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Nonlinear propagation of relativistically intense electromagnetic waves in a collisionless plasma

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Abstract. We discuss the nonlinear propagation of relativistically intense electromagnetic waves into collisionless plasmas with special emphasis on one dimensional plane wave solutions of the propagating, standing and modulated types. These solutions exhibit a rich variety of phenomena associated with relativistic electron mass variation and coupling between transverse electromagnetic and longitudinal fields. They have important applications to problems of laser propagation, self-focusing in overdense plasmas, particle and photon acceleration and to electromagnetic radiation around pulsars.

Keywords. Nonlinear waves; relativistically intense; plasma waves.

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1. Introduction

The propagation of intense electromagnetic waves in a collisionless plasma is a problem of considerable interest because of its applications to laser fusion, modern plasma based particle acceleration methods, laboratory experiments using T^3 (table top terawatt) lasers, low frequency electromagnetic radiation around pulsars etc. In this paper we shall present a general review of work on this topic with special emphasis on the work done by our group.

In a weak electromagnetic wave propagating in a plasma, electrons jitter under the influence of the oscillating electric field vector giving a shielding current which is responsible for the well known dielectric constant expression $\epsilon = 1 - \omega_p^2/\omega^2$ where $\omega_p \equiv (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency (n_0 is the electron number density, m is the electron mass and e is the electron charge). Since the electromagnetic waves are transverse in nature, the electron motion is normal to the propagation vector; this prevents the convective fluid nonlinearities, entering through the $(\mathbf{v}\cdot\nabla)$ terms from becoming effective for the electromagnetic wave. However, as the directed jitter electron velocity $\mathbf{v} \sim e\mathbf{E}/m\omega$ becomes comparable to the velocity of light c, nonlinear effects enter through relativistic mass variation of the electrons and the $\mathbf{v} \times \mathbf{B}$ forces where $\mathbf{B}$ is the magnetic field associated with the electromagnetic wave. The $\mathbf{v} \times \mathbf{B}$ forces drive the electrons along the propagation direction and give the possibility of coupling to longitudinal plasma oscillations. A relativistically intense electromagnetic wave ($v/c \sim 1$) is therefore always nonlinear and in general a mixed wave, with significant space charge fluctuations associated with it. It is only for highly specialized polarizations, such as circular, where the $\mathbf{v} \times \mathbf{B}$ forces exactly cancel that we have a pure nonlinear electromagnetic wave. The possibility of exciting space charge fields propagating nearly at the velocity of light (viz. the group velocity of an electromagnetic pulse) is the key to modern plasma based particle acceleration schemes, which are currently undergoing intense experimental investigation. We may also physically anticipate other interesting consequences of working with relativistically intense electromagnetic waves. It is obvious that as the electron jitter velocity approaches c, the effective mass of the electrons will increase. This would encourage penetration of intense electromagnetic waves into normally overdense plasmas, i.e., $\omega_p > \omega > \omega_{peff}$. Similarly, if we send a finite plane wave front with an intensity variation on it, the intense regions will propagate slower than the weakly intense regions, because $c/(1-\omega_{peff}^2/\omega^2)^{1/2} < c/(1-\omega_p^2/\omega^2)^{1/2}$; this causes the plane wave front to acquire curvature and leads to self-focusing and filamentation of the intense electromagnetic wave. We will consider examples of all the above physical effects in our subsequent discussions.

Our paper is organized as follows. We first discuss the basic equations of the plasma model and its range of validity. We next investigate some special solutions of the full nonlinear set of equations. These solutions assume the existence of nonlinear stationary plane waves and reduce the problem to a set of coupled ordinary differential equations. We then go on to discuss some exact solutions for circularly polarized waves and numerical/approximate analytical solutions for linearly polarized waves. Following Akhiezer and Polovin [1] we next show how the linearly polarized waves may be reduced to the problem of a Hamiltonian with two degrees of freedom and describe a few analytical results obtained by them for high phase velocity waves and waves close to c. We then present some of our own numerical results of the problem, which for the first time showed the rich variety of possible solutions [2]. The appropriate Hamiltonian involves square roots because of relativistic effects (unlike most commonly studied nonlinear Hamiltonians which tend to involve polynomials). The results of these investigations are therefore of general interest to nonlinear dynamics. Since the numerical results indicated an almost integrable system there was widespread interest for a while to obtain an analytical confirmation of integrability. It came as somewhat of a surprise when it was actually proved to be nonintegrable [3]. In § 4, we discuss the extension of these results to standing nonlinear waves. The next section is devoted to modulated nonlinear waves in which solutions are not stationary in a moving frame. This allows us to consider the possible excitation of one dimensional solitons in the coupled electromagneticwave-plasma wave problem. Many features of the numerical solutions of this soliton problem can be reproduced analytically using a WKB method. In the last section we summarise the applications of the above results to problems of laser propagation and self-focusing in overdense plasmas, the problems of particle and photon acceleration and to electromagneticfields around rotating pulsars. We also discuss questions related to stability of these solutions, the importance of multidimensional effects and the conditions under which movement of ions can significantly alter the conclusions.

2. Plasma model and basic equations

Consider a finite amplitude electromagnetic wave propagating along the z axis. If the wave amplitude is sufficiently large so that $eE/m\omega c \gg 1$, the electrons are accelerated to the speed of light in a fraction of the wave period. Two nonlinear effects immediately become important, namely, the relativistic variation of the electron mass and the excitation of longitudinal space charge fields by strong $\mathbf{v} \times \mathbf{B}$ forces driving electrons along the direction of propagation. On the time scale of the electron motion the ions are virtually immobile due to their larger mass and hence their dynamics can be ignored. Of course, at larger electric field strengths, this assumption can break down particularly when the ions also acquire large induced directed velocities. Thus one is restricting oneself to wave amplitudes where the electron directed velocity is large but the ion directed velocity is still small ($eE/M\omega \sim 0.1c$ where M is the ion mass). Likewise the electrons can be treated as a cold fluid since their induced velocity is close to the speed of light and therefore the electron thermal velocity can be neglected in comparison. The basic nonlinear propagation equations for the coupled electromagnetic and longitudinal waves can then be written down in terms of the following set of relativistic cold fluid equations.

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \cdot \mathbf{E} = -4\pi e(n - n_0), \tag{2}$$

$$\nabla \times \mathbf{B} = -\frac{4\pi e}{c} n\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{5}$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{p} = -e\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right), \tag{6}$$

where $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic field components of the wave, $\mathbf{v}$ is the fluid velocity, n is the density and $\mathbf{p} = m\mathbf{v}\gamma$ is the relativistic momentum (with $\gamma = 1/\sqrt{1 - v^2/c^2}$). Equations (1–4) are the full set of Maxwell's equations, (5) is the continuity equation and (6) is the relativistic fluid momentum equation. These model equations were first written down by Akhiezer and Polovin [1] and have formed the basis of most fundamental studies related to the propagation of intense electromagnetic waves in a plasma. While a fully time dependent solution of these coupled nonlinear partial differential set of equations is difficult to obtain analytically (or even handle numerically), various special solutions have been obtained from time to time. A large class of them pertain to so called one dimensional traveling wave solutions, which we discuss next.

3. Traveling wave solutions

We restrict ourselves to one dimension (z the direction of propagation) and look for solutions which are functions of the independent variable $\zeta = z - ut$. Such waves, if they exist, would be nonlinear waves that are stationary in a frame moving with the velocity u. We adopt the following normalisations,

$$\beta = \frac{u}{c}; \quad \rho = \frac{\mathbf{p}}{mc}; \quad \zeta = \frac{\omega_{\mathrm{p}}}{c}(z - ut).$$

Then combining (1–6) and eliminating the electric and magnetic fields, we obtain the following set of coupled differential equations:

$$\frac{d^2\rho_x}{d\zeta^2} + \frac{1}{(\beta^2-1)}\frac{\beta\rho_x}{\beta(1+\rho^2)^{1/2}-\rho_z} = 0, \tag{7}$$

$$\frac{d^2\rho_y}{d\zeta^2} + \frac{1}{(\beta^2-1)}\frac{\beta\rho_y}{\beta(1+\rho^2)^{1/2}-\rho_z} = 0, \tag{8}$$

$$\frac{d^2}{d\zeta^2}[\beta\rho_z - (1+\rho^2)^{1/2}] + \frac{\rho_z}{\beta(1+\rho^2)^{1/2}-\rho_z} = 0 \tag{9}$$

The coupled equations above are still difficult to solve analytically but some special exact solutions exist.

3.1 *Pure transverse waves*

If we set $\rho_z = 0$, then for bounded solutions (9) gives $\rho^2 =$ constant. Equations (7, 8) then admit the solution

$$\rho_x = \rho\cos\omega\left(t - \frac{z}{u}\right), \tag{10}$$

$$\rho_y = \rho\sin\omega\left(t - \frac{z}{u}\right), \tag{11}$$

where $\omega = \omega_p\beta(\beta^2-1)^{-1/2}(1+\rho^2)^{-1/4}$. The wave phase velocity u can be expressed in terms of β as

$$u = \beta c = c\epsilon^{-1/2},$$

where ϵ, the dielectric constant of the plasma, is given by

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}(1+\rho^2)^{-1/2}. \tag{12}$$

Substituting for ρ in the Maxwell's equations, it can be shown that

$$E_x = \left(\frac{mc\omega\rho}{e}\right)\sin\left(\frac{\omega}{u}\zeta\right), \tag{13}$$

$$E_y = \left(\frac{mc\omega\rho}{e}\right)\cos\left(\frac{\omega}{u}\zeta\right) \tag{14}$$

so that

$$\rho^2 = (e^2E_0^2/m^2c^2\omega^2),$$

where $E_0^2 = E_x^2 + E_y^2$. Combining these results one can now see that

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{e^2E_0^2}{m^2c^2\omega^2}\right)^{-1/2}. \tag{15}$$

Equations (13) and (14) indicate that only circularly polarized modes may propagate as pure transverse waves in the plasma. Equation (15) demonstrates that such waves can propagate into overdense plasmas provided their frequencies lie in the range

$$\omega_p\left(1+\frac{e^2E_0^2}{m^2c^2\omega^2}\right)^{-1/2} < \omega < \omega_p.$$

We will return to the physical implications of this effect in the final section.

3.2 *Pure longitudinal waves*

For $\rho_x = \rho_y = 0$, (7, 8) are identically satisfied and $\rho_z = \rho$ in (9). This can be integrated exactly to yield a periodic solution which is expressible in terms of elliptic functions. The frequency of the wave is given by

$$\omega = \omega_p\frac{\pi}{\sqrt{2}}\left[\int_0^{\alpha_m}\frac{d\alpha}{(1-\alpha^2)^{5/4}}\left(\frac{(1-\alpha^2)^{1/2}}{(1-\alpha_m^2)^{1/2}}-1\right)^{-1/2}\right]^{-1}, \tag{16}$$

where α_m is related to the electric field amplitude through Maxwell equations and is given by

$$eE = \sqrt{2}m\omega_p c\left(\frac{1}{(1-\alpha_m^2)^{1/2}}-\frac{1}{(1-\alpha^2)^{1/2}}\right)^{-1/2}. \tag{17}$$

In the limit $\alpha_m \ll 1$, these expressions can be simplified to

$$\omega = \omega_p\left(1-\tfrac{3}{16}\alpha_m^2\right)$$

with $\alpha_m = (eE_0/m\omega_p c)$ where E_0 is the maximum value of E. For $\alpha_m \to 1$, one gets

$$\omega \approx 2^{-3/2}\pi\omega_p(1-\alpha_m^2)^{1/4}$$

where $eE_0 = \sqrt{2}m\omega_p c(1-\alpha_m^2)^{-1/4}$ so that

$$\omega \approx \frac{\pi}{2}\left(\frac{\omega_p^2 mc}{eE_0}\right).$$

In both these limits one observes that pure longitudinal waves with frequencies less than ω_p can propagate in the plasma provided the electric fields associated with them are sufficiently high.

3.3 *Coupled longitudinal-transverse waves*

For this general case it is not possible to obtain exact solutions but some interesting analytical and numerical solutions have been obtained in limiting cases. It is also convenient in some cases to restrict oneself to plane polarized waves such that $\rho_y = 0$. One interesting limit, which was first pointed out by Akhiezer and Polovin [1] is that of $\beta \to \infty$. This corresponds to an almost transverse wave with $\rho_z \to 0$. In this case, (7, 9) can be simplified to

$$\frac{d^2\rho_x}{d\zeta^2} + \frac{1}{\beta^2}\frac{\rho_x}{(1+\rho^2)^{1/2}} = 0, \tag{18}$$

$$\frac{d^2\rho_z}{d\zeta^2} + \frac{1}{\beta^2}\frac{\rho_z}{(1+\rho^2)^{1/2}} = 0. \tag{19}$$

Equations (18, 19) are exactly integrable and yield periodic solutions with amplitudes oscillating between two limits which are determined by the initial values of the two constants of motion that the system permits. The other interesting limit is $\beta \to 1$, i.e., $\beta - 1 \ll 1$. In this case one can redefine the independent variable as $\eta = (\beta^2 - 1)^{-1/2}$ and rewrite (7—9) as

$$\frac{d^2\rho_x}{d\eta^2} + \frac{\rho_x}{(1+\rho^2)^{1/2} - \rho_z} = 0, \tag{20}$$

$$\frac{d^2\rho_y}{d\eta^2} + \frac{\rho_y}{(1+\rho^2)^{1/2} - \rho_z} = 0, \tag{21}$$

$$\frac{d^2}{d\eta^2}[\rho_z - (1+\rho^2)^{1/2}] + \frac{(\beta^2-1)\rho_z}{(1+\rho^2)^{1/2} - \rho_z} = 0, \tag{22}$$

where $\beta - 1 \ll 1$ has been used. In (22) the last term can be neglected to yield $(1+\rho^2)^{1/2} - \rho_z = C^2$ where C is a constant. One then obtains the following periodic solutions

$$\rho_x = A_x \cos(\omega\eta); \quad \rho_y = A_y \sin(\omega\eta),$$

$$\rho_z = \frac{A_x^2 - A_y^2}{4\sqrt{1 + \frac{1}{2}(A_x^2 + A_y^2)}} \cos(2\omega\eta),$$

where

$$\omega = [1 + \tfrac{1}{2}(A_x^2 + A_y^2)]^{-1/4}$$

and the amplitudes A_x and A_y are related to C by $A_x^2 + A_y^2 = 2(C^4 - 1)$. Kaw and Dawson [4] carried out a detailed numerical investigation of the original equations (7, 9) and demonstrated the existence of periodic propagating solutions. They also confirmed that when $\beta \gg 1$ (phase velocity $\gg$ velocity of light), the waves are almost transverse. Furthermore, for strongly relativistic regime J vs ζ (where J is the current density) is in the nature of a square wave and correspondingly the electric field E as a function of ζ has a saw-tooth shape. Physically this is expected because in a strongly relativistic regime, $v_x \to c$ in most of the period and so $J_x \to -n_0 ec$, giving the square wave form. One may now give a simple interpretation of propagation of strongly relativistic waves into overdense plasmas. Normally, the conduction current generated by the electron response to the wave fields acts as a shielding current giving the plasma a dielectric behaviour and preventing propagation of waves into overdense regions. Now since the shielding current saturates in the strongly relativistic regime, the waves can penetrate into overdense plasmas. Relativistic effects thus diminish the ability of the plasma to act as a dielectric.

An analytic description of the numerical results of Kaw and Dawson [4] was obtained by Max and Perkins [5] who also derived the propagation condition in the strongly relativistic regime as,

$$\frac{1}{\beta^2} = \frac{c^2k^2}{\omega^2} = 1 - \frac{\pi}{2}\frac{\omega_p^2}{\omega}\frac{mc}{eE} > 0. \tag{23}$$

Max and Perkins [5] were also the first ones to extend the concept of these solutions to propagation into weakly inhomogeneous plasmas. They used a WKB like condition for constancy of energy flux for waves propagating into reflectionless inhomogeneous plasma and pointed out that indefinite penetration into overdense plasmas is possible if the following condition is satisfied,

$$\left(\frac{\omega L}{c}\right)^{1/2}\frac{eE_i\omega}{mc} > \omega_p^2$$

where E_i denotes the electric field strength in a vacuum and L is the density scale length. Physically as the wave penetrates to a higher density, there is a WKB amplification of the electric fields which decreases the effective plasma frequency (due to increased effective mass) faster than its increase due to density changes. The above condition on L arises in ensuring that the WKB condition, viz. scale length of density variation $\gg$ the wavelength of the wave, continues to be satisfied indefinitely. The notion of indefinite wave propagation into overdense plasmas has some interesting physical applications in astrophysics particularly to radiation occuring in pulsars. Finally we would like to mention the article of DeCoster [6] in which a lot of the early work related to traveling wave solutions has been summarised. He has also carried out a weak coupling analysis and pointed out (without any amplification or discussion) the existence of nonlinear resonances whenever $2m = n(1 - 1/\beta^2)^{1/2}$.

3.4 *The nonlinear Hamiltonian*

For a more general overview of the linearly polarized traveling wave solutions we introduce the variables,

$$X = \rho_x(\beta^2 - 1); \quad Z = \beta\rho_z - (1 + \rho^2)^{1/2}$$

and rewrite (7, 9) in the following form

$$\ddot{X} + \frac{\beta}{\sqrt{\beta^2 - 1 + X^2 + Z^2}} X = 0, \tag{24}$$

$$\ddot{Z} + \frac{\beta}{\sqrt{\beta^2 - 1 + X^2 + Z^2}} Z + 1 = 0, \tag{25}$$

where the overdots refer to differentiation with respect to ζ. These equations can be derived from the Hamiltonian,

$$H = \frac{\dot{X}^2}{2} + \frac{\dot{Z}^2}{2} + \beta\sqrt{\beta^2 - 1 + X^2 + Z^2} + Z \tag{26}$$

which is a constant of motion. The coupled stationary problem is thus equivalent to that of the motion of a fictitious particle in a two dimensional potential. Note that for $\beta > 1$, the Hamiltonian is positive definite in the entire $X - Z$ plane and will yield bound solutions of X and Z. For $\beta < 1$, this property is destroyed and the possibility of unbounded runaway solutions exist [7].

The Hamiltonian (26) has one equilibrium point ($X = 0$, $Z = -1$) where $H = H_{\min} = \beta^2 - 1$. Physically this corresponds to $\rho_z = \rho_x = \dot{\rho}_z = \dot{\rho}_x = 0$, i.e., to the equilibrium plasma with no oscillatory motions. Linearisation of (24) and (25) around the equilibrium points, gives

$$\delta\ddot{X} + X = 0, \tag{27}$$

$$\delta\ddot{\tilde{z}} + \left(1 - \frac{1}{\beta^2}\right)\tilde{z} = 0, \tag{28}$$

where $\tilde{z} = Z + 1$. The 'frequency' $(1 - 1/\beta^2)^{1/2}$ corresponds to longitudinal oscillations at the plasma frequency, as can be verified by going back to the unnormalised variables. Similarly the 'unit' frequency of 'X' oscillation corresponds to linear electromagnetic waves. The limit $X = \dot{X} = 0$ corresponds to nonlinear longitudinal oscillations whose exact form has been discussed in an earlier section.

To explore the full range of nonlinear solutions, we carried out an extensive numerical investigation of (24, 25) using the Poincare surface of section technique. Basically, we solved for the 'particle orbits' of the Hamiltonian (26), and looked at the successive crossings of the orbits of the plane $X = 0$ (with $\dot{X} > 0$) for a given set of values of H and

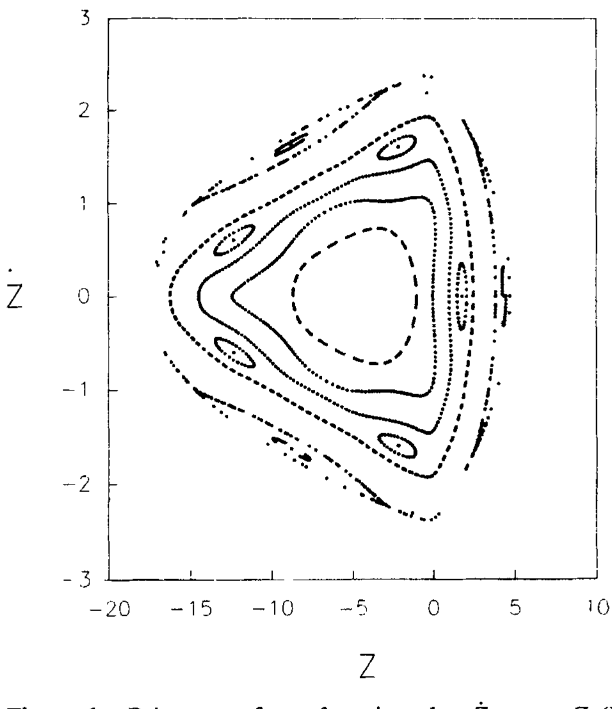

Figure 1. Poincare surface of section plot, $\dot{Z}$ versus Z ($X = 0, \dot{X} > 0$), for the Hamiltonian with $\beta = 1.1$, $\beta_b = 0.7$ and $H = 3.0$

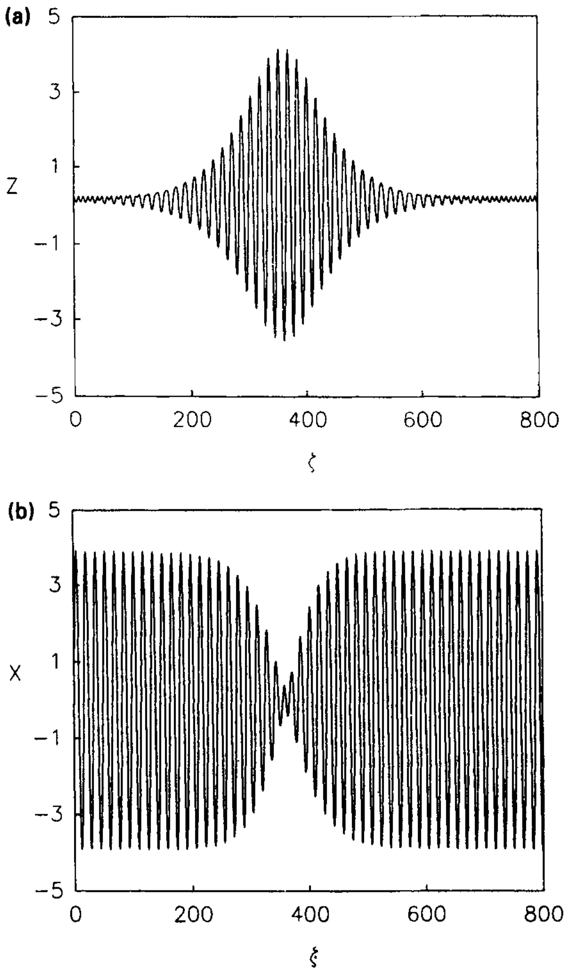

Figure 2. (a) A typical soliton solution of the Z oscillator corresponding to $H = 2, \beta = 1.3$ and $\beta_b = 0.8$, where the separatrix is between $(2.94, 0)$ and $(-3.305, 0)$. (b) A typical 'time' orbit curve of the X oscillator for the corresponding Z soliton.

β. Each orbit corresponds to a solution of the coupled equations. In general we saw the following kind of orbits: (i) Fixed points which map back on themselves after a finite number of iterations. These correspond to periodic orbits in which the ratio of the frequencies of the two oscillators is a rational number. (ii) Quasi-periodic orbits which ergodically fill a curve in the $Z - \dot{Z}$ plane and for which the ratio of the frequencies is an irrational number. (iii) Islands surrounding a fixed point which correspond to amplitude modulated waves. (iv) Separatrix orbits between chains of islands which correspond to amplitude modulated waves with the modulation period being infinite. Physically these correspond to soliton solutions. Another possibility which exists in principle (but which we did not find in our early numerical work) is stochastic orbits for which the orbit ergodically fills an area in space. These would correspond to aperiodic orbits which are

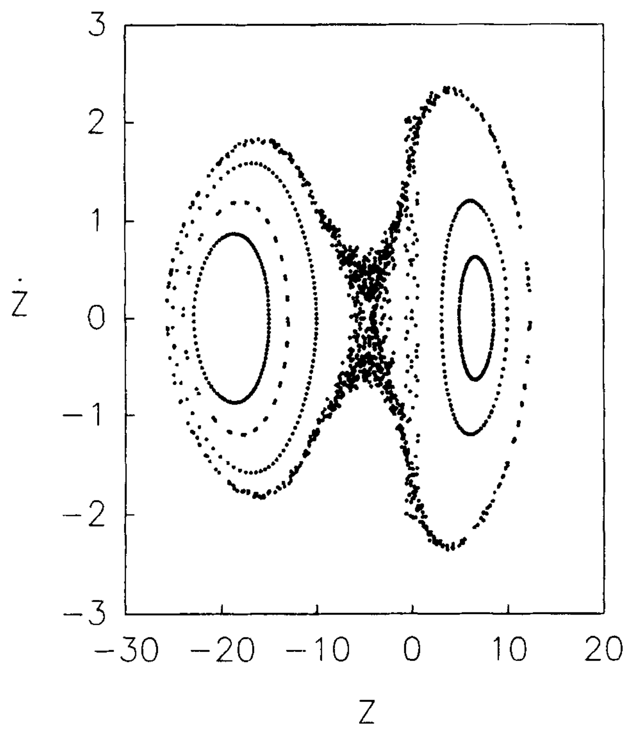

Figure 3. Chaotic orbits in Poincare surface of section plot, $\dot{Z}$ versus Z ($X = 0, \dot{X} > 0$), for the Hamiltonian with $\beta = 1.21$, $\beta_b = 0.7$ and $H = 10.0$

highly sensitive to initial conditions. The absence of numerical evidence for stochastic orbits and the results of local linear analysis of the trajectories led us to conjecture that perhaps the Hamiltonian was integrable. The analytic structure of the Hamiltonian, particularly the presence of the squareroot term in the potential, posed a challenge for application of standard integrability tests such as the Painleve criterion. There was thus widespread interest for a while to provide an analytic proof of its integrability. It came as somewhat of a surprise when Grammaticos, Ramani and Yoshida [3] proved that it was nonintegrable. Their proof was based on a clever application of Ziglin's theorem. Subsequently numerical evidence of stochastic orbits was also found by Romeiras [8]. In a recent work, Bisai, Sen and Jain [9] have carried out further detailed studies on a more generalised form of the Hamiltonian (26), namely

$$H = \frac{\dot{X}^2}{2} + \frac{\dot{Z}^2}{2} + |\beta - \beta_b| \sqrt{\beta^2 - 1 + X^2 + Z^2} + (1 - \beta\beta_b)Z \tag{29}$$

In this the plasma is assumed to have an equilibrium velocity v_b (beam velocity) which is represented by the normalised quantity β_b. Figure 1 shows a typical Poincare plot of orbits obtained from (29) and displays the different kind of solutions including the 'island' orbits. Figures (2a, b) display the 'soliton' orbits corresponding to the separatrices on the Poincare plots. In figure 3 we see the presence of chaotic orbits in the Poincare plot. Note that for $\beta = 1/\beta_b$, the Hamiltonian is integrable, since it admits $L \equiv Z\dot{X} - X\dot{Z}$ as another invariant (which is in involution to H, i.e., $[H, L] = 0$). This is reminiscent of the $\beta \to \infty$ case discussed by Akhiezer and Polovin [1]. The authors also carried out an analysis to calculate the nonlinear frequency shifts and the saturated amplitudes of the simplest periodic solutions (namely the lowest order fixed point solutions on the Poincare surface) to provide a detailed theoretical basis for the earlier conjectures of DeCoster [6] regarding the nonlinear resonances.

4. Relativistically intense standing waves

Marburger and Tooper [10] and Lai [11] found a new class of solutions to the Akhiezer–Polovin equations (1–6), which correspond to relativistically intense standing waves in an overdense plasma. They introduced a new ansatz, namely, $\mathbf{E}(z,t) = \mathbf{F}(z)f(t)$ and restricted themselves to circular polarizations only. The conservation of canonical momentum in the transverse direction then led to the condition $\mathbf{p}_\perp + e\mathbf{A}/c = 0$ and the longitudinal momentum p_z was found to vanish. Here the vector potential $\mathbf{A}$ follows the ansatz

$$\frac{e\mathbf{A}}{mc^2} = F(\zeta)(-\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)),$$

where $\zeta = \omega z/c$ and F satisfies the equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}\zeta^2} + F = \left(\frac{\omega_\mathrm{p}^2}{\omega^2}\right)\frac{F}{(1+F^2)^{1/2}}. \tag{30}$$

The electron density n and hence ω_p^2 is bunched in the longitudinal direction because the standing wave produces ponderomotive forces which push the electrons away from the field maxima. Since ions are assumed to be frozen on the time scales of interest, the electrons are held in balance by generation of static longitudinal fields $\Phi(z) = (mc^2/e)\phi(\zeta)$ where ϕ satisfies the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = \left(\frac{\omega_{\mathrm{p}0}^2 - \omega_\mathrm{p}^2}{\omega^2}\right) = \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}(1+F^2)^{1/2}. \tag{31}$$

One may now eliminate ω_p^2 from (30) and reduce it to that of a nonlinear one dimensional oscillator governed by the Hamiltonian

$$H = \frac{P^2}{2M(F)} + \tfrac{1}{2}F^2 - \alpha(1+F^2)^{1/2} \tag{32}$$

where $P = M(F)(\mathrm{d}F/\mathrm{d}\zeta), M(F) = (1+F^2)^{-1}$ and $\alpha = \omega_{\mathrm{p}0}^2/\omega^2$. For overdense plasmas ($\alpha > 1$), this leads to three types of nonlinear standing waves, viz,

(1) oscillations about zero when the maximum field F_m exceeds a critical value $F_T = 2(\alpha^2 - \alpha)^{1/2}$;
(2) oscillations about a bias field $F_B = (\alpha^2 - 1)^{1/2}$ when $F_m < F_T$ and
(3) motion with an infinite period (along the separatrix) when $F_m = F_T$.

For $F_m < F_a = 2(\alpha^2 - \alpha)^{1/2}$, the equations for F may be integrated explicitly in terms of Jacobian elliptic functions. Thus, when $F_T \le F_m \le F_a$, the quartic period is $\zeta_\mathrm{p} = K(m)/\gamma$

where K is the complete elliptic integral of the first kind, $m = (\epsilon_m - 1)(2\alpha + 1 - \epsilon_m)/4(\epsilon_m - \alpha)$, $\gamma^2 = \epsilon_m - \alpha$, $\epsilon_m^2 = 1 + F_m^2$ and the field amplitude is given by

$$F(\zeta) = \frac{2F_m cn(\gamma\zeta)}{[2 + (\epsilon_m - 1)sn^2(\gamma\zeta)]}.$$

For the important nonperiodic limit, $F_m = F_T$, $m = 1$ and

$$F(\zeta) = \frac{F_T \cosh(\gamma''\zeta)}{[1 + \alpha \sinh^2(\gamma''\zeta)]},$$

where $\gamma'' = (\alpha - 1)^{1/2}$.

An important physical effect which has to be carefully considered is the possibility of complete depletion of electrons from a part of the region occupied by the standing waves. The condition for complete vanishing of the electron density may be written down as

$$\frac{d^2}{d\zeta^2}(1 + F^2)^{1/2} < -\alpha.$$

If this condition is satisfied somewhere, part of the standing wave is in a vaccuum cavity; appropriate solutions are then obtained inside the cavity and their logarithmic derivatives are matched across the depletion boundary. One can show that for $F_m < F_d = \{(\alpha/2)[\alpha + (4 + \alpha^2)^{1/2}]\}^{1/2}$ complete electron depletion does not take place anywhere and the above simple analytic solutions are valid. Elsewhere, the solutions have to be obtained with greater care.

The most interesting application of these solutions is in estimating the properties of standing waves in inhomogeneous plasmas. Using the WKB condition that the phase integral $I = \oint P dF$ is an invariant for propagation into an inhomogeneous plasma with $\alpha = \alpha(\zeta)$, one finds the condition

$$I_0 = \pi F_0^2 = I_T = 8\alpha \sin^{-1}[(\alpha - 1)/\alpha]^{1/2} - 8(\alpha - 1)^{1/2}.$$

Here F_0 is the vacuum field strength and I_T, the phase integral corresponding to the turning point, is obtained by using the solution for the case $F_m = F_T$ (since in this case the period $\zeta_p \to \infty$). Note that in contrast to the linear WKB solution, the wave amplitude remains finite at the turning point: $F_T = 2(\alpha^2 - \alpha)^{1/2}$. The above condition may be used to find out the maximum density upto which an incident wave with field amplitude F_0 may penetrate.

5. Modulated nonlinear waves

In this section, we will discuss another class of exact one dimensional nonlinear solutions of the relativistic cold plasma equations which represent envelope solitons of light waves [12, 13], in which the modulation envelope propagates as a large amplitude plasma wave in the medium. These solutions are a step beyond the well known traveling wave solutions in a cold plasma because the envelope and the phase propagate with different speeds so that the nonlinear relationships between the phase and group velocities can be investigated. For simplicity we again restrict our analysis to circularly polarized waves which couple to longitudinal disturbances only because the amplitude is modulated.

We start as usual with the relativistic set of fluid equations (1–6 for a cold plasma (in one dimension) and introduce the change of variables $x - \beta t = \xi$, $t = \tau$. Instead of the fluid momenta (ρ) we now work with A (the vector potential) and ϕ (the electrostatic potential) which are normalised as eA/mc^2, $e\phi/mc^2$. Time is normalised as $\omega_p t$ and space by $\omega_p x/c$, where the direction of propagation is taken as x. We now make the key ansatz that in the moving frame, the vector potential is circularly polarized and has a sinusoidal phase variation, i.e., $\mathbf{A} = [\{\hat{y}a(\xi) + \hat{z}ia(\xi)\}\exp(-i\lambda\tau) +$ c.c$]$. The introduction of a frequency parameter λ in the phase factor is the basic deviation from earlier stationary wave solutions; this allows us to identify β with the group velocity of the light pulse and also to distinguish between the group and phase speeds. The choice of circular polarization allows us to avoid the generation of harmonics in all wave fields. For plasma oscillations, which form the modulation envelope, this leads to $\partial/\partial\tau = 0$ in the moving frame. Integrating the fluid equations we then get $n(\beta - u) = \beta$ and $\gamma(1 - \beta u) - \phi = 1$. Using these relations and further writing $a(\xi) = R\exp(i\theta)$, we can reduce the Poisson's equation and the wave equation to obtain the following set of coupled nonlinear equations

$$\phi'' = \frac{u}{(\beta - u)} \tag{33}$$

with u given by

$$u = \frac{\beta(1 + R^2) - (1 + \phi)[(1 + \phi)^2 - (1 - \beta^2)(1 + R^2)]^{1/2}}{(1 + \phi)^2 + \beta^2(1 + R^2)} \tag{34}$$

and

$$R'' + \frac{R}{1 - \beta^2}\left[\left(\lambda^2 - \frac{M^2}{R^4}\right)\frac{1}{1 - \beta^2} - \frac{\beta}{\beta - u}\frac{1 - \beta u}{1 + \phi}\right] = 0, \tag{35}$$

where $M = R^2\{(1 - \beta^2)\theta' - \lambda\beta\}$, is a constant of integration. In our representation the amplitude $R = (A_y^2 + A_z^2)^{1/2}$ of the circularly polarized electromagnetic wave exhibits a modulation propagating at the group speed β. The question of the definition of phase speed is a little more complex. For small amplitudes R is always positive and the exponential factor contains the entire phase information. The phase speed may then be shown to be simply $1/\beta$ satisfying the conventional relation $V_p V_g = 1$. However, for arbitrary amplitudes R oscillates because of the strong $R - \phi$ coupling and the phase speed has to be defined by directly determining the z, t variation of $R\cos(\theta - \lambda\tau)$. Equations (33–35) admit of one exact integral of motion which may be written as

$$K = \frac{R'^2}{2} - \frac{\phi'^2}{2(1 - \beta^2)} + V(R, \phi), \tag{36}$$

where

$$V(R, \phi) = \frac{\lambda^2}{(1 - \beta^2)^2}\frac{R^2}{2} + \frac{M^2}{2R^2(1 - \beta^2)^2} - \frac{\phi}{(1 - \beta^2)}$$
$$- \frac{\beta}{(1 - \beta^2)^2}[\beta(1 + \phi) - \{(1 + \phi)^2 - (1 - \beta^2)(1 + R^2)\}^{1/2}]. \tag{37}$$

Since $\beta < 1$ (group velocity less than c), the problem is similar to a Hamiltonian of coupled anharmonic oscillators with two degrees of freedom (R, ϕ) where the effective mass for one of the anharmonic oscillators is negative. In the limit of weak density response, the problem can be made one dimensional by the substitutions $\phi \approx (1+R^2)^{1/2} - 1$ and $n \approx (\sqrt{1+R^2})''$. Expanding in R^2 and taking $M = 0$, $K = 0$ we get the well known envelope soliton solution

$$A_y = R_m \operatorname{sech}\left[\frac{\sqrt{1-\beta^2-\lambda^2}}{1-\beta^2}(x-\beta t)\right]\cos\left[\frac{\lambda\beta}{1-\beta^2}\left(x-\frac{t}{\beta}\right)\right], \tag{38}$$

where

$$R_m = 4\sqrt{\frac{(1-\beta^2)(1-\beta^2-\lambda^2)}{4\lambda^2-(1-\beta^2)(3+\beta^2)}}. \tag{39}$$

$K \neq 0$ gives modulated periodic wave train solutions with the envelope factor described by an elliptic function. Equation (38) represents an isolated light pulse with a frequency $\omega = \lambda/(1-\beta^2)$, wavenumber $k = \lambda\beta/(1-\beta^2)$, phase velocity $1/\beta$, group velocity β (note $V_p V_g = c^2$), an envelope scale length (in units of c/ω_p) equal to $(1-\beta^2)/\sqrt{(1-\beta^2-\lambda^2)}$ and an amplitude-group velocity relationship given by (40). Note that the envelope scale length is real only when $\lambda^2 < (1-\beta^2)$. This may be physically interpreted as follows. The wave-frequency in the frame moving with the group velocity has the Doppler shifted value $\bar{\omega} = (\omega - k\beta)/\sqrt{1-\beta^2} = \lambda/\sqrt{1-\beta^2}$. Thus the inequality $\lambda < \sqrt{1-\beta^2}$ corresponds to the situation when the Doppler shifted wave frequency $\bar{\omega}/\omega_p < 1$, i.e. the electromagnetic wave finds itself in an overdense plasma and is totally trapped; this is why we get a soliton solution and the wave does not leak out. The effective scale-length for trapping may now be seen to be $\sqrt{1-\beta^2}[c/\sqrt{\omega_p^2 - \bar{\omega}^2}]$ which is quite reasonable. Similarly the amplitude-group velocity relationship may be written in terms of physical parameters as

$$1-\beta^2 = \frac{-(R_m^2 - 4 + 4R_m^2\omega^2) + \sqrt{((R_m^2 - 4 + 4R_m^2\omega^2)^2 + 64R_m^2\omega^2)}}{8\omega^2}. \tag{40}$$

For arbitrary amplitudes we have numerically solved equations (33–35) for soliton solutions. For λ very close to $\sqrt{1-\beta^2}$ we get small amplitude solitons which are well described by the analytic solutions (38). As λ is decreased the soliton amplitude increases and acceptable solutions occur only at discrete values of λ. In other words, for a fixed value of β finding soliton solutions turns out to be an eigenvalue problem in λ. The sizes and shapes of these solitons also vary as a function of λ. Typically ϕ has a characteristic bell shape whereas R has a number of nodes. Figure 4 shows a typical soliton solution for $\beta = 0.97$ and $\lambda = 0.224445$. For applications such as particle acceleration or photon acceleration, the regime of interest is $\beta \to 1$, where the group velocity is close to c. We have carried out a detailed investigation of soliton solutions in this regime. Figure 5 is a plot of normalized group velocity V_g versus the normalized carrier frequency $\Omega (= \omega/\omega_p)$. The solid curve corresponds to soliton pulse results obtained from our numerical work. For comparison we have also plotted the linear group velocity (dashed line)

$V_{gL} = \sqrt{(1 - 1/\Omega^2)}$ and the nonlinear group velocity for the infinite plane wave (dotted line), $V_{g\infty} = \beta[1 - (\gamma_\infty - 1)/(2\Omega^2\gamma_\infty(\gamma_\infty + 1)]$. This last expression has been recently obtained by Mori *et al* [14] and depends on the infinite wave γ viz. $\gamma_\infty = \sqrt{1 + A^2}$. The nonlinear group velocities are closer to c than the linear group velocity. This is physically understandable because the nonlinearity makes the electrons heavier and thereby weakens the plasma dielectric effects. Finite width soliton pulses propagate slower than the infinite plane waves. Physically this is because coupling to plasma waves acts as a drag on the electromagnetic waves and slows them down. The deviations between $V_{g\infty}$ and V_{gs} can be substantial; in our numerical work, we have observed (not shown) deviations upto 25 per cent of the difference between V_g and 1.

To understand these solutions better we consider another approximate analytic limit to equations (33–35) particularly applicable when the ϕ soliton structure is large. When $\phi_{\max}$ is large, the longitudinal velocity $u \to -1$, $n \to \beta(1 + \beta)^{-1}$ and $n/\gamma \to 0$. In this case, the light wave propagates essentially in a plasma free region (because the residual electrons have become infinitely massive) and we can approximate the effective Hamiltonian as

$$K_{\text{eff}} = \frac{R'^2}{2} + \frac{\lambda^2}{(1-\beta^2)^2}\frac{R^2}{2} - \frac{\phi'^2}{2(1-\beta^2)} - \frac{\phi}{(1-\beta^2)(1+\beta)} + \frac{\beta}{(1+\beta)(1-\beta^2)} + \cdots, \tag{41}$$

where the $\cdots$ refers to terms of order $\leq \phi^{-1}$ which are neglected. From (41) we get $R = R_0 \sin(\lambda\xi/(1 - \beta^2))$. Furthermore, the structure of ϕ near its maximum can be

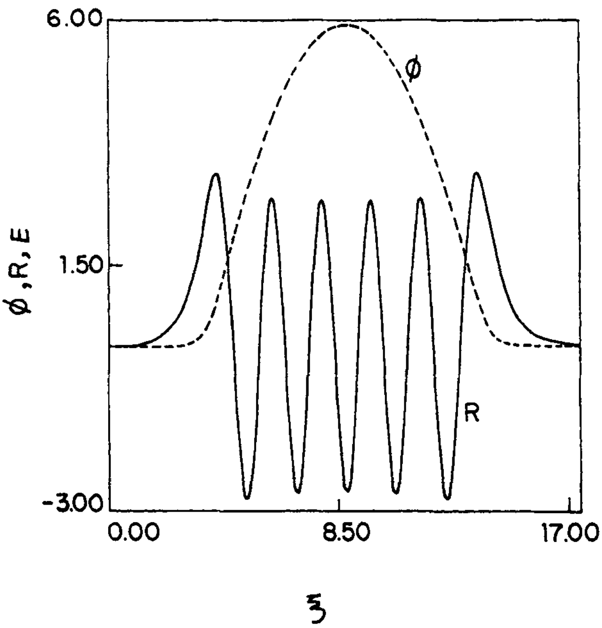

Figure 4. Electrostatic potential ϕ (dashed line), and electromagnetic wave amplitude (R) (solid line) profiles for a soliton pulse with $p = 6$, $\beta = 0.97$ and $\lambda = 0.224445$.

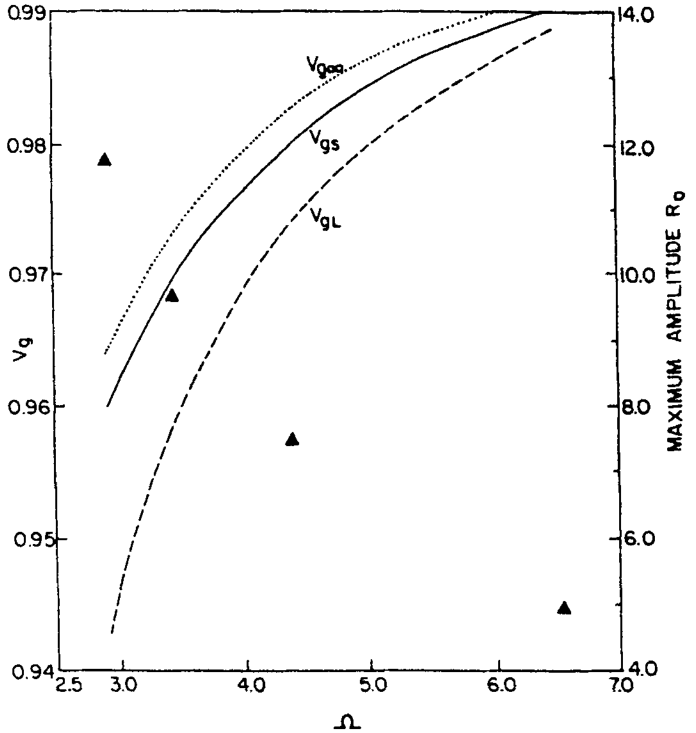

Figure 5. Normalized group velocity vs normalized frequency curves for soliton pulses (solid line), infinite pulses (dotted line) and linear pulses (dashed line). The maximum electromagnetic wave amplitude R_0 for the soliton pulse (for $p = 18$) are indicated with solid triangles.

approximated by the parabolic equation ($u \to -1$ in (33)) $\phi = \phi_{\max} - \xi^2/2(1+\beta)$, which gives an estimate of the spatial scale size of the soliton as $\xi_{\max} \simeq \sqrt{2(1+\beta)\phi_{\max}}$. From (36, 37) we know that $K = 0$ for a soliton solution. Noting that $K \simeq K_{\text{eff}}$ for $\phi = \phi_{\max}$, $\phi' = 0$, we get $\phi_{\max} = \beta + \lambda^2 R_0^2/2(1-\beta)$. Furthermore, if the soliton region involves p wavelengths of the light wave, we have $p\pi(1-\beta^2)/\lambda \simeq \xi_{\max}$. The above expressions lead to the following approximate analytic relationship between λ, R_0 and β as

$$R_0^2 = \frac{2(1-\beta)}{\lambda^2}\left[\frac{\pi^2 p^2}{2\lambda^2}(1-\beta^2)(1-\beta) - \beta\right]. \tag{42}$$

Equation (42) is largely corroborated by the numerical work shown in figures 4 and 5.

Recently, Kuehl *et al* [15] have carried out a WKB analysis in the weak amplitude limit of eqs (33–35) and have obtained a more refined estimate of the discrete spectrum. They have also shown that the discrete nature of the eigenvalue spectrum persists as long as the dispersion term in the Poisson equation is retained. If one neglects the dispersion then $\phi \propto R^2$ in the small amplitude limit and substituting into the R equation leads to the nonlinear Schroedinger equation which has a continous spectrum of solitons. Thus the coupled electromagnetic-plasma wave solitons have a unique characteristic. The importance of the dispersive term in the formation of these solitons has also been pointed out by Chen and Sudan [16] who have derived a general set of equations in three dimensions for the coupled wave problem. The importance of determining the group velocity of the finite nonlinear pulse in a more accurate fashion is highlighted by the recent work of Decker and Mori *et al* [17] who have carried out particle simulations for this purpose.

6. Discussion

We have discussed the nonlinear propagation of relativistically intense electromagnetic waves into collisionless overdense plasmas putting a special emphasis on 1-d plane wave solutions of the propagating, standing and modulated (i.e. localized) types. These solutions already exhibit a rich variety of phenomena associated with relativistic electron mass variations and coupling between transverse electromagnetic and longitudinal fields. However, it must be admitted that whereas these special solutions are excellent candidates for elucidating the physics of the important nonlinear phenomena, their relevance to truly experimental situations may be determined by questions related to their accessibility. Thus one may need very special initial conditions for setting them up and one may have to worry about their stability to noise perturbations in one and more dimensions. Many of these questions have not yet been addressed in much detail.

One question which has received some attention is that of self-focusing and self-modulational instability of relativistically intense electromagnetic wave [18, 19]. Self-focusing instability occurs when the field power exceeds a critical value and leads to filamentation of intense electromagnetic waves [20, 18] Similarly a self-modulational instability could be responsible for converting a propagating plane wave solution (§ 3) into one with localized modulational envelopes (§ 5). If both instabilities are simultaneously occuring, the final object would be a multi-dimensional localized region ('bubble') of intense electromagnetic waves interacting with longitudinal fields as they propagate. The analytic description of such 'bubbles' still remains an outstanding problem in nonlinear plasma physics. The strong low frequency electromagnetic radiation believed to be present in the pulsar environments [21, 22] is the most important astrophysical problem where these results have been applied. Retaining relativistic effects in the ion motions also, it has been concluded that the pulsar radiation can penetrate to large distances in the Crab nebula, thus allowing the possibility of an energy source in these volumes [5].

In laboratory plasmas, the most important application of this work is in laser fusion and plasma based accelerators. In the inertial fusion schemes, a point of view has recently developed that it would be of interest to create a hot spot in the central core of a compressed inertial fusion pellet by some independent means [23]. The use of a relativistically intense light beam might be one possible method of reaching such overdense regions. Similarly, for plasma based accelerator schemes the creation of space charge fields propagating with velocities close to that of light is of considerable interest [24–26]. A proper nonlinear description of the group velocity of such coupled stationary waves is therefore under intense investigation [13, 15, 14]. The solitary light pulses which are mostly empty of plasma in the central region, but have large plasma accumulation at the edges, have consequently large changes in n/γ inside the pulse. One can hope to get a very large frequency multiplication factor from using such pulses as photon accelerators [27]. The recently initiated table top terawatt experimental programme [28, 29] on the interaction of intense laser beams with matter has been motivated by these and other similar applications.

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