PRAMANA
— journal of
physics

Vol. 48, No. 3,
March 1997
pp. 775–786

The Larmor nutation

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MS received 10 June 1996; revised 22 November 1996

Abstract. The classical non-relativistic problem of the motion of a charged particle in an external central force field and a weak uniform magnetic field is revisited to show that the motion of the kinetic angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of the particle, in the so-called Larmor approximation, is not a simple precession but is actually a composite motion involving precession as well as a high frequency nutation. The precession-nutation motion of $\mathbf{L}$ is discussed in the Larmor approximation when the Larmor-frame-orbit of the charged particle is an ellipse (or a circle) for the case of the two central forces namely the Coulomb and the Hooke-law-force, which are the only two central forces known to permit closed orbits.

Keywords. Larmor precession and nutation; charged-particle orbits in electromagnetic fields; atom in a magnetic field.

PACS Nos 03·20; 32·30; 31·90; 32·60; 41·90; 46·90

1. Introduction

It appears that only one exact solution is available for the classical non-relativistic problem of a charged particle moving under the combined influence of an attractive central electric field and a uniform magnetic field. This special solution describes the circular orbit of a charged particle in a plane perpendicular to the applied uniform magnetic field. More general exact solutions for this problem do not exist although a variety of approximate solutions may be obtained through Larmor's theorem. Larmor's theorem as applied to this problem [1–5] implies that every bounded-orbit [6] of the charged particle moving in an inertial frame under the action of the central force field and a sufficiently weak magnetic field is approximately a bounded orbit in the same central field, but without the magnetic field, when viewed from an appropriate rotating frame called the Larmor frame. If we assume that the motion of the charged particle in such a precessing orbit generates a 'pointlike magnetic dipole-moment whose magnitude is not affected by the motion it undergoes' [4], then it follows that the magnetic dipole precesses uniformly with the Larmor frequency [7]. The assumption that the orbit is a pointlike magnetic moment is appropriate to permanent magnets or systems on an atomic or smaller scale and obviously is not valid for macroscopic charged particle orbits. Therefore, it is of some interest to study the motion of the magnetic moment $\boldsymbol{\mu}$ or the orbital angular momentum $\mathbf{L} = \boldsymbol{\mu}/\gamma$, where γ is the gyromagnetic ratio, in the case of macroscopic orbits.

Since the Larmor theorem is valid for macroscopic orbits also, we may expect that under the conditions of validity of the Larmor theorem, the motion of the vector **L** may be more complex than a simple precession unlike in the case of microscopic orbits. Secondly, since a precessing charged particle orbit is essentially a 'magnetic-top', we may expect the Larmor motion of **L** to involve nutation also in complete analogy with the problem of a massive top in a gravitational field. In fact, Landau and Lifshitz [8] show that in the case of a point charge executing bounded periodic motion under the influence of a central electrostatic field and a weak magnetic field, a suitably time-averaged motion of **L** is the familiar Larmor precession. In this paper, by restricting ourselves to a study of the motion of **L** in the Larmor approximation, we show that the detailed motion of **L** consists of a precession as well as a high frequency nutation.

2. Equation of motion for L and its solution in the Larmor approximation

The equation of motion of a negatively-charged particle [9] of charge $-q<0$ and mass m, moving in an inertial frame under the combined influence of an external central force field $f(r)\hat{\mathbf{r}}$ and a uniform magnetic field $\mathbf{B}$ is given by

$$\mathrm{d}(m\mathbf{v})/\mathrm{d}t = f(r)\hat{\mathbf{r}} - q\mathbf{v}\times\mathbf{B}/c, \tag{2.1}$$

where $\mathbf{r}$ is the position vector of the particle, $\hat{\mathbf{r}}$ is a unit vector along $\mathbf{r}$, $\mathbf{v}=\mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity of the particle and c is the speed of light in vacuum. (Note: We work in Gaussian CGS units.) On cross-multiplying this equation by $\mathbf{r}$, we get

$$\mathrm{d}\mathbf{L}/\mathrm{d}t = -2m\mathbf{r}\times(\mathbf{v}\times\boldsymbol{\omega}), \tag{2.2}$$

where $\mathbf{L}=\mathbf{r}\times m\mathbf{v}$ is the kinetic angular momentum of the particle,

$$\boldsymbol{\omega}\equiv q\mathbf{B}/2mc, \tag{2.3}$$

and $|\boldsymbol{\omega}|$ is the 'Larmor frequency'. Now, if we write the term $-2m\mathbf{r}\times(\mathbf{v}\times\boldsymbol{\omega})$ as the sum of two equal halves, express one of them using the Jacobi identity as

$$-m\mathbf{r}\times(\mathbf{v}\times\boldsymbol{\omega}) = m\mathbf{v}\times(\boldsymbol{\omega}\times\mathbf{r}) + m\boldsymbol{\omega}\times(\mathbf{r}\times\mathbf{v}),$$

and add it to the other in equation (2.2), we obtain

$$\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\omega}\times\mathbf{L} + \mathrm{d}\boldsymbol{\Lambda}/\mathrm{d}t, \tag{2.4}$$

where we have defined

$$\boldsymbol{\Lambda}\equiv m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{2.5}$$

Although (2.2) and (2.4) are the same, the latter is more convenient in discussing the motion of **L**. In fact (2.4) shows at once [10] that the motion of **L** would be a pure precession around **B** only when the second term $\mathrm{d}\boldsymbol{\Lambda}/\mathrm{d}t$ on the right hand side of (2.4) is either zero or is negligible in comparison with the first term $\boldsymbol{\omega}\times\mathbf{L}$. Since both these terms are of the same order in **B**, it is clear that the motion of **L** can never be a simple precession however weak the magnetic field **B** may be.

The vector $\boldsymbol{\Lambda}$ above has a simple interpretation from the standpoint of the so called Larmor frame, i.e., a reference frame which rotates with the angular velocity $\boldsymbol{\omega}$ relative to the inertial frame in which the problem is being studied. If we denote quantities referred to the Larmor frame by an asterisk (*) and the time-rates-of-change with respect to the inertial and Larmor frames by $\mathrm{d}/\mathrm{d}t$ and $\mathrm{d}^*/\mathrm{d}t$ respectively, then we have [5]

$$\mathrm{d}\mathbf{A}/\mathrm{d}t = \mathrm{d}^*\mathbf{A}/\mathrm{d}t + \boldsymbol{\omega} \times \mathbf{A},$$

which is a general relation valid for all vector functions $\mathbf{A}(t)$. Also, we assume that the origins of the two frames coincide, so that we have $\mathbf{r}(t) = r^*(t)$ and this relation together with the fact that $\boldsymbol{\omega}$ is a constant, leads to the following well-exploited relations for velocity and acceleration:

$$\mathbf{v} = \mathbf{v}^* + \boldsymbol{\omega} \times \mathbf{r}; \quad \mathbf{v} \equiv \mathrm{d}\mathbf{r}/\mathrm{d}t; \quad \mathbf{v}^* \equiv \mathrm{d}^*\mathbf{r}/\mathrm{d}t; \tag{2.6}$$

$$\mathbf{a} = \mathbf{a}^* + 2\boldsymbol{\omega} \times \mathbf{v}^* + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}); \quad \mathbf{a} \equiv \mathrm{d}\mathbf{v}/\mathrm{d}t; \quad \mathbf{a}^* \equiv \mathrm{d}^*\mathbf{v}^*/\mathrm{d}t. \tag{2.7}$$

In the same manner, we obtain the following relation connecting kinetic angular momenta in the two frames of reference

$$\mathbf{L} = \mathbf{L}^* + \boldsymbol{\Lambda}; \quad \mathbf{L} \equiv \mathbf{r} \times m\mathbf{v}; \quad \mathbf{L}^* \equiv \mathbf{r} \times m\mathbf{v}^*. \tag{2.8}$$

This relation, somehow, seems to have received little attention in the literature. The vector $\mathbf{L}^*$ in the above equation may be interpreted as the angular momentum relative to the rotating frame and the vector $\boldsymbol{\Lambda}$ as the angular momentum of transport. To appreciate the significance of $\boldsymbol{\Lambda}$ we may note that, for example, in the case of a rigid body, the vector $\mathbf{L}^* = \sum \mathbf{r} \times m\,\mathbf{v}^*$ (where the summation extends over all the constituent particles) vanishes in the body rest frame and therefore the entire angular momentum appears as the transport angular momentum $\boldsymbol{\Lambda}$ in this frame.

On using (2.6)–(2.8), (2.1) and (2.4), we obtain

$$m\mathbf{a}^* = m\mathrm{d}^{*2}\mathbf{r}/\mathrm{d}t^2 = f(r)\,\hat{\mathbf{r}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{2.9}$$

and

$$\mathrm{d}^*\mathbf{L}^*/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{\Lambda}, \tag{2.10}$$

which are exact equations corresponding to (2.1) and (2.2), but written in the rotating Larmor frame. In the so-called Larmor approximation which corresponds to situations in which the magnetic field $\mathbf{B}$ is sufficiently weak and orbit of the particle is bounded in the sense described earlier, we neglect the second term on the right hand side of (2.9) in comparison with the first and obtain

$$m\,\mathrm{d}^{*2}\mathbf{r}/\mathrm{d}t^2 \approx f(r)\,\hat{\mathbf{r}}. \tag{2.11}$$

Taking the vector product of this equation with $\mathbf{r}$ then yields

$$\mathrm{d}^*\mathbf{L}^*/\mathrm{d}t \approx 0, \tag{2.12}$$

which is the equation of motion of $\mathbf{L}^*$ in the Larmor approximation. Thus, in the Larmor approximation, $\mathbf{L}^*$ is a constant as viewed from the rotating frame, or, equivalently, it is a precessing vector satisfying

$$\mathrm{d}\mathbf{L}^*/\mathrm{d}t \approx \boldsymbol{\omega} \times \mathbf{L}^*, \tag{2.13}$$

as viewed from the inertial frame. It is gratifying to note that there is at least one angular momentum vector in the problem, namely $\mathbf{L}^*$, that executes a simple precessing motion in the Larmor approximation. However, it is **L**, and not $\mathbf{L}^*$, which is the angular momentum of the particle in the inertial frame and in view of (2.8), the motion of **L** (in the inertial frame) is more complex than a simple precession because **L** is the sum of a precessing vector $\mathbf{L}^*$ and the (moving) vector $\mathbf{\Lambda}$.

In passing, we make an interesting observation on the canonical angular momentum $\mathbf{L}_c \equiv \mathbf{r} \times (\mathbf{p} - q\mathbf{A}/c)$ of the charged particle in this problem. In a suitable gauge, the vector potential for a uniform magnetic field **B** may be written as $\mathbf{A} = -\mathbf{r} \times \mathbf{B}/2$. With this **A**, we obtain $\mathbf{L}_c = \mathbf{L} - \mathbf{\Lambda}$. Comparing this with (2.8) then shows that $\mathbf{L}_c = \mathbf{L}^*$. In view of this relation, (2.13) may also be interpreted as describing the Larmor precession of $\mathbf{L}_c$. Larmor's theorem in this form, for $\mathbf{L}_c$, may also be obtained directly without using the rotating frame [11].

We now proceed to determine the motion of **L** in the inertial frame. First, we mention very briefly about the only exact solution available for equation (2.1). This solution describes the uniform circular motion of a particle of charge $-q$ and mass m in the Coulomb force field of a positive charge q' at rest at the origin of coordinates of the inertial frame. With a denoting the radius of the circle and $|\omega|$ denoting the constant angular frequency, the solution mentioned is given by $\mathbf{r} = (a\cos\omega' t, a\sin\omega' t, 0)$, where $\omega' \equiv \omega \pm \sqrt{\omega_0^2 + \omega^2}$; $\omega = q|\mathbf{B}|/2mc$; $\omega_0 = \sqrt{qq'/ma^3} > 0$ and the z-axis of the inertial frame has been chosen such that $\mathbf{B} = (0, 0, B)$. A simple calculation then yields $\mathbf{L} = (0, 0, ma^2\omega') =$ constant. (For a further discussion of this solution see the following references: Purcell [12], Reitz *et al* [13] or Matveev [14].) Leaving aside this exact solution which anyway has a constant **L** and is therefore not of interest to us, we consider approximate solutions of equation (2.1) obtained through the Larmor theorem. (In fact, our interest in these solutions is restricted to a study of the motion of the angular momenta rather than the orbits themselves.) We have a simple prescription for obtaining the orbital angular momentum vector associated with any such solution of (2.1) obtained through the Larmor theorem: First, we pick up an arbitrary bounded-orbit solution $\mathbf{r} = \mathbf{r}_0(t)$ of the problem (2.11). (It is important to note that $\mathbf{r}_0(t)$ so chosen is in fact an exact central force orbit and not an approximate one as required by (2.11).) Then, using this vector $\mathbf{r}_0(t)$, we calculate vector field $\mathbf{\Lambda}(t)$ defined in (2.5). Next, we calculate the conserved orbital angular momentum vector $\mathbf{L}^* = \mathbf{r}_0 \times (m\,\mathrm{d}\mathbf{r}_0/\mathrm{d}t)$ of this orbit and use it to obtain the vector field $\mathbf{L} = \mathbf{L}^* + \mathbf{\Lambda}$, which is the required orbital angular momentum in the inertial frame (in the Larmor approximation).

We now use this prescription and calculate the **L** associated with an arbitrary, exact solution $\mathbf{r}_0(t)$ of (2.11). Let $\mathbf{r}_0(t)$ expressed in the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of the inertial frame be given by

$$\mathbf{r}_0(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{2.14}$$

Further, let the z-axis of the inertial frame be chosen such that **B** is parallel or antiparallel to **k**. Then

$$\boldsymbol{\omega} = \omega\mathbf{k}, \tag{2.15}$$

where $\omega > 0$ when **B** is parallel to **k** and negative otherwise. Next, using (2.14), (2.15) and the definition (2.5), we obtain

$$\mathbf{\Lambda} = m\mathbf{r}_0 \times (\boldsymbol{\omega} \times \mathbf{r}_0) = -m\omega\left\{xz\mathbf{i} + yz\mathbf{j} - (x^2 + y^2)\mathbf{k}\right\}. \tag{2.16}$$

Since we know that $\mathbf{L}^*$ is a precessing vector as seen from the inertial frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, and hence satisfies (2.13) (now exactly, as $\mathbf{r}_0(t)$ is an exact solution of (2.11)), we may take it to be the general solution of (2.13) given below:

$$\mathbf{L}^* = L_0(\sin\theta\sin\varphi\mathbf{i} - \sin\theta\cos\varphi\mathbf{j} + \cos\theta\mathbf{k}). \tag{2.17}$$

Here L_0 and θ are two constants of integration and [15]

$$\varphi = \omega t. \tag{2.18}$$

The angles $\theta =$ constant and $\varphi - \pi/2 = \omega t - \pi/2$ are evidently the polar and azimuthal angles of $\mathbf{L}^*$ with respect to the inertial frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and these angles clearly show that the vector $\mathbf{L}^*$ precesses around the z-axis with angular frequency $|\omega|$. Now using (2.8), (2.14) and (2.17), we get $\mathbf{L}$ to the desired Larmor approximation as

$$\mathbf{L} = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}, \tag{2.19}$$

where

$$L_x = L_0\sin\theta\sin\varphi - m\omega xz;\quad L_y = -L_0\sin\theta\cos\varphi - m\omega yz;$$

$$L_z = L_0\cos\theta + m\omega(x^2 + y^2). \tag{2.20}$$

These equations yield $\mathbf{L}$ once the central force orbit $\mathbf{r}_0(t)$ is specified as in (2.14). This solves the problem in principle. However, in actual calculations it is more convenient to specify the planar orbit $\mathbf{r}_0(t)$ in terms of two variables rather than in terms of the three variables x, y, z as done above. We do this by introducing the rotating Larmor frame $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ with corresponding coordinates, X, Y, Z, as follows [16]. The plane of $\mathbf{r}_0(t)$ is taken to be the $X - Y$ plane so that $\mathbf{L}^*$ is always along the Z-axis. Further, we choose the X-axis to lie along the line of nodes formed by the $X - Y$ plane with the plane of the orbit $\mathbf{r}_0(t)$ as shown in figure 1. Then, constructing the rotation matrix connecting the

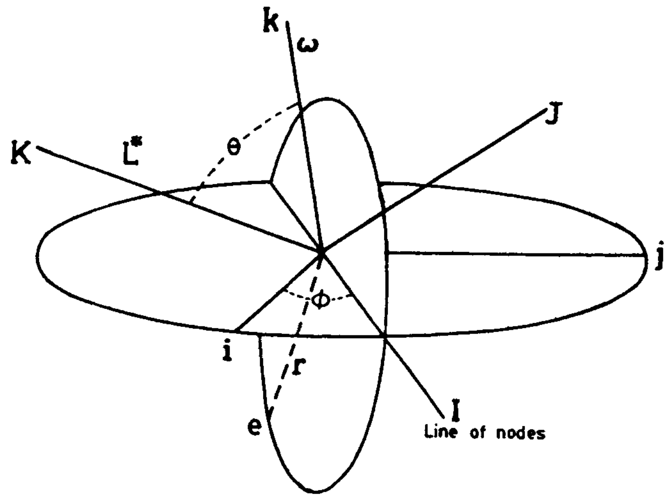


Figure 1. Euler angles specifying the orientation of the Larmor frame relative to the laboratory (inertial) frame.

two frames in terms of the Euler angles, we arrive at the coordinate transformation relating the two frames:

$$x = X\cos\varphi - Y\cos\theta\sin\varphi,\ \ y = X\sin\varphi + Y\cos\theta\cos\varphi,\ \ z = Y\sin\theta. \quad (2.21)$$

Substituting these in equation (2.20), we finally obtain the required formulae

$$L_x = L_0\sin\theta\sin\varphi - m\omega XY\sin\theta\cos\varphi + m\omega Y^2\sin\theta\cos\theta\sin\varphi,$$

$$L_y = -L_0\sin\theta\cos\varphi - m\omega XY\sin\theta\sin\varphi - m\omega Y^2\sin\theta\cos\theta\cos\varphi,$$

$$L_z = L_0\cos\theta + m\omega X^2 + m\omega Y^2\cos^2\theta. \quad (2.22)$$

The components of **L** relative to the rotating frame (**I**, **J**, **K**) are similarly found to be given by

$$\mathbf{L} = m\omega\sin\theta(-XY\mathbf{I} + X^2\mathbf{J}) + (L_0 + m\omega r_0^2\cos\theta)\mathbf{K}, \quad (2.23)$$

where $r_0^2 \equiv (X^2 + Y^2)$.

Before concluding this section, we restate the Larmor approximation in terms of a small dimensionless parameter. We may recall that for a bounded orbit $\mathbf{r} = \mathbf{r}(t)$, $r \equiv |\mathbf{r}|$ lies in the interval $0 < r_{\min} \leqslant r \leqslant r_{\max} < \infty$. Now, let the magnitude of the central force $f(r)\hat{\mathbf{r}}$ have its smallest value on the orbit, say, when $r = r_1$. Then, evidently, $|f(r)\hat{\mathbf{r}}| \geqslant |f(r_1)|$. We also note that $|m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| \leqslant m\omega^2 r_{\max}$. Thus, the Larmor approximation condition $|f(r)| \gg |m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})|$ may be expressed as

$$|f(r_1)| \gg m r_{\max}\omega^2. \quad (2.24)$$

If we now define a characteristic angular frequency $\Omega > 0$ associated with the bounded orbit of the charged particle by

$$m\Omega^2 r_{\max} = |f(r_1)|, \quad (2.25)$$

the condition (2.24) reads

$$\Omega \gg |\omega|. \quad (2.26)$$

Thus, the Larmor approximation describes a situation in which the dimensionless parameter $|\omega|/\Omega$ is 'small'. Incidentally, the modulus of ω appears in (2.26) above because ω, we may recall, could be positive as well as negative depending on whether **B** is parallel or antiparallel to **k**.

3. Tracing the motion of L in the case of elliptic Larmor frame orbits

In this section, we study the motion of the unit vector [17] $\hat{\mathbf{L}}$ along **L** in the case of such inertial frame orbits $\mathbf{r}(t)$ for which the corresponding Larmor frame orbit (LFO) $\mathbf{r}_0(t)$ is an ellipse. We know that (Bertrand's theorem: see Goldstein [4], p. 93, or Landau and Lifshitz [1], p. 32) the attractive inverse square law force and the Hooke law force (also called the space oscillator force) are the only two central forces which permit closed, and hence elliptic orbits. Hence our discussion would cover the cases of both these forces.

To study the motion of $\hat{\mathbf{L}}$, we determine the azimuth φ_L and the polar angle θ_L of the vector **L** as functions of time. The angle φ_L is given by $\tan\varphi_L = L_y/L_x$. On using (2.22) and performing some elementary trigonometric rearrangement, we may invert this relation as

$$\varphi_L = \varphi - \frac{\pi}{2} + \eta = \omega t - \frac{\pi}{2} + \eta, \tag{3.1}$$

where we have defined $\eta = \eta(t)$ by

$$\tan\eta = -(m\omega/L_0)XY(1 + \alpha Y^2 \cos\theta)^{-1}. \tag{3.2}$$

On the other hand, the polar angle θ_L of $\mathbf{L}$ may be calculated from the relation

$$\cos\theta_L = L_z/|\mathbf{L}|. \tag{3.3}$$

Now, if the LFO $\{X(t), Y(t)\}$ is specified, we may calculate the angles θ_L and φ_L as functions of time by using equations (3.1)–(3.3). Then, the motion of $\hat{\mathbf{L}}$ may be studied by considering the vector

$$\mathrm{L}_\perp \equiv \sin\theta_L \cos\varphi_L \mathbf{i} + \sin\theta_L \sin\varphi_L \mathbf{j}, \tag{3.4}$$

which is the projection of $\hat{\mathbf{L}}$ onto the x–y plane of the inertial frame. The tip of this vector would trace a circle if the motion of $\hat{\mathbf{L}}$ is a pure precession. Any non-circular tracing clearly indicates the presence of nutation. This is as far as we can go with a general central force field. Thus, we consider the two specific cases of interest.

3.1 *Coulomb law elliptic orbits*

We begin by recalling some essential properties of an elliptic orbit ([1], pp. 32–39) of a negatively-charged particle of charge $-q < 0$ and mass m moving in the Coulomb field of another positively-charged particle of charge $q' > 0$ at rest at the coordinate orgin (which also happens to be one of the two foci of the elliptic orbit). The parametric equations describing such an elliptic orbit are

$$X = a\rho;\ \ Y = b\sigma;\ \ \rho \equiv (\cos\xi - e);\ \ \sigma \equiv \sin\xi;\ \ t = (T_0/2\pi)(\xi - e\sin\xi), \tag{3.5}$$

where $X = a\rho(t)$ and $Y = b\sigma(t)$ are the Cartesian coordinates of the particle tracing the ellipse. The constants a, b and e are respectively the semi-major axis, semi-minor axis and the eccentricity of the ellipse. The eccentric angle $\xi = \xi(t)$ increases by 2π for one complete passage of the charged particle round the ellipse. The origin of time has been chosen such that $\xi = 0$ at $t = 0$ and $\xi = 2\pi$ at $t = T_0$, where T_0 is the period of the elliptic motion. The angular frequency ω_0 associated with T_0, is given by $\omega_0 = 2\pi/T_0 = L_0/mab = \sqrt{-qq'/ma^3}$, where the positive constant L_0 is the magnitude of the conserved angular momentum associated with the elliptic orbit. Lastly, we note that the variables ρ and σ in (3.5) are dimensionless and have the ranges $-(1+e) \leqslant \rho \leqslant (1-e)$ and $-1 \leqslant \sigma \leqslant 1$.

Now, using (3.5), (3.3) and (2.22), we obtain

$$\cos\theta_L = \frac{k + \varepsilon\{(a/b)\rho^2 + (bk^2/a)\sigma^2\}}{\sqrt{1 + 2k\varepsilon\{(a/b)\rho^2 + (b/a)\sigma^2\} + \varepsilon^2\{(a^2/b^2)\rho^4 + (1+k^2)\rho^2\sigma^2 + (b^2k^2/a^2)\sigma^4\}}} \tag{3.6}$$

where we have denoted $\cos\theta$ by k and have introduced the dimensionless parameter $\varepsilon \equiv \omega/\omega_0$. If we now observe that ε and θ are constants and the variables ρ and σ are periodic under $\xi \to \xi + 2\pi$ which corresponds to $t \to t + T_0$ (see remarks following (3.5)), it follows that $\cos\theta_L$ given by (3.6) is a periodic function of time having the period T_0. This means that, in the Larmor approximation, **L** nutates with a time-period which is precisely the orbital period T_0 of the charged particle in its elliptic LFO.

Next, since the function $\eta(t)$ defined through (3.2) is also periodic under $t \to t + T_0$, we note that whenever $T \equiv 2\pi/|\omega| = NT_0$ where N is an integer, the azimuth φ_L of **L** changes by 2π in a time T during which θ_L, which has a time period $T_0 = T/N$, would go through exactly N cycles. In other words, when $T = NT_0$, the tip of the unit vector $\hat{\mathbf{L}}$ would trace a closed curve. When ω and ω_0 are not commensurate, the tip of $\hat{\mathbf{L}}$ does not trace a closed curve. In any case, the motion of $\hat{\mathbf{L}}$ in the Larmor approximation is a composite motion involving both precession and nutation.

Regarding the nutation frequency, we may note that it is given by ω_0 for elliptic paths of non-zero eccentricities. Interestingly, when e becomes zero, i.e., when the LFO is a circle, the nutation frequency changes to $2\omega_0$. This happens because, when $e = 0$, $\cos\theta_L$ given by (3.6) becomes increasingly periodic; its period changes from 2π to π for ξ (or equivalently, T_0 to $T_0/2$ for t), as only even powers of ρ and σ appear in it. We also note that since the maximum separation $r_{\max}$ between q' and $-q$ on the ellipse (3.5) is $a(1+e)$, the characteristic frequency Ω of (2.25) becomes $\Omega = \omega_0(1+e)^{3/2}$. Thus the Larmor approximation condition (2.26) now requires $\omega_0 \gg (1+e)^{3/2}|\omega|$. As a consequence, the nutation frequencies (ω_0 or $2\omega_0$) are very large compared to the Larmor frequency $|\omega|$.

An estimate of the magnitude of $\cos\theta_L$, may be obtained as follows. We note that if we retain only terms of the order $O(\varepsilon)$ in (3.6), and substitute back for ε as ω/ω_0, we obtain $\cos\theta_L = \cos\theta + (\omega a/\omega_0 b)\sin^2\theta\rho^2 + O(\varepsilon^2)$. Incidentally, this formula shows that

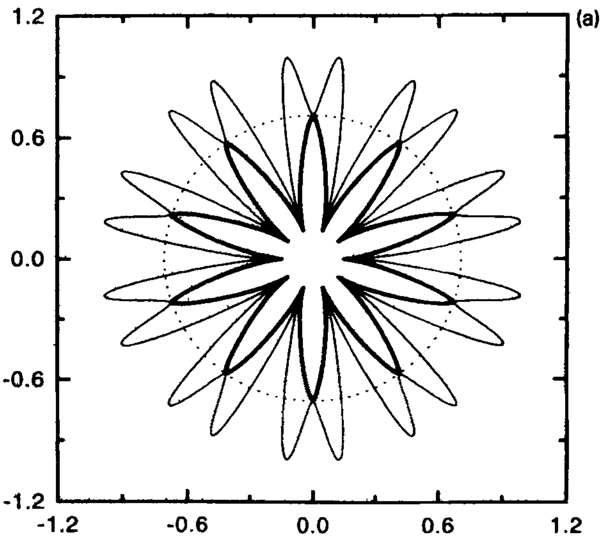
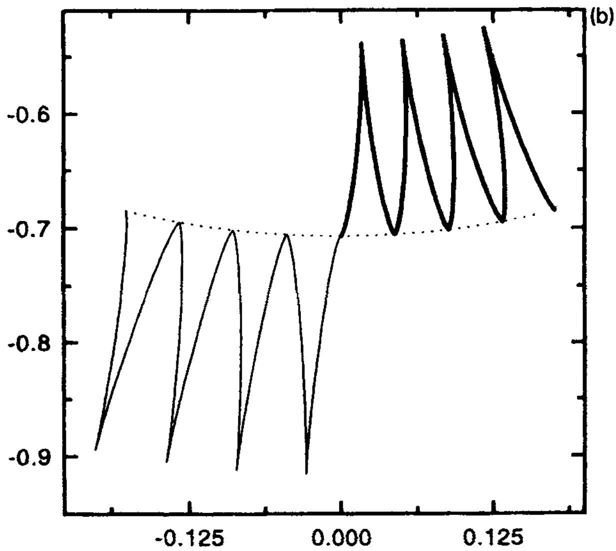
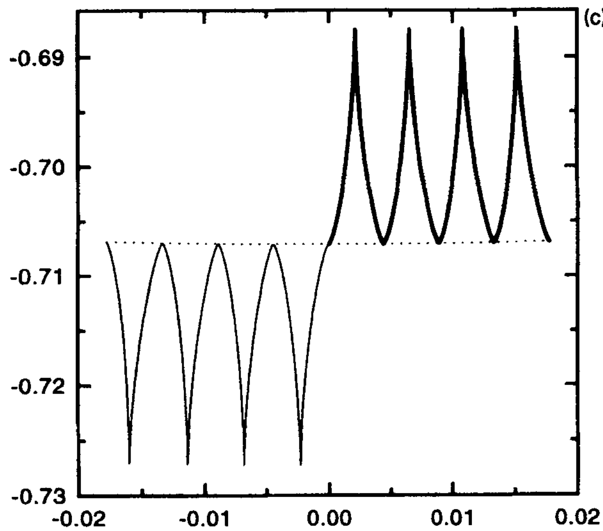


Figure 2a.





Figures 2b, c.

$\cos\theta_L > \cos\theta$ when $\omega > 0$ and $\cos\theta_L < \cos\theta$ when $\omega < 0$. Also, observing that $(\rho^2)_{\max} = (1+e)^2$ and $(\rho^2)_{\min} = 0$, we obtain

$$(\cos\theta_L)_{\max} - (\cos\theta)_{\min} = \pm(\omega a/\omega_0 b)(1+e)^2 \sin^2\theta + O(\omega^2), \tag{3.7}$$

where the + sign corresponds to $\omega > 0$ and the − sign to $\omega < 0$. It is important to note that the leading term on the right hand side of (3.7) is linear in ω and hence in the magnetic field strength $|\mathbf{B}|$. Equation (3.7), however, does not yield an estimate of the nutation amplitude $\equiv (\theta_L)_{\max} - (\theta)_{\min}$. It may be obtained numerically by calculating $\cos\theta_L$ as a function of the angle ξ and inverting.

In figure 2, we have shown some model precession-nutation curves for the Coulomb case. These curves have been drawn by setting $L_0 = 1$, $\omega = \pm 1$ and $\theta = \pi/4$. When $\theta = \pi/4, \theta_L$ also lies in $(0, \pi/2)$ so that $\sin\theta_L$ is positive. Therefore it may be calculated as $\sqrt{1-\cos^2\theta_L}$ through equation (3.6). The angle φ_L has been calculated from (3.1) and (3.2). Further, we have chosen three convenient values for ω_0 namely 10, 100 and 1000 which yield closed nutation curves. In each of the figures 2(a), 2(b) and 2(c), we have shown three different curves. The heavy-solid curve corresponds to $\omega = 1$, the light-solid curve corresponds to $\omega = -1$ whereas the (dashed) circle of radius $\sin\theta = 1/\sqrt{2}$ does not represent any real motion of **L**, but has been drawn only for the sake of reference. (Of course, it corresponds to the circle of $(\cos\theta_L)_{\min} = \cos\theta = 1/\sqrt{2}$ when $\omega > 0$ and to the circle of $(\cos\theta_L)_{\max} = \cos\theta = 1/\sqrt{2}$ when $\omega < 0$). The computed values of the nutation amplitude are as follows. For $\omega = 1$, it is 36·4, 12·4 and 1·6 degrees when $\omega_0 = 10, 100$, and 1000 respectively. Similarly, for $\omega = -1$, it is given by 122·8, 21·4 and 1·6 degrees when $\omega_0 = 10, 100$ and 1000 respectively. These values clearly show that the nutation amplitude decreases as $|\varepsilon|$ decreases. In fact, the precession slows down and the nutation dies out as $|\varepsilon|$ decreases and eventually when $|\varepsilon|$ touches zero, i.e., when $\mathbf{B} = 0$, **L** gets arrested at $\theta_L = \theta$, and $\varphi_L = -\pi/2$ (see (3.1), (3.6) and the remarks following (2.18)).

3.2 *Hooke law elliptic orbits*

Writing the Hooke law central force as $f(r)\hat{\mathbf{r}} = -m\omega_0^2\mathbf{r}$, we obtain the following parametric equations to the elliptic orbit (with the centre of attraction $\mathbf{r} = 0$ at the centre of the ellipse) ([1], p. 70)

$$X = a\cos\xi;\quad Y = b\cos(\xi + \delta);\quad \xi = \omega_0 t, \tag{3.8}$$

where we have set $\xi = 0$ at $t = 0$. The other constant of integration δ, when chosen appropriately, gives the various familiar Lissajous figures. We have specifically

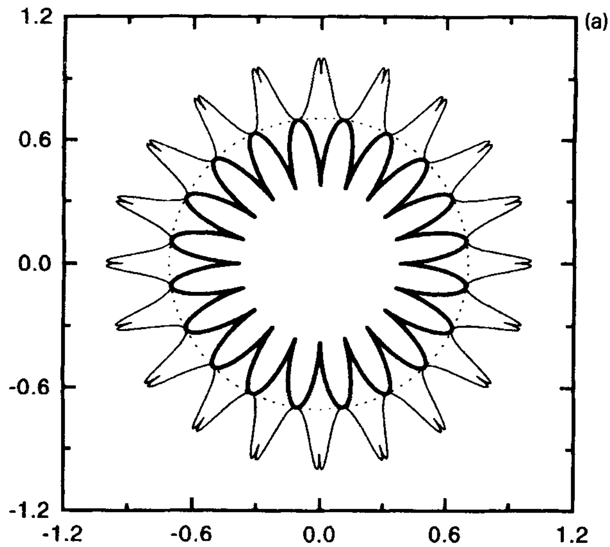
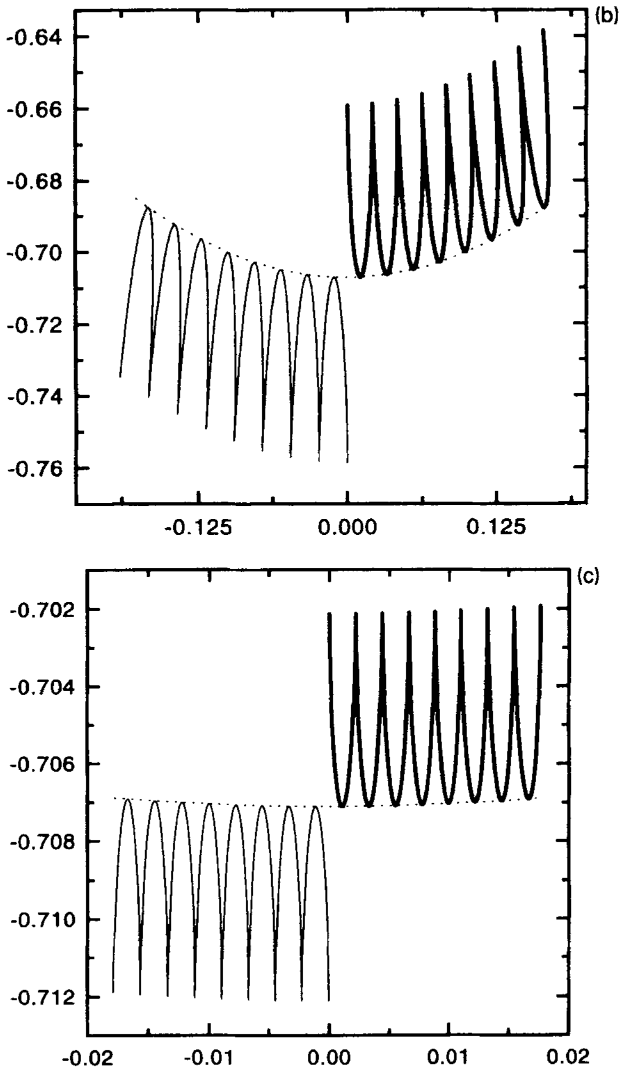


Figure 3a



Figures 3b, c.

Figures 2 and 3. Precession-nutation curves in the Coulomb (figure 2) and Hooke cases (figure 3): These are curves traced by the tip of the angular momentum vector projected on to the plane perpendicular **B** (i.e., the x–y plane) of the inertial frame. **L** has been scaled such that its magnitude when $\mathbf{B} = 0$, i.e. L_0, is unity. The curves show nutation of considerable amplitude besides precession.

considered the case $\delta = -\pi/2$ which yields $X = a\cos\omega t$ and $Y = b\sin\omega t$ representing a central ellipse. Figures 3(a), to 3(b) show some model precession nutation curves drawn with $a = 10$ units and $b = 1$ unit. Further $\omega_0 = 10, 100$ and 1000 radians/second respectively in the figures 3(a), 3(b) and 3(c). As before, in each figure, the heavy-solid curve corresponds to $\omega = 1$, the light-solid curve corresponds to $\omega = -1$ whereas the (dashed) circle of radius $\sin\theta = 1/\sqrt{2}$ has been drawn for the sake of reference. Lastly, we note that the computed values of the nutation amplitude are as follows. For $\omega = 1$, it is 22·5, 3·8 and 0·4 degrees when $\omega_0 = 10, 100$ and 1000 respectively. Similarly, for $\omega = -1$, it is given by 67·5, 4·4 and 0·4 degrees when $\omega_0 = 10, 100$ and 1000 respectively.

Acknowledgements

The authors wish to thank the referee for his comments which have helped in improving the presentation of the paper considerably. They also thank Dr M A Sridhar for generating the nutation curves on a computer.

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[6] We call an orbit $\mathbf{r} = \mathbf{r}(t)$ in which $|\mathbf{r}|$ remains finite and non-zero, i.e., $0 < r_{\min} \leqslant |\mathbf{r}| \leqslant r_{\max}$, a bounded-orbit
[7] See equation (2.3) for definition
[8] L D Landau and E M Lifshitz, *The classical theory of fields*, third edition (Pergamon Press, Oxford, 1971) pp. 105–107
[9] Although we specifically consider a particle of charge $-q$, $q > 0$, in our discussion, obviously with the electron in mind, the discussion may be easily modified to cover the case of a positively-charged particle by allowing q to be negative
[10] We may note that, in general, a vector function $\mathbf{A} = \mathbf{A}(t)$ which precesses around another constant vector $\mathbf{C}$ with the angular frequency $|\mathbf{C}|$ satisfies the differential equation $\mathrm{d}\mathbf{A}/\mathrm{d}t = \mathbf{C} \times \mathbf{A}$
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[15] Actually, we get $\varphi = \omega t +$ constant, but we have chosen the constant to be zero in (2.18)
[16] Note the deviation in notation here: As per the convention adopted in arriving at equations (2.6) to (2.8), we must rather have denoted ($\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$) and X, Y, Z respectively by $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ and $\mathbf{x}^*$, $\mathbf{y}^*$, $\mathbf{z}^*$. But we have deliberately chosen the former set which is simpler. With the exception of these, all other quantities in the Larmor frame are marked by an asterisk as before
[17] We consider the unit vector $\hat{\mathbf{L}}$ as we are interested in studying the change in the orientation of $\mathbf{L}$ with time, rather than in its length