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Noisy nonlinear coupled equations — Some new insights

J K BHATTACHARJEE, ANITA MEHTA* and J M LUCK†
Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700 032, India
*Present Address: The Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 0HE, England
*Permanent Address: S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Calcutta 700 091, India
†C.E.A. Saclay, Service de Physique Théorique, 91191 Gif-sur-Yvette cedex, France

Abstract. We discuss the use of coupled nonlinear stochastic differential equations to model the dynamics of complex systems, and present some analytical insights into their critical behaviour. These concern in particular the role of infrared divergences which show up in a self-consistent resummation of perturbation theory (mode-coupling approximation), and their effects on critical exponents obtained in earlier work.

Keywords. Self organised criticality; Langevin equations; self consistent mode coupling.

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1. Introduction

The theory of self-organized criticality (SOC) [1] provided the original impetus for the study of systems with inherent complexity – systems, therefore, which could be observed in many metastable states, with, frequently, strong hysteretic behaviour. The chosen paradigm for SOC was the sandpile; it turned out, however [2, 3], that real sandpiles did not manifest the scale invariance postulated in that theory, and this proved to be the motor for a significant body of work on granular media [4].

The absence of scale invariance was postulated [5] to be due to the presence of competing dynamical mechanisms, one independent-particle and one collective, for the reorganisation of grains in a sandpile. In addition to numerous simulations [6, 7] which have supported this picture, there have been experiments recently [8] which have shown its validity.

This picture, however, necessitates the invention of new classes of models which are able to capture its intricacies. We have been involved with the formulation of discrete models which have quantified this picture, firstly, in the language of cellular automata [9] and then in that of coupled-map lattice models [10]; each of these has a representation for the two relaxation modes referred to above which turns out to be intimately related to the presence of times and lengths characteristic of the system, and hence the absence of scale invariance. At the continuum end of things, we came up with, first, a system of noisy nonlinear coupled equations at the macroscopic level [11] and next the equations [4, 12] that form the subject of the present study, which were designed to be a local representation of sandpile dynamics. The description of these equations forms the subject matter of § 2, which also contains a summary of the main results obtained thus far on their critical behaviour [12].

Section 3 concerns the analysis of these equations using a self-consistent resummation of perturbation theory (mode-coupling approximation), which provides a satisfactory explanation for the numerical results recalled in § 2. Finally, in § 4 we make some concluding remarks.

2. Coupled local equations for a driven sandpile – the MLN equations

There have been a number of recent approaches [13–17] which have attempted to model the surfaces of driven sandpiles; starting from the most basic dynamical mechanism, that of diffusion [13], these have incorporated various subtleties, ranging from lateral growth [14] to disorder [17]. However, they have nearly always involved one variable, the local height $h(\mathbf{x}, t)$ of the surface, rather than any form of grain-cluster coupling. In our approach [12], however, the underlying picture is of currents of grains moving down the slope, knocking out bumps and filling in holes in clusters on the landscape. The effective coordinate representing clusters is the local height $h(\mathbf{x}, t)$, while the moving grains are represented by their local density $\rho(\mathbf{x}, t)$ – in a driven sandpile these coordinates will clearly be coupled, and the specific form we have chosen for this coupling follows later.

We now review some general facts about rough interfaces [18, 19]. Three critical exponents, α, β, and z, characterize the spatial and temporal scaling behaviour of a rough interface. They are conveniently defined by considering the (connected) two-point correlation function of the heights

$$S(\mathbf{x}-\mathbf{x}', t-t') = \langle h(\mathbf{x}, t)h(\mathbf{x}', t')\rangle - \langle h(\mathbf{x}, t)\rangle\langle h(\mathbf{x}', t')\rangle. \tag{2.1}$$

We have

$$S(\mathbf{x}, 0) \sim |\mathbf{x}|^{2\alpha} \quad (|\mathbf{x}| \to \infty), \qquad S(0, t) \sim |t|^{2\beta} \quad (|t| \to \infty), \tag{2.2}$$

and more generally

$$S(\mathbf{x}, t) \approx |\mathbf{x}|^{2\alpha} F(|t|/|\mathbf{x}|^{z}) \tag{2.3}$$

in the whole long-distance scaling regime ($\mathbf{x}$ and t large). The scaling function F is universal in the usual sense; α and $z = \alpha/\beta$ are respectively referred to as the roughness exponent and the dynamical exponent of the problem.

Our stochastic dynamical equations have the following general form in one space dimension [12]

$$\partial h/\partial t = D_h \nabla^2 h - T + \eta_h(x, t), \tag{2.4}$$

$$\partial \rho/\partial t = -\nabla j + D_\rho \nabla^2 \rho + T + \eta_\rho(x, t). \tag{2.5}$$

For the time being we set

$$T = -\kappa\rho\nabla^2 h - \lambda\rho(\nabla h)_+ - \mu\rho\nabla h - \nu(\nabla h)_- \tag{2.6}$$

and

$$j = -\gamma\rho(\nabla h)_- \tag{2.7}$$

We have introduced the notations $\nabla = \partial/\partial x$, and

$$z_+ = z\,\Theta(z) = \begin{cases} z & \text{for} \quad z \geq 0, \\ 0 & \text{for} \quad z \leq 0, \end{cases} \quad z_- = z(1-\Theta(z)) = \begin{cases} 0 & \text{for} \quad z \geq 0, \\ z & \text{for} \quad z \leq 0, \end{cases} \tag{2.8}$$

with $\Theta(z)$ being Heaviside's step function.

The content of our dynamical equations is as follows:

- The first term on the right hand side of (2.4) represents the rearrangement of clusters in the presence of an applied noise and is the conventional EW [13] term, representing diffusion.
- The second block of terms, T, represents grain-cluster exchange, where:
  - — The term $\kappa\rho\nabla^2 h$ represents diffusive motion of the heights mediated by the flowing grains.
  - — The term $\mu\rho\nabla h$ also represents intercluster motion of grains, but in a slope-dependent fashion.
  - — The term $\nu(\nabla h)_-$ represents the spontaneous generation of flowing grains whenever the local slope is larger than critical; this exists even in the absence of flowing grains and is meant to represent the effect of tilting a stationary sandpile.
  - — The term $\lambda\rho(\nabla h)_+$ is a representation of the effect of the boundary layer, in the sense that it enables the reconversion of moving to stationary grains; it compensates for the effect of the tilt term by limiting the release of flowing grains. Although originally conceived chiefly as a regulator [12], it has a definite physical meaning and, as we shall see, is responsible for the anomalous roughening of the interface.
- The first term in (2.5), $-\nabla j$, represents the variation in ρ due to the nonuniformity of the current of flowing grains, in such a way that the total number of particles is conserved. The current $j(x,t)$ has been written in (2.7) as the product of the number of mobile grains by their velocity, the latter being assumed to read $v = -\gamma(\nabla h)_-$.
- The second term in (2.5) represents the diffusive relaxation of the flowing grains, and is a crude way of representing intergrain collisions.
- The source terms $\eta_h(x,t)$ and $\eta_\rho(x,t)$ are taken to be two independent Gaussian white noises, such that

$$\begin{cases} \langle \eta_h(x,t)\eta_h(x',t')\rangle = 2\Delta_h\delta(x-x')\delta(t-t'), \\ \langle \eta_\rho(x,t)\eta_\rho(x',t')\rangle = 2\Delta_\rho\delta(x-x')\delta(t-t'). \end{cases} \tag{2.9}$$

  Pouring grains onto a sandpile should be represented by noise only in ρ, while shaking it would be represented by noise in h.

We describe below the numerical results obtained in earlier work [12] on some special cases of (2.4), (2.5).

2.1 *The asymmetric situation (cases 1 to 3)*

This situation is the most commonly encountered one, of a (sloping) sandpile with a preferred direction of flow. The dynamical equations read, on setting $D_h = D_\rho = \lambda = 1$ and $\kappa = \mu = 0$,

$$\text{Cases 1 to 3}: \quad \begin{cases} \partial h/\partial t = \nabla^2 h - T + \eta_h(x,t), \\ \partial \rho/\partial t = \nabla^2 \rho + T + \eta_\rho(x,t), \\ \quad T = -\rho(\nabla h)_+ - \nu(\nabla h)_-. \end{cases} \tag{2.10}$$

We have considered [12] the following three cases:

- Case 1: noise in h ($\Delta_h > 0$, $\Delta_\rho = 0$).
- Case 2: noise in ρ ($\Delta_h = 0$, $\Delta_\rho > 0$).
- Case 3: noise in h and ρ ($\Delta_h > 0$, $\Delta_\rho > 0$).

Non-trivial long-range spatial and temporal critical fluctuations are observed for both species h and ρ in these three cases. Case 1 is of particular interest, as it is the most directly comparable with the well-known EW [13] and KPZ [14] models. We observe a rougher behaviour of the h-profile than the two aforenamed. This effect is very pronounced in the spatial direction ($\alpha_h = 0.94 \pm 0.07$ is to be compared with $1/2$ in both cases), and still appreciable in the temporal one ($\beta_h = 0.43 \pm 0.04$ is to be compared with $1/3$ and $1/4$). This anomalous roughening is due, as we will see by the analysis in the next section, to the effect of the $\lambda\rho(\nabla h)_+$ term.

We have also investigated numerically [12] a simpler variant of the above (referred to therein as case 1a) – these were essentially identical, apart from notational differences, to the equations analysed by another group [20]. They read

$$\text{Case 1a}: \quad \begin{cases} \partial h/\partial t = \nabla^2 h - T + \eta_h(x,t), \\ \partial \rho/\partial t = \nabla^2 \rho + T + \eta_\rho(x,t), \\ \quad T = -\mu\rho\nabla h. \end{cases} \tag{2.11}$$

The critical exponents for this case are also presented in table 1.

2.2 *The symmetric situation (cases 4 to 6)*

We now turn to the description of the symmetric version of our dynamical equations – these have the $x \leftrightarrow -x$ symmetry, and describe the surface of a sandpile which is flat on average. The dynamical equations now read in full generality

$$\text{Cases 4 to 6}: \quad \begin{cases} \partial h/\partial t = D_h\nabla^2 h - T + \eta_h(x,t), \\ \partial \rho/\partial t = -\nabla j + D_\rho\nabla^2 \rho + T + \eta_\rho(x,t), \\ \quad T = -\kappa\rho\nabla^2 h - \lambda\rho|\nabla h| + \nu(|\nabla h| - C)_+, \\ \quad j = \gamma\rho \,\text{sign}\,(\nabla h)\,(|\nabla h| - C)_+, \end{cases} \tag{2.12}$$

where C is a slope threshold. In order to investigate the critical regime, we simplify the above equations by setting $\gamma = C = 0$. We again consider three cases:

- Case 4: noise in h ($\Delta_h > 0$, $\Delta_\rho = 0$).
- Case 5: noise in ρ ($\Delta_h = 0$, $\Delta_\rho > 0$).
- Case 6: noise in h and in ρ ($\Delta_h > 0$, $\Delta_\rho > 0$).

In cases 4 and 6 [12], we observe $\alpha_h \approx \beta_h \approx 0.40$ and $z_h = 1$ within error bars, just as in case 1a, whereas $\rho(x,t)$ does not exhibit divergent fluctuations: the structure factors $S_\rho(q,0)$ and $S_\rho(0,\omega)$ rather saturate to constant values for small enough values of wavevector q or frequency ω, implying the fast decay of $\rho - \rho$ correlations at long separations in space and in time. In case 5, where noise is present only in the equation for ρ, we find that h gets frozen, soon after the initial period has elapsed, into a nontrivial rough landscape, entirely inherited from the transient period, and therefore characterised by a roughness exponent $\alpha \approx 0.40$. The evolution of ρ which then takes place, with this frozen height configuration as a background, is effectively linear, and the EW exponents are observed, in spite of the background.

3. Mode-coupling analysis

In this section we provide some analytical insight into the scaling behaviour of the various cases of coupled Langevin equations discussed in the previous section. Our analysis is based on a self-consistent resummation of perturbation theory at the one-loop level, introduced long ago in the context of fluid mechanics [21], and usually referred to as the mode-coupling approximation. The main outcome will be that those models exhibit infrared singularities that control their scaling behaviour.

We illustrate this point by analysing case 1a in detail. The two coupled equations for the variables h and ρ are given in (2.11) and (2.9). We denote by $S_h(k,\omega)$ the double Fourier transform of the correlation function (2.1), such that

$$\langle h(k,\omega)h(k',\omega')\rangle = \delta(k+k')\delta(\omega+\omega')S_h(k,\omega) \tag{3.1}$$

and by $G_h(k,\omega)$ the Green's function, or propagator, of (2.11) for h, averaged over the noise, and similar notations for the ρ-field.

In the absence of coupling ($\mu = 0$), (2.11) degenerate into two independent free (EW) models in the variables h and ρ. The Green's functions and correlation functions then read

$$G_h^{(0)-1}(k,\omega) = -i\omega + k^2,$$
$$S_h^{(0)}(k,\omega) = \frac{2\Delta_h}{\omega^2 + k^4}, \tag{3.2}$$

and similar expressions for $G_\rho^{(0)}$ and $S_\rho^{(0)}$.

The effect of the coupling μ is to dress the propagator, namely to change it from the free one, given in (3.2), to the full one, $G_h(k,\omega)$. Dyson's equation yields

$$G_h^{-1}(k,\omega) = G_h^{(0)-1}(k,\omega) + \Sigma_h(k,\omega), \tag{3.3}$$

where $\Sigma_h(k,\omega)$ is the self-energy, or mass operator [22].

Within the self-consistent one-loop (mode-coupling) approximation, where the internal lines of the self-energy graph are fully dressed, we have

$$\Sigma_h(k,\omega) = \mu^2 \int \frac{\mathrm{d}k'}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} S_\rho(k',\omega') k(k-k') G_h(k-k',\omega-\omega'). \tag{3.4}$$

Now we make the following scaling ansatz about $\Sigma_h(k,\omega)$ and $S_h(k,\omega)$

$$\begin{aligned} \Sigma_h(k,\omega) &\approx k^{z_h}\sigma(\omega/k^{z_h}), \\ S_h(k,\omega) &\approx k^{-1-2\alpha_h-z_h} S(\omega/k^{z_h}), \end{aligned} \qquad (k>0) \tag{3.5}$$

and similarly for $\Sigma_\rho(k,\omega)$ and $S_\rho(k,\omega)$, with exponents z_ρ and α_ρ. The determination of z_h involves the computation of the static (zero-frequency) self-energy $\Sigma_h(k) = \Sigma_h(k,\omega=0)$. We assume $z_h < 2$, so that $\Sigma_h(k)$ dominates k^2 in the scaling region [this hypothesis will be checked a posteriori]. Hence in (3.3) we have

$$G_h(k,\omega)^{-1} \approx -i\omega + \Sigma_h(k,\omega). \tag{3.6}$$

Accordingly, we explore the integral on the right-hand-side of (3.4) at zero external frequency. The integral has an infrared divergence when the momentum k' of the correlation function goes to zero. Indeed in this limit $k(k-k') \approx k^2$, and

$$\begin{aligned} G_h^{-1}(k-k',-\omega') &\approx i\omega' + \Sigma_h(k-k') \\ &\approx i\omega' + \Sigma_h(k) \\ &\approx \Sigma_h(k). \end{aligned} \tag{3.7}$$

Because of the scaling laws (3.5) we effectively explore the low-frequency behaviour when we explore low values of k'. The self-energy of (3.4) now becomes

$$\begin{aligned} \Sigma_h(k) &\approx \frac{\mu^2 k^2}{\Sigma_h(k)} \int \frac{\mathrm{d}k'}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} S_\rho(k',\omega') \\ &\sim \frac{\mu^2 k^2}{\Sigma_h(k)} \int \frac{\mathrm{d}k'}{2\pi} |k'|^{-1-2\alpha_\rho}, \end{aligned} \tag{3.8}$$

since

$$\int \frac{\mathrm{d}\omega}{2\pi} S_h(k,\omega) \sim |k|^{-1-2\alpha_h}, \tag{3.9}$$

and a similar formula for the ρ-field, as a consequence of (3.5). The integral on the right-hand-side of (3.8) clearly exhibits an infrared divergence. For $\alpha_\rho \geq 0$, the integral diverges at small k'. If we regularise the integral by introducing as an infrared cutoff a smallest wavevector k_0, (3.8) yields

$$\Sigma_h(k) \sim \mu |k| k_0^{-\alpha_\rho}. \tag{3.10}$$

The above result at zero frequency implies $z_h = 1$; an identical analysis yields $z_\rho = 1$.

Now, in order to determine α_h, we investigate the correlation function $S_h(k,\omega)$. Again within the self-consistent one-loop approximation, we have

$$S_h(k,\omega)\approx\frac{1}{\omega^2+k^4}+\mu^2\int\frac{\mathrm{d}k'}{2\pi}\int\frac{\mathrm{d}\omega'}{2\pi}S_\rho(k',\omega')(k-k')^2S_h(k-k',\omega-\omega')$$

$$\approx\frac{1}{\omega^2+k^4}\left[1+\mu^2\int\frac{\mathrm{d}k'}{2\pi}\frac{(k-k')^2}{|k-k'|^{1+2\alpha_h}}\frac{1}{|k'|^{1+2\alpha_\rho}}\frac{\Sigma_\rho(k')+\Sigma_h(k-k')}{\omega^2+(\Sigma_\rho(k')+\Sigma_h(k-k'))^2}\right]. \tag{3.11}$$

Using (3.9), we have

$$\frac{1}{k^{1+2\alpha_h}}\approx\frac{1}{k^2}+\mu^2\int\frac{\mathrm{d}k'}{2\pi}\frac{(k-k')^2}{(k-k')^{1+2\alpha_h}}\frac{1}{k'^{1+2\alpha_\rho}}. \tag{3.12}$$

The low-momentum divergence of the integral on the right-hand-side of (3.12) implies that the integral scales as $|k|^{1-2\alpha_h}k_0^{-2\alpha_\rho}$, which is much smaller than $k^{-1-2\alpha_h}$ in the long-wavelength limit. Thus α_h remains at its EW value, namely $\alpha_h=1/2$. An identical argument on the ρ-correlation yields $\alpha_h=\alpha_\rho=1/2$ and $z=1$ in case 1a.

A variant of case 1a is the model studied by Bouchaud *et al* [20]. In our notation, this model reads

$$\begin{cases}\partial h/\partial t=\nabla^2h+c\nabla h+\mu\rho\nabla h+\eta_h(x,t),\\ \partial\rho/\partial t=\nabla^2\rho-c\nabla h-\mu\rho\nabla h.\end{cases} \tag{3.13}$$

The zeroth-order (free) Green's function and correlation function for h are respectively

$$G_h^{(0)}(k,\omega)=\frac{1}{-i\omega+k^2+ick},$$

$$S_h^{(0)}(k,\omega)=\frac{2\Delta_h}{(\omega-ck)^2+k^4}. \tag{3.14}$$

The coupling between both fields can be viewed as a coloured noise acting on ρ. The zeroth-order correlation function for ρ is

$$S_\rho^{(0)}(k,\omega)=\frac{2c^2\Delta_hk^2}{((\omega-ck)^2+k^4)(\omega^2+k^4)}, \tag{3.15}$$

hence

$$\int S_\rho^{(0)}(k,\omega)\frac{\mathrm{d}\omega}{2\pi}=\frac{2c^2\Delta_h}{k^2(4k^2+c^2)}. \tag{3.16}$$

The full Green's function is then again given by (3.3), with the self-energy as in (3.4). The low-momentum region produces a divergent integral, because of (3.9), so that the conclusion $z_h=1$ is still valid. The analysis of the correlation function follows in a manner similar to that done for case 1a, and leads to $\alpha_h=1/2$.

Bouchaud *et al* [20] applied standard perturbation theory to their equations (3.13). They did not meet the infrared singularity described above, since this phenomenon can only appear through treatments which go beyond strict perturbation theory, such as our self-consistent resummation procedure. Hence the conclusion that the model (3.13) is free (EW-like) on a perturbative basis is incorrect.

To close up this analysis, we come back to our case 1. We observe that the transfer term

$$T = -\nu(\nabla h)_- - \lambda\rho(\nabla h)_+ \tag{3.17}$$

can be thought of as a formal infinite series by invoking a suitable representation for Heaviside's Θ-function. We are thus led to consider the following more general structure for the transfer term

$$T = -\nu\nabla h - \lambda\rho\nabla h - \sum_{n=1}^{\infty}\nu_n(\nabla h)^{n+1} - \rho\sum_{n=1}^{\infty}\lambda_n(\nabla h)^{n+1}. \tag{3.18}$$

Dimensional analysis along the lines of ref. [12] shows that, among the terms in the sums, only $\nu_1(\nabla h)^2$ is relevant in one space dimension. Now consider the calculation of $\Sigma_h(k)$, within the above self-consistent one-loop approach. Two contributions are to be taken into account. We have already dealt with one of them in (3.4). That forces $z_h = 1$, independently of α_h. The other contribution is of the KPZ variety [14], for which a symmetry property, referred to as Galilean invariance, requires $\alpha_h + z_h = 2$ [18,19]. This in turn forces $\alpha_h = 1$ and $\beta_h = 1$.

The numerical work [12] for case 1 suggests $\alpha_h \approx 1$ and $\beta_h \approx 1/2$, consistent with $z_h \approx 2$. One possible, although rather exotic, explanation for the difference between the analytical prediction $z_h = 1$ and the numerical estimate $z_h \approx 2$ is the phenomenon observed recently [23] in a suitably regularized mean-field version of the KPZ problem. These authors have shown that the dynamics of the approach to the stationary state is much slower than the linear-response dynamics in the stationary state.

Finally, for our cases 4 and 6, which bear a close resemblance to the models studied by Krug and Spohn [19], the present work yields $\alpha_\rho = 1/2$ and $z_h = 1$, as was indeed found numerically in [12]. The situation of case 5 was already discussed in § 2.

4. Discussion

In the above, we have presented analytical insights on the behaviour of a system of coupled noisy nonlinear equations put forward earlier [4, 12] as a model of sandpile dynamics. The anomalous roughening seen in case 1, in particular, is seen to be due predominantly to the effect of the term $\lambda\rho(\nabla h)_+$; this has a simple physical explanation in the context of the observed agreement of our results with the experimental work of Kurnaz *et al* [24]. The authors of that work found, for the case of a rotated and shaken sandpile, exponents $\alpha_h = 0.92 \pm 0.05$ and $\beta_h = 0.48 \pm 0.16$ (in close agreement with case 1 in our table), and attributed this to the effects of interstitial fluids in their system. The effect of the $\lambda\rho(\nabla h)_+$ term is precisely to put back "roughness" on the profile of the pile at a rate proportional to the density ρ of mobile grains. We see therefore that our analysis above, among other things, provides a rather appealing picture of an experimental scenario.

Table 1. Critical exponents α, β, and z for both fields $h(x,t)$ and $\rho(x,t)$, measured from numerical simulations of the coupled stochastic equations in the various cases described in text (after ref. [12]). The exact EW and KPZ values are recalled for comparison. A hyphen means that the corresponding quantity is in our opinion not critical. An asterisk means that the exponent z cannot be accurately evaluated from the available data on α and β.

Model	Species	α	β	z
EW	h	1/2	1/4	2
KPZ	h	1/2	1/3	3/2
(1) asymmetric noise in h	h	0.94 ± 0.07	0.43 ± 0.04	2.2 ± 0.4
	μ	0.22 ± 0.08	0.07 ± 0.02	*
(1a) no-tilt noise in hand in ρ	h	0.36 ± 0.02	0.41 ± 0.08	0.9 ± 0.2
	ρ	0.39 ± 0.05	0.27 ± 0.04	1.4 ± 0.4

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