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Some recent applications of cell dynamical modelling to phase ordering dynamics

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Abstract. We study experimentally relevant effects in phase ordering dynamics using Cell Dynamical System (CDS) models. In particular, we present representative numerical results for phase ordering in random magnets and phase separation in binary fluids.

Keywords. Cell dynamical modelling; phase ordering dynamics; disordered systems.

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1. Introduction

There has been much recent interest in the problem of phase ordering dynamics, viz., the temporal evolution of a homogeneous two-phase mixture which has been rendered thermodynamically unstable by a sudden quench below its critical temperature [1]. Typically, the evolving system separates into domains which are rich in one or the other constituent of the mixture. These domains coarsen with time and are characterised by a single length scale $R(t)$ at time t (for isotropic systems). A direct consequence of the existence of a unique length scale is the so-called dynamical scaling of the time-dependent correlation function $g(\vec{r})$, i.e., $g(\vec{r}) = G(r/R(t))$, where $\vec{r}$ is the spatial variable and $G(x)$ is a master function, which is independent of time [2]. The nature of the domain growth law; and the scaling form of the correlation function are of great experimental relevance in disciplines as diverse as physics, chemistry, materials science and metallurgy. They depend critically on a variety of factors as follows:

(a) whether the evolution is characterised by a nonconserved order parameter (e.g., ordering of a ferromagnet) or conserved order parameter (e.g., segregation of a binary alloy);
(b) whether or not hydrodynamic effects are relevant;
(c) the nature of disorders (quenched or annealed) present in the phase ordering system, etc.

Theoretical descriptions of phase ordering dynamics are usually in terms of nonlinear partial differential equations, which are analytically intractable – especially in the case with conserved order parameter. Thus, it is necessary to have computationally efficient numerical models which enable us to access the asymptotic behaviour of phase ordering dynamics with limited numerical effort. Some time ago, we formulated discrete

space-time models of phase ordering dynamics, which served exactly such a purpose [3]. These are now referred to as Cell Dynamical System (CDS) models and have played an important role in elaborating the late-stage behaviour of phase ordering dynamics. Apart from their relevance in phase ordering dynamics, our CDS models have also served as an important example of the relevance of discrete space-time models in the modelling of nonequilibrium phenomena.

In this paper, we would like to briefly discuss two experimentally relevant problems of phase ordering dynamics, which we have recently tackled using CDS models. This paper is organised in the following fashion. In § 2, we discuss conventional models of phase ordering dynamics and also describe our CDS models. In § 3, we study the first problem of interest to us here, viz., phase ordering dynamics in systems with quenched disorder. Section 4 of this paper examines the second problem of interest to us here, viz., the role of hydrodynamic effects in phase separation dynamics. Section 5 ends this paper with a summary and discussion.

2. Conventional and cell dynamical modelling of phase ordering dynamics

At the coarsegrained level, phase ordering dynamics is conventionally modelled via stochastic partial differential equations. Typically, systems with a nonconserved order parameter (e.g., ordering of ferromagnets) are described by the time-dependent Ginzburg-Landau (TDGL) equation:

$$\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\delta H[\psi(\vec{r},t)]}{\delta\psi(\vec{r},t)} + \sigma(\vec{r},t), \tag{1}$$

where $\psi(\vec{r},t)$ is the order parameter of the system at point $\vec{r}$ at time t. For pure and isotropic systems, $H[\psi(\vec{r},t)]$ is usually taken to be the coarse-grained ϕ^4-free energy functional:

$$H[\psi(\vec{r},t)] = \int \mathrm{d}\vec{r}\,[-\tfrac{1}{2}\psi(\vec{r},t)^2 + \tfrac{1}{4}\psi(\vec{r},t)^4 + \tfrac{1}{2}(\vec{\nabla}\psi(\vec{r},t))^2], \tag{2}$$

where we assume that all quantities have been rescaled into appropriate dimensionless units. The Gaussian white noise $\sigma(\vec{r},t)$ in (1) has zero expectation value and satisfies the usual fluctuation-dissipation relation. In the classification scheme of Hohenberg and Halperin [4], Eq. (1) is referred to as Model A.

Of course, real experimental systems are neither pure nor isotropic and the free-energy functional in (2) constitutes an oversimplification. For example, in ferromagnets, quenched disorder typically arises in the form of nonmagnetic impurities. At the phenomenological level, this is modelled by assigning a random spatial dependence to the coefficients of various terms in the free-energy functional of (2) [5, 6]. The distribution of this random fluctuation is usually taken to be either Gaussian or uniform. For studies of phase ordering dynamics, the precise nature of this distribution is not relevant [6].

Phase ordering dynamics in systems with a conserved order parameter but no hydrodynamic effects (e.g., segregation of a binary alloy AB) is described by the

Cahn–Hilliard–Cook (CHC) equation [7]:

$$\frac{\partial\psi(\vec{r},t)}{\partial t} = \nabla^2\left[\frac{\delta H[\psi(\vec{r},t)]}{\delta\psi(\vec{r},t)}\right] + \sigma(\vec{r},t), \tag{3}$$

where symbols have the same meaning as previously. In the case with conserved order parameter, $\int d\vec{r}\psi(\vec{r},t)$ is a constant in time, as is clear on integrating Eq. (3) over space. Eq. (3) is also referred to as Model B [4].

The simplest model which incorporates hydrodynamic effects in segregation dynamics is known as Model H [4] and consists of a conserved scalar density (the order parameter) coupled to the hydrodynamic velocity field. We will consider here a variant of Model H, with the following form in three dimensions (again in appropriate dimensionless units):

$$\begin{aligned}
\frac{\partial\psi(\vec{r},t)}{\partial t} &= \nabla^2\left[\frac{\delta H[\psi(\vec{r},t)]}{\delta\psi(\vec{r},t)}\right] - \bar{\alpha}\vec{\nabla}\cdot[\psi(\vec{r},t)\vec{J}(\vec{r},t)] + \sigma(\vec{r},t), \\
\frac{\partial J_i(\vec{r},t)}{\partial t} &= \bar{\eta}\nabla^2 J_i(\vec{r},t) + \bar{\sigma}\sum_{k=1}^{3}\nabla_i\nabla_k J_k(\vec{r},t) \\
&\quad - \bar{\alpha}\psi(\vec{r},t)\nabla_i\left[\frac{\delta H[\psi(\vec{r},t)]}{\delta\psi(\vec{r},t)}\right] + \nu_i(\vec{r},t).
\end{aligned} \tag{4}$$

In (4), $\psi(\vec{r},t)$ and $J_i(\vec{r},t)$ $(i = 1,2,3)$ are, respectively, the order parameter and the dimensionless velocity fields as a function of dimensionless space and time. The Gaussian white noises $\sigma(\vec{r},t)$ and $\nu_i(\vec{r},t)$ $(i=1,2,3)$ are the noises for the order parameter field and the velocity field, respectively. They satisfy the appropriate fluctuation-dissipation relations. The rescaled parameters in (4) are the coupling constant $\bar{\alpha}$ ($\bar{\alpha}=0$ corresponds to the usual case of binary alloys, viz., Model B); and the transport coefficients $\bar{\eta}$ and $\bar{\sigma}$, which are the viscosities. We should remark that (4) reduces to the standard form of Model H [4] if we impose the additional constraint $\vec{\nabla}\cdot\vec{J}(\vec{r},t) = 0$. However, as was pointed out by Farrell and Valls [8], this extra constraint causes considerable numerical complications and we do not impose it here.

Given the numerical effort involved in successfully simulating the stochastic partial differential equation models in Eqs. (1)–(4) or performing Monte Carlo (MC) simulations of appropriate microscopic models, it is desirable to have computationally efficient models to study the scaling regime. We have proposed CDS models of phase ordering dynamics, which are highly efficient in terms of computer time usage [3]. These models impose the space-time coarsening, implicit in conventional phenomenological models, by using a discretised space-time lattice. An order parameter variable is associated with each cell of the spatial lattice, which can be considered to be the result of coarse-graining the microscopic model. The underlying principles of our modelling have already been discussed extensively in the literature and we refer the interested reader to ref. [3]. Here, we merely confine ourselves to presenting the resultant models, which are formulated on a lattice whose cells are labelled by, say, n. We associate an order parameter value at (discrete) time t with each cell and refer to it as $\psi(t,n)$. Our CDS model for the case with nonconserved order parameter in d-dimensional space takes the

following form [3]:

$$\psi(t+1,n) = f(\psi(t,n)) + \frac{D}{2d}\Delta_D\psi(t,n)$$
$$\equiv \mathcal{F}[\psi(t,n)], \tag{5}$$

where $f(\psi)$ is a monotonic function with an unstable fixed point at $\psi = 0$ (which mimics the unstable homogeneous state); and two symmetrically located stable fixed points at $\psi = \pm\psi_0$ (which mimic the stable spontaneously ordered states). In (5), D is a positive constant proportional to the phenomenological diffusion constant; and Δ_D is the isotropically discretised Laplacian operator at the site n. As the local order parameter takes continuous values, this CDS model falls into the subclass of Coupled Map Lattice (CML) models [9, 10].

We can also incorporate thermal noise terms in the deterministic model of Eq. (5). However, we have demonstrated numerically that thermal noise is asymptotically irrelevant for phase ordering dynamics – at least in the context of pure systems [3]. Therefore, the deterministic model in (5) suffices if one is interested in universal asymptotic behaviour. Furthermore, we should stress that any function $f(\psi)$ with the specified properties provides a reasonable CDS model for phase ordering with a nonconserved order parameter. This is a consequence of the strong universality of the ordering process and is discussed at length in Ref. [3].

The corresponding CDS model for the case with quenched disorder, e.g., ordering in random magnets, is obtained as a simple extension of (5) by associating a random spatial dependence with the parameters of the function $f(\psi)$, so that the local fixed points are randomly distributed around $\pm\psi_0$ [6]. In our subsequent discussion, the term "amplitude of disorder" refers to the amplitude of fluctuations in the local fixed points of the function $f(\psi)$.

The corresponding CDS model for the conserved case is easy to obtain. Naively, it may be obtained analogously to the continuum case by appending an extra Laplacian operator to the chemical potential for the nonconserved case as follows [3]:

$$\psi(t+1,n) = \psi(t,n) - \frac{1}{2d}\Delta_D[\mathcal{F}[\psi(t,n)] - \psi(t,n)]. \tag{6}$$

Our procedure for obtaining CDS models can also be interpreted as a novel method of discretising the corresponding partial differential equations, which enables the use of rather coarse mesh sizes, without any loss of numerical stability [3]. As a matter of fact, we can use this interpretation to derive the CDS model equivalent to (4), i.e., for the case with conserved order parameter when hydrodynamic effects are relevant. The CDS model we obtain from (4) (again neglecting noise terms) has the following form in three dimensions [11]:

$$\psi(t+1,n) = \psi(t,n) - \tfrac{1}{6}\Delta_D[\mathcal{F}[\psi(t,n)] - \psi(t,n)] - \alpha\vec{\nabla}_D \cdot [\psi(t,n)\vec{J}(t,n)],$$
$$J_i(t+1,n) = J_i(t,n) + \eta\Delta_D J_i(t,n) + \sigma\sum_{k=1}^{3}\nabla_{D,i}\nabla_{D,k}J_k(t,n)$$
$$+ \alpha\psi(t,n)\nabla_{D,i}[\mathcal{F}[\psi(t,n)] - \psi(t,n)], \tag{7}$$

where α, η and σ are phenomenological parameters. They are analogous to the corresponding parameters in the continuum model (4). In (7), $\vec{\nabla}_D$ is the symmetrically discretised gradient operator. The values of the parameters are dictated by the requirements that the scheme be stable and that the results be physically reasonable [3, 11].

We will next describe some recent results obtained using the CDS models we have just discussed. As mentioned earlier, we focus on two experimentally relevant problems, viz., phase ordering in random magnets; and phase separation in binary fluids.

3. Phase ordering dynamics in disordered magnets

We have studied phase ordering in random magnets by implementing the disordered version of the CDS model defined by (5) on a 2-dimensional lattice of size N^2. Periodic boundary conditions were applied to both edges of the lattice. The parameter values for our simulations and the manner in which we introduce disorder in $f(\psi)$ are discussed at length in Ref. [6] and we do not replicate the discussion here.

The initial condition for a single run corresponds to small-amplitude uniformly-distributed random fluctuations about a zero background, mimicking the disordered state before the quench. The statistical quantities of interest in phase ordering dynamics are the time-dependent structure factor, which is the Fourier transform of the correlation function, and the characteristic domain length scale $R(t)$. The time-dependent structure factor is defined as

$$S(\vec{k}, t) = [\langle \psi(\vec{k}, t) \psi(\vec{k}, t)^* \rangle], \tag{8}$$

where $\psi(\vec{k}, t)$ is the Fourier transform of the order parameter (spontaneous magnetisation, in this case) with wave-vector $\vec{k}$; and $\psi(\vec{k}, t)^*$ is the complex conjugate of $\psi(\vec{k}, t)$. In (8), the angular brackets refer to an averaging over independent initial conditions and the square brackets refer to an averaging over disorder configurations. In the dynamical scaling regime, the structure factor has the scaling form $S(\vec{k}, t) = R(t)^d F(kR(t))$, where $F(x)$ is a time-independent master function.

All numerical results presented below are obtained on 128^2 systems as averages over 20 disorder configurations, with 20 independent initial conditions for each disorder configuration. Because of the discrete nature of our lattice, the wavevectors $\vec{k}$ take discrete values $(2\pi/N)(k_x, k_y)$, where k_x and k_y range from $-N/2$ to $(N/2) - 1$. As the system of interest is isotropic, we can improve statistics by spherically averaging the vector function $S(\vec{k}, t)$ to obtain a scalar function $S(k, t)$. This is the quantity we show in subsequent figures.

As mentioned earlier, the second quantity of interest is the time-dependent length scale $R(t)$. We define $R(t)$ as the reciprocal of the first moment of the scalarised structure factor, i.e., $R(t) = \langle k \rangle^{-1}$, where

$$\langle k \rangle = \frac{\int_0^{k_m} \mathrm{d}k k S(k, t)}{\int_0^{k_m} \mathrm{d}k S(k, t)}. \tag{9}$$

In (9), the upper cut-off k_m is taken to be half the magnitude of the largest wavevector lying in the Brillouin zone of the lattice. We have confirmed that the structure factor has

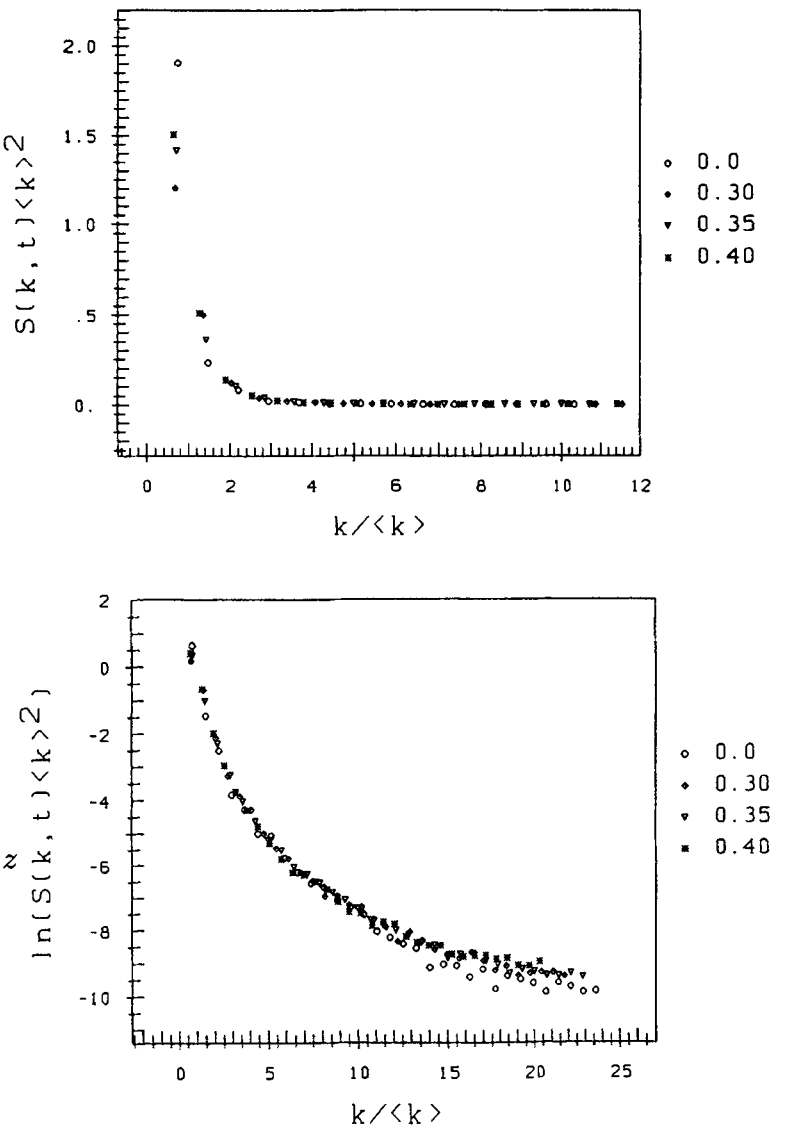

Figure 1. (**a**) Superposition of data for the scaled structure factor $S(k,t)\langle k\rangle^2$ vs. $k/\langle k\rangle$ obtained from CDS studies of phase ordering dynamics in random magnets. All structure factor data is obtained on 128^2 systems as an average over 20 disorder configurations, with 20 independent initial conditions for each disorder configuration. We present data for the pure case (denoted by circles); and a range of disorder amplitudes (denoted by the symbols indicated). The manner in which disorder is introduced into the CDS models is detailed in ref. [6]. The term "disorder amplitude" refers to the amplitude of random spatial fluctuations introduced into the local fixed points of the function $f(\psi)$ in Eq. (5). Data for the pure case (disorder amplitude $C = 0.0$) is from dimensionless time $t = 1000$. Data for the disordered cases ($C = 0.30, 0.35$ and 0.40) is from dimensionless time $t = 3000$. (From Ref. [6].) (**b**) Semi-log plot of data in figure 1(a), i.e., we plot $\ln(S(k,t)\langle k\rangle^2)$ vs. $k/\langle k\rangle$. (From ref. [6].)

decayed sufficiently upto this value of k_m and that numerical results for $R(t)$ are unchanged by considering larger values of k_m. The length scale thus defined is measured in units of the lattice spacing. We should stress that this definition of $R(t)$ is not unique and one can also define $R(t)$ by using higher moments of the structure factor or various zero-crossings of the correlation function. However, in the dynamical scaling regime, all these definitions are equivalent with the only difference being in numerical prefactors [2].

Our extensive numerical investigations of phase ordering dynamics with quenched disorder revealed the following facts. Firstly, we found that the coarsening systems exhibit dynamical scaling of the structure factor as in the pure case, indicating thereby that the presence of quenched disorder does not affect the existence of a unique domain length scale. Furthermore, we found that the functional form of the scaled structure factor appears to be independent of the amplitude of disorder and is the same as that for the pure case – at least within the rather wide range of disorder amplitudes considered. We have referred to this important property as "superuniversal" scaling of the structure factor [6]. Figure 1 exhibits the superuniversal scaling of the structure factor for phase ordering in random magnets. In figure 1(a), we plot the scaled structure factor $S(k,t)\langle k\rangle^2$ vs. $k/\langle k\rangle$ for the pure case and for a range of disorder amplitudes. Clearly, the functional form of the scaled structure factor appears to be unaffected by the presence of quenched disorder. In figure 1(b), we show the data of figure 1(a) on a semi-log scale, i.e., we plot $\ln(S(k,t)\langle k\rangle^2)$ vs. $k/\langle k\rangle$. It is clear that the coincidence of the scaled structure factors extends into the intermediate tail region also. The data in the extreme tail is not very reliable, in any case, as large wave-vector values are affected by the nonzero interfacial thickness. Superuniversal scaling of the structure factor has an important physical implication in that it suggests that the morphology of growing domains is unchanged inspite of trapping by disorder sites. Of course, we have only considered the lowest-order structure factor and it is not necessary that higher-order structure factors also exhibit superuniversal scaling. Nevertheless, the fact remains that the lowest-order structure factor is the experimentally relevant one and the higher-order ones are primarily of theoretical relevance. Therefore, our numerical observation has important experimental implications.

Our numerical results are not as conclusive in regards to the time-dependence of the domain length scale $R(t)$. Figure 2 plots the square of the length scale $R(t)^2$ vs. t for the pure case and for the disordered cases discussed previously. (It should be kept in mind that the length scale is measured in units of the lattice spacing.) In this figure, we plot $R(t)^2$ vs. t rather than $R(t)$ vs. t because the data for the pure case is well-known to exhibit the Lifshitz-Cahn-Allen (LCA) growth law $R(t) \sim t^{1/2}$ and this serves as a benchmark for discussing data for the disordered cases. It is evident that domain growth with quenched disorder does not exhibit the LCA growth law, except in the very early stages where domains are too small to be trapped by disorder sites. The crossover time from the LCA growth law is inversely proportional to the amplitude of disorder. Furthermore, if we plot the data of figure 2 on a log-log scale (not shown here), it is also evident that data for the disordered cases does not show a power-law behaviour at all. Unfortunately, these are the only conclusive statements we can make in this context. Inspite of extensive simulations by us [6, 12] and other authors [13], there is still no understanding of the functional form of the asymptotic growth law, or even whether there is a universal functional form valid for a wide range of disorder amplitudes. There have also been theoretical investigations of

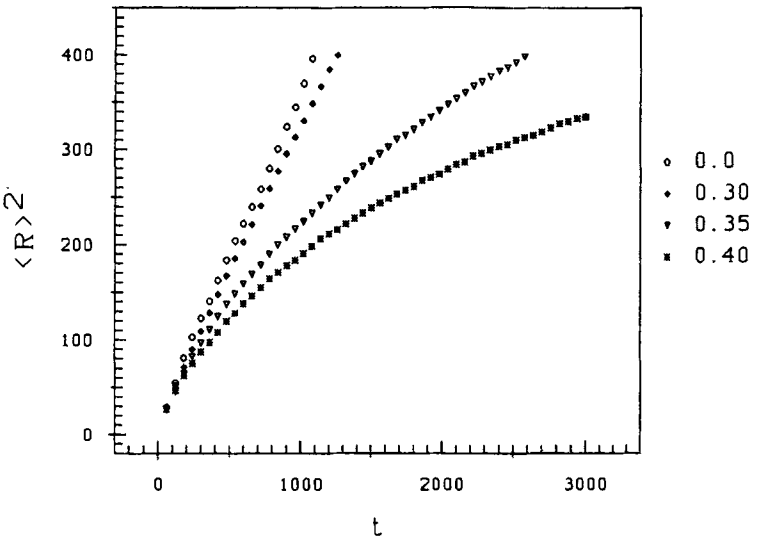

Figure 2. Square of the characteristic domain side $R(t)^2$ as a function of update time t for phase ordering dynamics in random magnets. We show data for the pure case and the disordered cases of figure 1. Symbols have the same meaning as in figure 1. (From ref. [6].)

this problem [14], which have proposed a domain growth law of the form $R(t) \sim (\ln t)^x$, where x is a universal exponent. These "logarithmic" domain growth laws are argued on the basis of thermally-activated hopping of domains over potential barriers set up by the trapping sites. However, extant numerical results do not support such domain growth laws, except in the case with quenched random fields [15].

We have also investigated the dynamics of phase separation in binary alloys with quenched disorder [16]. Our results for that case show similar trends to those discussed above for the nonconserved case, *i.e.*, the structure factor exhibits superuniversal scaling; and the asymptotic domain growth law is nonalgebraic, but we are again unable to ascertain the precise form of the universal growth law, if any.

4. Phase separation dynamics in binary fluids

We studied phase separation in binary fluids by implementing the CDS model defined by (7) on a 3-dimensional lattice of size N^3 with periodic boundary conditions. The results presented below are for the local relaxation function $f(\psi) = A \tanh \psi$ (with $A = 1.5$) and $D = 0.3$ [3]. We can associate mesh sizes with these parameter values by comparing the CDS scheme with the usual Euler discretisation scheme for partial differential equations [3]. The corresponding values are $\Delta t = 0.5$ and $\Delta x = 2.45$. In (4), we set $\bar{\eta} = 1$ and $\bar{\sigma} = 2$, following Farrell and Valls [8]. The phenomenological parameters η and σ in (7) are then fixed as $\eta = \bar{\eta}\Delta t/(\Delta x)^2 = 0.08$ and $\sigma = \bar{\sigma}\Delta t/(\Delta x)^2 = 0.17$. This should not be interpreted as a rigorous prescription for fixing parameters but rather as a rule of thumb to associate parameter values in our CDS model with those in the corresponding partial

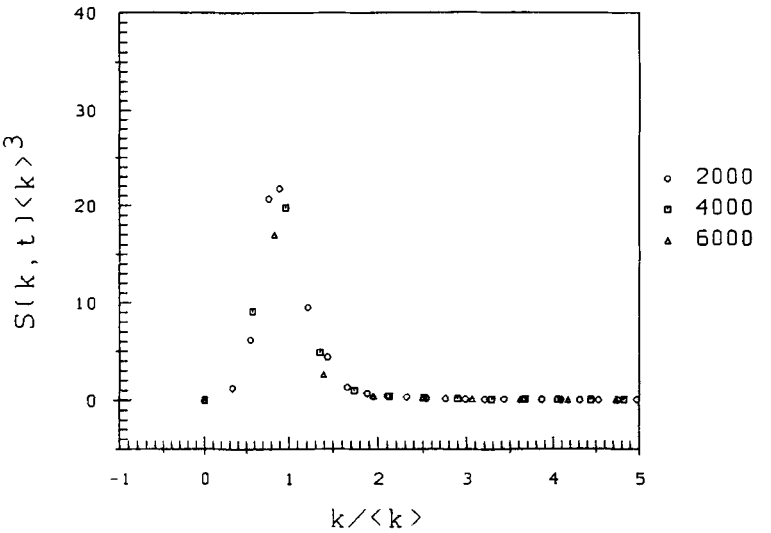

Figure 3. Superposition of data for the scaled structure factor $S(k,t)\langle k\rangle^3$ vs. $k/\langle k\rangle$ for phase separation in binary fluids. Structure factor data is obtained on 80^3 systems as an average over 40 independent runs. We superpose data from dimensionless times 2000, 4000 and 6000, denoted by the symbols indicated. (From Ref. [11].)

differential equation model. We should stress that the choice of values for viscosity parameters does not affect the asymptotic behaviour of phase separation in binary fluids and only changes the time of crossover to the asymptotic regime. Typically, higher values of viscosity suppress hydrodynamic effects and delay the onset of the asymptotic hydrodynamic regime.

Finally, we discuss the most important parameter in (7), viz., α, which fixes the strength of the coupling between the order parameter and the velocity field. We have studied domain growth as modelled by (7) for a range of values of α. In the limit $\alpha \to 0$, we recover the Lifshitz-Slyozov growth law $R(t) \sim t^{1/3}$, which characterises domain growth in Model B, as expected. In this paper, we present results for $\alpha = 0.41$, which corresponds (in the prescription defined above) to $\bar{\alpha} = 2$ in (4).

All results presented here are for the case of a critical quench, where there are equal concentrations of both components in the mixture. The structure factor is calculated for systems with $N = 80$ and as an average over 40 runs from independent initial conditions, each of which consists of the order parameter and velocity field uniformly and randomly distributed about a zero background with small amplitudes. The time-dependent characteristic length scale $R(t)$ is defined as in Eq. (9).

For purposes of comparison, we have also performed a CDS simulation for the case without hydrodynamics (i.e., Model B) on an 80^3 lattice. We use the same functional form for $f(\psi)$ and the same values for A and D as in the case with hydrodynamics.

Figure 3 superposes data for the scaled structure factor $S(k,t)\langle k\rangle^3$ vs. $k/\langle k\rangle$ from dimensionless times 2000, 4000 and 6000. The different data sets collapse neatly onto a master curve, verifying the validity of dynamical scaling in this case. We next turn our attention to the temporal behaviour of the characteristic length scale. Figure 4 plots $R(t)$

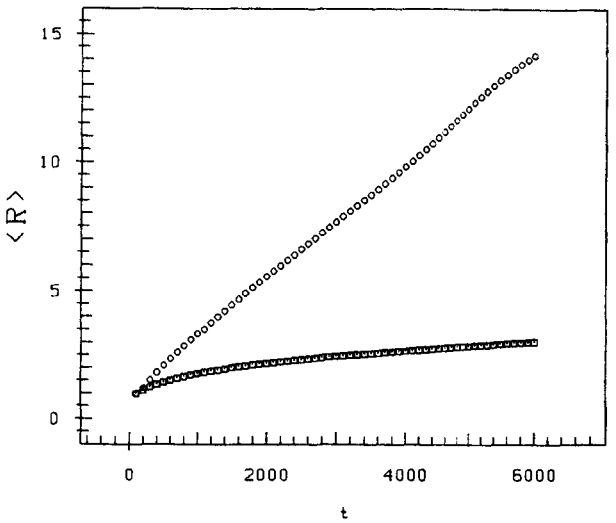

Figure 4. Characteristic domain size $R(t)$ vs. update time t for phase separation with and without hydrodynamics, denoted by circles and squares, respectively.

vs. t for our CDS model in (7) (marked by circles); and for the case without hydrodynamics (marked by squares). After an initial transient regime (which can be extended by using a smaller value of α), the domain size for the hydrodynamic case grows linearly in time, viz., $R(t) \sim t$. At times somewhat beyond those shown in figure 4, freezing sets in because of the finite size of the system. To ensure that our data is not affected by finite-size effects, we have also performed (less thorough) simulations on lattices of size 64^3 and 100^3. The results are the same as those presented here with the only difference being that the onset of freezing is delayed in larger systems. Domain growth for the case without hydrodynamics is much slower and can be easily confirmed to obey the usual Lifshitz-Slyozov growth law $R(t) \sim t^{1/3}$, though we do not show this explicitly here. Our numerical observation of temporally linear domain growth in the phase separation of binary fluids confirms a long-standing theoretical prediction by Siggia [17]. Similar results have also been obtained by Koga and Kawasaki [18] using a CDS model in which they solve for the instantaneous configuration of the fluid velocity field using the Oseen tensor approximation. There are also a number of related works, some of which we cite in Ref. [19], suggesting that there are a variety of growth regimes depending on the dimensionality and system parameters.

5. Summary and discussion

In this paper, we have attempted to acquaint the reader with some recent investigations of experimentally relevant problems in phase ordering dynamics. Initial numerical studies of idealised (i.e., pure and isotropic) phase ordering systems were greatly facilitated by Cell Dynamical System (CDS) models [3]. It is our belief that similar modelling will also help elaborate late-stage behaviour in experimentally realistic problems. To this end, we have

described numerical results from CDS models of phase ordering dynamics with quenched disorder; and segregation dynamics with hydrodynamic effects.

In the context of domain growth with quenched disorder, we find that the scaled structure factor, which characterises the morphology of the coarsening system, appears to be very similar to that for the pure case. This should be experimentally verifiable and we urge experimentalists to carefully examine this aspect of the problem. Unfortunately, however, we are unable to make a conclusive statement about the asymptotic nature of the domain growth law in this problem. This is an outstanding problem which needs to be resolved but it is not at all clear that one merely needs to perform "larger" and "longer" simulations to obtain a deeper insight. As a matter of fact, we believe that some novel theoretical input is necessary to enable a better understanding of this problem.

In the context of phase separation in binary fluids, our numerical results confirm a theoretical prediction due to Siggia [17], viz., hydrodynamic modes on the surfaces of growing domains greatly enhance domain growth in the late stages, resulting in a crossover in the domain growth law from $R(t) \sim t^{1/3}$ to $R(t) \sim t$. This crossover has also been observed in a large number of experiments [1] and it is gratifying to verify it numerically.

Models similar to those described in this paper have also been used to investigate a variety of other problems in nonequilibrium statistical mechanics. Typical examples include the segregation dynamics of block copolymers [20]; population dynamics [21]; dynamics of vortices and other topological defects [22]; and the dynamics of pattern recognition [23]. In all these cases, the enhanced computational efficiency of CDS models has enabled the study of truly asymptotic behaviours. It is our belief that CDS modelling will provide us with a quantitatively accurate, and yet efficient, means of studying a host of physical phenomena.

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