal criterion, such as, energy of the particles should be very high or that the de Broglie wavelength must be small compared to the relevant linear dimension of the problem. In fact there are various physical conditions for realizing the classical limit formula. Let us give an example. In the case $\alpha = (2\gamma m)^{1/2}/\hbar$ for $V = \gamma/r^2$ potential it is apparent that a strong interaction strength (γ) alone keeping $\hbar$ and m constant may lead to classical result.

In conclusion we would like to draw attention to the remarks made after equations (3.5) and (4.4). In all works connected with the classical limit of quantum mechanics, attempt is made to derive a particular classical formula from quantum mechanics using a suitable limiting procedure. It is overlooked that by this procedure we do not retrieve the entire classical phenomenon. For example in the present work we derive the classical expression for the differential cross-section but we do not recover from quantum mechanics an ensemble of particle tracks which are the fundamental elements in the classical theory of scattering. This problem raises an important question. Why can we retrieve only some classical formulae without actually recovering the entire classical phenomenon? It seems the answer to this puzzle is not known at present.

Appendix A

Using the value of δ_l in eq. (2.4) it can be shown that

$$\sum_0^{l_1}(2l+1)P_l(\cos\theta)e^{i2\delta_l}=0 \tag{A1}$$

and, using the asymptotic expression for $P(\cos\theta)$ for large l

$$\sum_{l_2}^{\infty}(2l+1)P_l(\cos\theta)e^{i2\delta_l}=\sum_{l_2}^{\infty}(2l+1)\frac{2}{\sqrt{2\pi l\sin\theta}}\sin\left\{(l+\tfrac{1}{2})\theta+\frac{\pi}{4}\right\}e^{i2\delta_l}=0. \tag{A2}$$

because in (A2) the sine-term in the numerator is rapidly oscillating with l and the other l-dependent terms are slowly varying. The vanishing of the first sum (A1) can be shown from the following results. The generating function for $P_l(\cos\theta)$ gives the relations

$$\frac{1-S^2}{(1-2S\cos\theta+S^2)^{1/2}}=\sum_0^{\infty}(2l+1)P_l(\cos\theta)S^l \tag{A3}$$

and

$$\left.\begin{aligned}\sum_0^{\infty}(2l+1)P_l(\cos\theta)=0\\ \sum_0^{\infty}(-1)^l(2l+1)P_l(\cos\theta)=0\end{aligned}\right\}\quad \text{for } \theta\neq 0. \tag{A4}$$

It follows from above that $\Sigma_0^{l_1}(-1)^l(2l+1)P_l(\cos\theta)=0$ because the sum over high l-values vanishes due to rapid oscillation of $P_l(\cos\theta)$ (its asymptotic expression). It may be noted that $\theta=0$ angle is excluded.

Appendix B

In order to project clearly the assumptions under which the expression (3.4) is valid, we briefly outline the derivation of the same and check whether the assumptions have been satisfied in deriving equation (3.5).

As has been shown in Appendix A, in the expression for $f(\theta)$ given by (1), the sum over region I and region III vanish. We have to evaluate the sum for region II only. In this region all the l's are much greater than 1 and we can use the asymptotic expression for $P_l(\cos\theta)$ for large l. We use the method of stationary phase to evaluate this sum. We break up the $\sin\{(l+(1/2))\theta+(\pi/4)\}$ term in (A2) in exponential functions and then sum over l leads to two separate sums with different phase terms. One of the phases will be stationary for $l=l_\theta$ which is given by eq. (3.3). We now expand δ_l about l_θ. Setting $x=l-l_\theta$ we get

$$\delta_l=\delta_\theta+x\delta'_\theta+\frac{x^2}{2}\delta''_\theta+\frac{x^3}{6}\delta''_\theta, \tag{B1}$$

where $\delta_\theta=\delta_{l=l_\theta}$ and

$$\frac{2l+1}{\sqrt{l}}\simeq 2\sqrt{l}=2(l_\theta+x)^{1/2}=2l_\theta^{1/2}\left(1+\frac{x}{2l_\theta}+\ldots\right). \tag{B2}$$

Using the above result in the expression (3.4), we get

$$f(\theta)\simeq -k^{-1}(l_\theta/2\pi\sin\theta)^{1/2}\int_{l_1}^{l_2}\exp i\left[x^2\delta''_\theta+\frac{x^3}{3}\delta'''_\theta+\ldots\right]\times\left(1+\frac{x}{2l_\theta}+\ldots\right)\mathrm{d}x. \tag{B3}$$

From (3.3) we find that $l_\theta\sim\alpha$ and hence $\delta''_\theta\sim 1/\alpha$ and $\delta'''_\theta\sim 1/\alpha^2$. The range of l between l_1 and l_2 is assumed to be small and of the order of $\sqrt{\alpha}$. Hence x will be less than $\sqrt{\alpha}$. Under these condition $x^3\delta'''_\theta$ term will be much less than the $x^2\delta''_\theta$ term in (B3) as $\sqrt{\alpha}\gg 1$. Similarly $x/2l_\theta\ll 1$. Neglecting these terms and changing the limits $l_1\to -\infty$ and $l_2\to +\infty$ (as most of the contribution to the integral comes from a small region around l_θ), we get

$$f(\theta)\simeq -k^{-1}(l_\theta/2ki\delta''_\theta\sin\theta)^{1/2}. \tag{B4}$$

From (3.2) we can write

$$\delta''_\theta=\mp\frac{1}{2(\mathrm{d}l_\theta/\mathrm{d}\theta)}. \tag{B5}$$

Using these results we get (3.4) for $|f(\theta)|^2$.

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