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Fractal characteristics of critical and localized states in incommensurate quantum systems

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Abstract. Incommensurate quantum systems with two competing periodicities exhibit metallic (with Bloch-type extended wave functions), insulating (with exponentially localized wave functions) as well as *critical* (with fractal wave functions) phases. An exact renormalization method, which takes into account the inherent incommensurability, is used to obtain the phase diagram of various quantum models for the two-dimensional electron gas and for quantum spin chains in a magnetic field. In this approach, the scaling properties of the fractal eigenstates are characterized by a fixed point or a strange invariant set of the renormalization flow. One of our novel results is the existence of self-similar fluctuations in the localized states once the exponentially decaying envelope is factorized out. In almost all cases under investigation here, the universality classes can be broadly classified as those of the nearest-neighbor square or triangular lattices.

Keywords. Incommensurate systems; localization; fractal.

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1. Introduction

Incommensurate or quasiperiodic (QP) systems with two irrationally related frequencies provide a class of aperiodic models which have been of great interest to experimentalists, theoreticians as well as mathematicians. Incommensurate structures appear in many physical systems such as quasicrystals, two-dimensional electron systems, magnetic superlattices, spin and charge-density waves as well as in one-dimensional ionic conductors. [1] The mathematical interest in QP systems is due to the small-divisor problems [2] which arise in that context. Theoretically, the subject is of great interest because these systems are considered to be in between periodic and random. Periodic systems exhibit Bloch-type extended (E) states and a continuous spectrum while for a random potential all states are localized (L) in one dimension. However, QP systems exhibit the metal-insulator (or E-L) transition in one dimension [1]. At the transition point the eigenfuntions are neither extended nor exponentially localized and are referred to as *critical* (C). The spectrum in QP systems can be either absolutely continuous, point-like, singular continuous, or a mixture of these [3].

In this paper we describe the results of our new, *exact* nonperturbative renormalization group (RG) to study the exotic fractal characteristics of QP systems modelled by a discrete Schrödinger equation. With the exception of some results in § § 4 and 5, most of our results have appeared in our previous publications [4–8]. This review brings out the commonality between a variety of systems and hence puts our results on a more general perspective. Furthermore, it proves the usefulness of our RG methodology in quasiperiodic systems. We hope that this review will serve as a useful source in the literature on incommensurate systems.

We will describe a variety of systems which can be written in the tight binding form

$$a(i)\psi_{i+1} + b(i)\psi_{i-1} + c(i)\psi_i = 0. \tag{1}$$

Here, ψ_i is the fermion wave function and a, b, and c are site-dependent real or complex functions determined by the potential and the eigenvalue. We assume the quasiperiodicity is such that it can be described by a single frequency σ relative to the underlying lattice, i.e., a, b, and c are essentially functions of the fractional part of $i\sigma$ where σ is irrational. Our formulation is valid also for the case where ψ_i is a multi-component vector and a, b, and c are matrices.

Our exact decimation scheme takes into account the inherent quasiperiodicity of the system. The resulting formulation provides a very precise criterion to distinguish the E, C and L phases and hence obtaining the phase diagram. Furthermore, different QP potentials may be described by the same asymptotic RG flow which implies universality for the scaling of the fractal spectra and wave functions. In general, the scaling of the wave functions is determined by the asymptotic RG fixed point or limit cycle whereas the scaling exponents for the spectrum are obtained from the eigenvalues of the linearized RG transformation. The existence of such finite RG attractors provides a new and very accurate method to compute e.g., eigenvalues or phase boundaries. As we will show, in some cases the RG flow is attracted by an infinite RG strange set. Thus, the nonlinearity of the RG transformation opens the way to chaotic dynamics even though the original tight binding model (TBM) is linear.

One of the key questions that we discuss here is the fluctuations of localized wavefunctions. Localization of electrons in disordered materials has been one of the central problems in condensed matter physics. Various recent publications [9] have addressed the problem of scaling of localized wave functions for the case of random disorder and have speculated that exponentially localized wave functions may exhibit fractal fluctuations. However, the issue has remained controversial as these numerical studies have been plagued by finite size effects. We study the fluctuations of the localized wave functions in QP systems using our exact RG approach. The motivation for studying such deterministic disorder is that in the case of random disorder additional complexity arises from distinguishing between the space fluctuations of a single wave function and the disorder fluctuations coming from various different realizations of the randomness. Furthermore, our new RG scheme for QP models can be implemented upto machine precision and is not handicapped by finite size effects encountered in the previous studies.

In this paper, we concentrate on the Harper (10) and its various generalizations which exhibit metallic, insulating, and *critical* phases. Our main focus is the universal scaling behavior in the C phase as well as on the universal self-similar fluctuations of the L phase. We discuss two different generalizations of the Harper equation: one is the TBM associated with two-dimensional electron gas with next nearest neighbor (NNN) interaction and the other is the vector TBM associated with anisotropic quantum XY model in a transverse field. In these generalized models the critical point is replaced by a

C phase existing in a finite parameter interval leading to additional richness and various new universality classes.

The main features of the RG method are summarized in § 2. In § 3, we briefly review the Harper equation and describe the scaling associated with critical and localized wavefunctions. Section 4 deals with the generalized Harper equation in the two-dimensional NNN coupling case. The resulting TBM can be either real or complex depending on the isotropy properties of the NNN couplings, and we discuss the phase diagram in both cases. The real TBM exhibits a fat C phase described by a strange attractor of the RG flow. On the other hand, the complex TBM provides an interesting example of a reentrant metallic phase. In § 5, we discuss the vector TBM for the spinless fermion representation of the anisotropic quantum XY model in an inhomogeneous transverse field. The exchange anisotropy causes this model to deviate from the Harper equation and fattens the critical point into a C phase. We also elaborate on the relationship between the TBM for an electron on the triangular lattice and the quantum Ising model in a transverse field. Finally, in § 6 we summarize our conclusions.

2. Decimation method

We have recently developed a new *exact* RG approach which takes into account the internal frequency σ of the system. For details of the RG approach, we refer the readers to our earlier papers [4–8]. Our methodology is somewhat similar to that of Ostlund *et al* [11] as it describes the scaling properties of the wave functions for a specific eigenvalue. However, the method of Ostlund *et al* which was based on transfer matrices, had limited success due to the fact that at the localization threshold infinite products of transfer matrices diverged. We propose a scheme where instead of multiplying transfer matrices, the TBM itself is decimated. The main advantage is the reduction in the number of functions needed to carry out the renormalization. The price we have to pay is that our recursion relations will be slighly more complicated. However, it turns out that with fewer functions we are able to eliminate directions which lead to divergences. This not only helps in approaching the RG problem but also provides practical means of calculating various essential quantities like the localization threshold and eigenvalues.

The decimation method can be implemented for any irrational σ but here we consider the simplest case where σ is given by the inverse golden ratio $\sigma = (\sqrt{5} - 1)/2$. In this case, it is appropriate to decimate out all sites except those labelled by the Fibonacci numbers F_n (which are the best rational approximants of the golden ratio). At the nth decimation level, the TBM is expressed in the form

$$f_n(i)\psi(i + F_{n+1}) = \psi(i + F_n) + e_n(i)\psi(i). \tag{2}$$

The additive property $F_{n+1} = F_n + F_{n-1}$ of the Fibonacci numbers provides exact recursion relations for the decimation functions e_n and f_n:

$$e_{n+1}(i) = -\frac{Ae_n(i)}{1 + Af_n(i)}, \tag{3}$$

$$f_{n+1}(i) = \frac{f_{n-1}(i + F_n)f_n(i + F_n)}{1 + Af_n(i)}, \tag{4}$$

$$A = e_{n-1}(i + F_n) + f_{n-1}(i + F_n)e_n(i + F_n).$$

There are a number of reasons why the iteration of the recursion relations is superior to the direct iteration of the TBM:

(1) Often the iteration of the recursion relations is numerically more stable than that of ([TBM]).

(2) The possible self-similarity observed by monitoring the behaviour of ψ_i over the range $i \in [-F_n, F_n]$ with increasing n can be captured by a simple asymptotic limit cycle for the decimation functions e_n and f_n. This in turn determines the universal scaling ratios

$$\zeta_j = \lim_{n\to\infty} |\psi(F_{pn+j})/\psi(0)|; \quad j = 0, \ldots, p-1, \tag{5}$$

where p is the cycle length.

(3) An unknown or inaccurate parameter in the TBM and the asymptotic limit cycle can be simultaneously determined self-consistently. This is based on using the secant method to calculate the parameter value for which e.g., $f_n(0) = f_{n-p}(0)$ for some very high n. In other words, the decimation equations themselves provide a new method to compute eigenvalues or phase boundaries up to machine precision.

3. Harper equation

The Harper equation [10], which describes a wide variety of condensed matter problems including the two-dimensional electron gas in a transverse magnetic field, has been in the forefront of theoretical research for the last two decades. In addition to various numerical and RG studies [1], the model has also been solved using the Bethe-ansatz [12].

The Harper equation has the simple form

$$\psi_{i+1} + \psi_{i-1} + 2\lambda\cos[2\pi(i\sigma + \phi)]\psi_i = E\psi_i. \tag{6}$$

The parameter σ describing the magnetic flux in the two-dimensional electron problem introduces quasiperiodicity into the system and is taken as the inverse golden ratio $\sigma = (\sqrt{5} - 1)/2$. Although the phase diagram is the same for all diophantine σ, the choice of the golden mean simplifies the RG analysis. Furthermore, it is assumed that the phase ϕ is chosen so that the main peak of the wave function ψ_i is at $i = 0$ [5, 11]. The variation in λ leads to the metal-insulator transition. At the onset of transition $\lambda = 1$, the model has been shown to be self-dual [13] with critical states and a fractal butterfly spectrum [14].

Figure 1 shows the wave function with $E = 0$ at the critical point $\lambda = 1$. The wavefunction repeats itself at every third Fibonacci site. In fact, this repetition of the wave function takes place at all sites: i.e., for every given site, there exists a whole sequence of sites displaced by Q_k (related to the rational approximants of σ^3), where the wave function approaches the same amplitude. [5, 6] This type of translational symmetry in the wave function on a *Fibonacci-lattice* or *Q_k-lattice* is a characteristic feature of the QP systems. Therefore, the wave function does not vanish asymptotically but there are an infinite set of universal numbers ζ (some of them are defined in 5) describing the amplitude of the wave function at points which are spaced further and further apart from

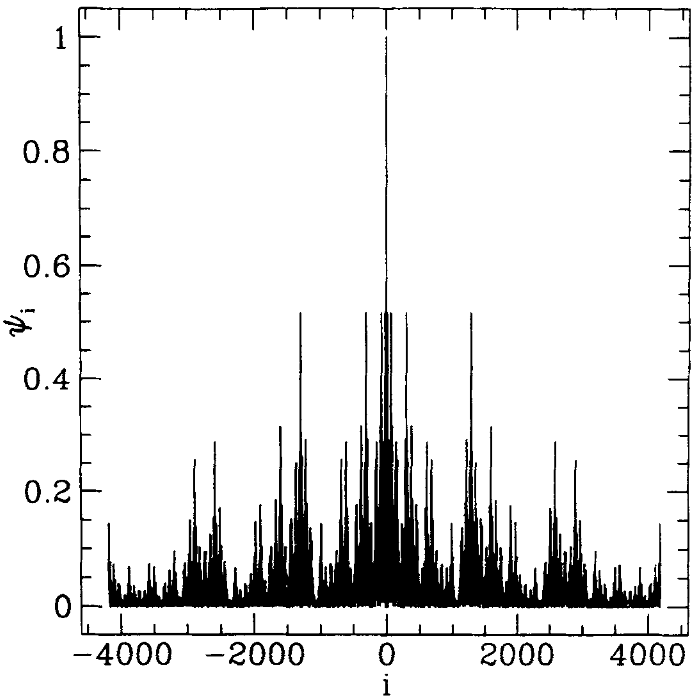

Figure 1. Wave function of the Harper equation at the critical point $\lambda = 1$ with $E = 0$ and $\phi = 1/4$.

the central peak. In addition, the structure around the subpeaks mimics the behavior seen on a larger length scale resulting in self-similarity in the wave function.

In the RG approach, the self-similarity results in a nontrivial asymptotic p-cycle for the decimation functions and the scaling ratio. In the E phase, the RG cycle is trivial with $\zeta = 1$. It should be noted that eigenstates with different energies are characterized by different asymptotic behavior of the RG trajectories. In addition to $E = 0$, the eigenstates at the band edges are described by a limit cycle while almost all eigen energies are believed to correspond to a strange invariant set.

In order to investigate the localized states, the wave function ψ_i is written as

$$\psi_i = \mathrm{e}^{-\gamma|i|}\eta_i, \tag{7}$$

where γ is the Lyapunov exponent which vanishes in the E and C phase. The L phase is characterized by a positive Lyapunov exponent $\gamma = \ln(\lambda)$ corresponding to an exponential decay of the wave function [AA]. Therefore, the function η_i describing the fluctuations of the localized wave function satisfies the following TBM for $i > 0$:

$$\frac{1}{\lambda}\eta_{i+1} + \lambda\eta_{i-1} + 2\lambda\cos[2\pi(i\sigma + \phi)]\eta_i = E\eta_i. \tag{8}$$

We study this TBM using the RG method. Figure 2 shows the fluctuations η_i at a band maximum (the band center with $E = 0$ is described by a period-3 RG cycle). The fluctuations for $\lambda > 1$ possess the same complexity and richness as the critical states. Furthermore, it turns out that the self-similar fluctuations are universal throughout the L

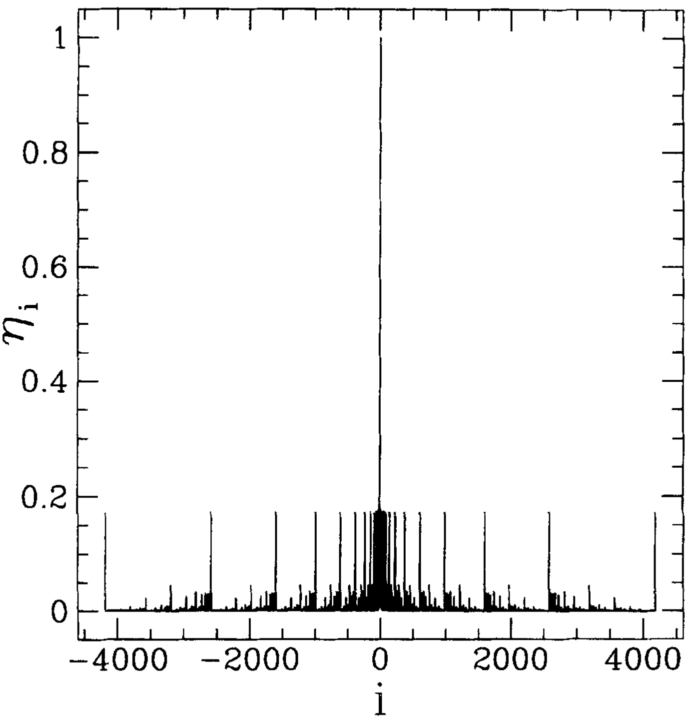

Figure 2. Absolute value of self-similar universal fluctuations in the supercritical phase of the Harper equation.

phase and are characterized by a unique strong coupling RG fixed point with $\zeta = 0.172586\cdots$. In particular, after writing the RG transformation and the decimation functions in the form where the discrete lattice index i is replaced by a continuous renormalized variable, we are able to solve for a power series expansion of the fixed point by the Newton method. Details of the RG fixed point and the eigenvalue analysis are discussed elsewhere. [8] It should be noted that although the fluctuations bear a qualitative resemblance to the wave functions in the C phase, quantitatively they belong to different universality classes.

4. Generalized Harper equation: Bloch electrons with NNN interaction

The Harper equation models a two-dimensional Bloch electron on a square lattice in a transverse magnetic field when only the hopping to nearest-neighbor (NN) lattice sites is taken into account. Including the hopping also to the NNN sites one obtains the following TBM:

$$\begin{aligned} & t_a(\psi_{k+1} + \psi_{k-1}) + 2t_b \cos[2\pi(k\sigma + \phi)]\psi_k \\ & \quad + \exp[i2\pi(k\sigma + \phi)]\{t_{ab}e^{i\pi\sigma}\psi_{k+1} + t_{a\bar{b}}e^{-i\pi\sigma}\psi_{k-1}\} \\ & \quad + \exp[-i2\pi(k\sigma + \phi)]\{t_{ab}e^{i\pi\sigma}\psi_{k-1} + t_{a\bar{b}}e^{-i\pi\sigma}\psi_{k+1}\} = E\psi_k\,. \end{aligned} \tag{9}$$

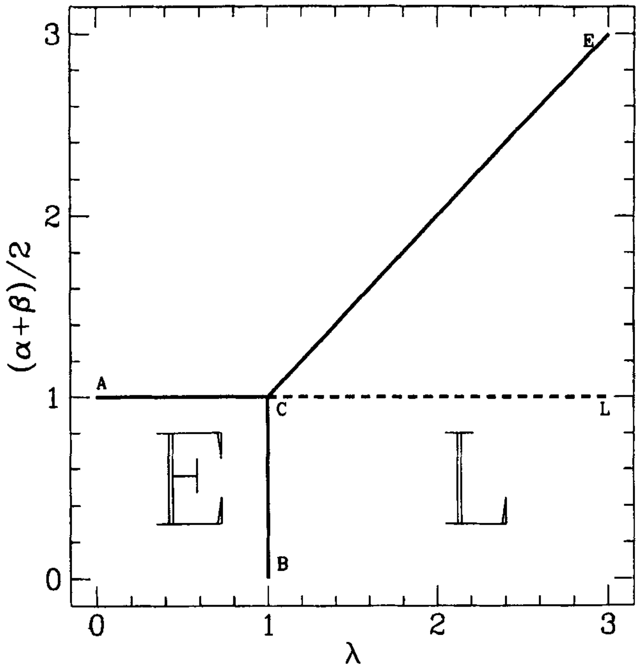

Figure 3. Phase diagram of the two-dimensional electron gas with NNN interaction. For $t_{ab} = t_{a\bar{b}}$, the region bounded by the AL and CE lines is critical. However, as soon as these two NNN couplings become different from each other, the C phase is replaced by a reentrant E phase. The solid lines BC, AC, and CE always remain critical. The extension of the line AC (line CL) separates the L phase into two different regions with different scaling properties.

Here t_a and t_b are the NN couplings while t_{ab} and $t_{a\bar{b}}$ are the diagonal NNN couplings which are taken to be zero in the case of the Harper equation. In the isotropic limit $t_{ab} = t_{a\bar{b}}$, the TBM becomes real and has been studied in various recent papers [15, 7, 8]. The two cases $t_{ab} = t_{a\bar{b}}$ and $t_{ab} \neq t_{a\bar{b}}$ will be discussed separately below.

Case I. $t_{ab} = t_{a\bar{b}}$. In this limit, the above TBM can be written as

$$\{1 + \alpha\cos[2\pi(\sigma(i + \tfrac{1}{2}) + \phi)]\}\psi_{i+1} + \{1 + \alpha\cos[2\pi(\sigma(i - \tfrac{1}{2}) + \phi)]\}\psi_{i-1}$$
$$+ 2\lambda\cos[2\pi(\sigma i + \phi)]\psi_i = E\psi_i\,. \qquad (10)$$

Here $\lambda = t_b/t_a$ and $\alpha = 2t_{ab}/t_a$. The Harper equation corresponds to the limit $\alpha = 0$ where the NNN coupling term is zero.

Figure 3 shows the phase diagram of the model in the $\lambda - \alpha$ plane [15, 7]. By carrying out the RG analysis at the band minimum, we find that the model belongs to the universality class of the Harper equation for $\alpha < 1$ [7]. Therefore, along the line BC (except at the point C), the universal scaling is identical to that of the Harper equation. An interesting feature of the phase diagram is that the C phase exists in a finite region of parameter values. Unlike the scaling at the critical point of the Harper equation given by limit cycles of the RG equation, the scaling in this

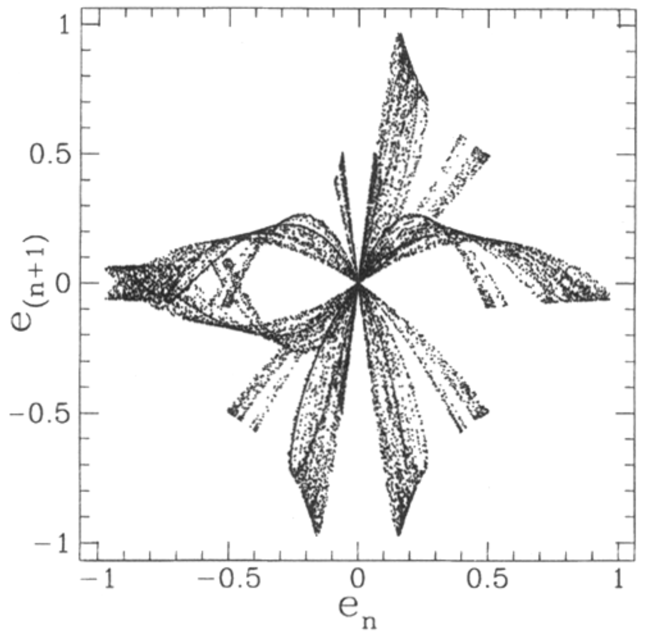

Figure 4. A two-dimensional projection of the attractor of the renormalization flow in the strong coupling limit $\lambda \to \infty$ with $\alpha = \beta > \lambda$. For about 4000 different parameter values, decimation equations were iterated 35 times and the first 10 iterates were ignored as transients.

C phase was found to be characterized by an infinite strange set of the RG flow [7]. Therefore, the fractal wave functions do not exhibit self-similarity and are described by an ergodic set which brings out the order underlying the chaotic RG trajectories.

It turns out that the qualitative behavior of the RG trajectories describing the fluctuations in the L phase are similar to that of the C phase discussed above. The universality class for the fluctuations in the L phase turns out to be unaltered as long as $\alpha < 1$ [8]. Above the threshold $\alpha = 1$ (above the line CL), the situation is again much more complicated. Almost all parameter values in this region are described by the same infinite ergodic set of scaling ratios. In analogy with the C phase discussed above, the ergodic set of scalings, when projected on two dimensions, exhibit an intriguing order manifested in the form of an orchid-flower (see figure 4).

Another intriguing feature of the C and L phase is that there appears to be a special set of parameter values $\alpha = 1/|\cos(2\pi r)|$ with rational r where the RG trajectories in both the L and C phase converge on a limit cycle of a high period (for a fixed α, the same limit cycle is observed for all λ in the L phase whereas in the C phase also λ has to be chosen carefully to obtain a finite cycle). We conjecture that these unstable fixed points are dense on the strange attractor of the RG flow.

Detailed numerical studies of RG flow show that the lines in the phase diagram describing the boundaries between the E, C and L phases exhibit fractal wave functions which differ quantitatively from the wave functions in the C phase discussed above. The lines AC (E-C transition) and CE (C-L transition) appear to correspond to their own RG strange sets [7]. The attractor associated with the line CE seems to envolope the strange set of the C phase enclosed by AC and CE.

Case II. $t_{ab} \neq t_{a\bar{b}}$. In this case, the TBM is complex and has not been fully investigated previously. Using the duality property of the model, Han *et al* [15] calculated the Lyapunov exponent of the model analytically in the space of the three parameters $\lambda = t_b/t_a$, $\alpha = 2t_{ab}/t_a$, and $\beta = 2t_{a\bar{b}}/t_a$. They concluded that the system is localized for $\lambda > 1$ if $(\alpha+\beta)/2 < 1$ and for $\lambda > (\alpha+\beta)/2$ otherwise. Apart from the existence of the metal-insulator transition, nothing was known about the scaling properties of the complex model. In particular, one of the important questions is how the inequality in the NNN couplings affects the scaling properties in the C and L phases described by the strange invariant sets.

With the decimation methodology, we found that the phase diagram changed discontinuosly as $\alpha - \beta$ became different from zero. As soon as α and β differ, $\alpha - \beta$ was found to be an irrelevant parameter and the phase diagram is determined solely by two parameters: λ and $(\alpha+\beta)/2$.

In the parameter range $(\alpha+\beta)/2 < 1$, the decimation functions become asymptotically real approaching the same universal cycles as for the Harper equation $\alpha = \beta = 0$. For $\alpha + \beta \geq 1$, $\alpha \neq \beta$, the decimation functions stay complex also asymptotically. Here the C phase of the case $\alpha = \beta > 1$ is replaced by a reentrant E phase. Moreover, the E-L transition belongs to the universality class of the corresponding transition in the Harper equation upto a complex phase factor in ζ. The invariant strange sets of the renormalization describing the fluctuations in the L phase for the limit $\alpha = \beta$ degenerates into the strong coupling Harper fixed point as soon as α and β become different [16].

On the line $(\alpha+\beta)/2 = 1$, the model can be explicitly shown to be related to the TBM for the triangular lattice, upto a complex phase factor [16]. The TBM describing Bloch electrons on a triangular lattice is [17]

$$\cos\left[\pi\left(i\sigma + \left(\phi + \frac{\sigma}{2}\right)\right)\right]\psi_{i+1} + \cos\left[\pi\left(i\sigma + \left(\phi - \frac{\sigma}{2}\right)\right)\right]\psi_{i-1} + \lambda\cos[2\pi(i\sigma + \phi)]\psi_i = E\psi_i. \tag{11}$$

Our detailed RG analysis shows that unlike the E-L transition of the TBM describing the square lattice, the TBM describing the triangular lattice exhibits the C-L transition: i.e., the line AC remains critical (described by a unique fixed point except at the point C which is characterized by its own fixed point) while the wave functions along the line CL are localized. The fractal characteristics on the line AC, point C and the fluctuations on the line CL define new universality classes (of the triangular lattice) which are quantitatively different from those of the square lattice, namely the Harper universality classes. The reentrant E phase can be understood in the anisotropic limit, $\alpha \to \infty$, $\beta/\alpha \to 0$, and $\lambda/\alpha \to 0$.

The resulting TBM at a site k is related to the weak coupling limit of the Harper model by a complex phase factor $\exp(i2\pi\phi k)\exp(i\pi\sigma k^2)$ [16].

5. Quantum spin chains

5.1 *Vector tight binding model*

We next describe a related problem, namely the XY quantum spin chain, described by a vector TBM which can be viewed as a perturbed Harper equation

$$J(y,x)\Psi_{i+1} + J(x,y)\Psi_{i-1} + 2h_i\Psi_i = \bar{E}\Psi_i, \tag{12}$$

where $h_i = \lambda\cos(2\pi(i\sigma + \phi))$, $J(x,y)$ and $\bar{E}$ are 2×2 matrices and Ψ is a two-dimensional vector describing the coupled system

$$J_y\psi_{i+1} + J_x\psi_{i-1} + 2h_i\psi_i = E\eta_i, \tag{13}$$

$$J_x\eta_{i+1} + J_y\eta_{i-1} + 2h_i\eta_i = E\psi_i. \tag{14}$$

The above model is the fermion representation of the quantum XY spin chain in a transverse field with the Hamiltonian [18]

$$H = -\sum_i [\tfrac{1}{2}(J_x s_i^x s_{i+1}^x + J_y s_i^y s_{i+1}^y) + h_i s_i^z]. \tag{15}$$

Here s are the Pauli matrices describing the spin 1/2. J_x and J_y describe ferromagnetic exchange interactions. The model can also be written as

$$(H_h - 2gH_g)(H_h + 2gH_g)\Psi = E^2\Psi, \tag{16}$$

where H_h denotes the Harper Hamiltonian and H_g denotes the part of the Hamiltonian which exists only at a non-zero value of the anisotropy g defined as $J_x = 1 - g$ and $J_y = 1 + g$. We see that the anisotropy perturbs the "squared" Harper model.

Figure 5 shows the phase diagram in the $\lambda - g$ plane [19, 20]. Whenever λ and one of the exchange interactions become equal either the E-C or C-L transition takes place. The E-C transition corresponds always to a smaller absolute value of λ, i.e. to the exchange interaction whose absolute value is smaller. Whenever J_x and J_y differ a fat C phase is observed in the phase diagram. Therefore, the fattening of the critical point is due to the breaking of the $U(1)$ symmetry in the fermion Hamiltonian which is a consequence of the broken $O(2)$ symmetry in spin space.

An interesting consequence of the spectral and magnetic interplay is the fact that the onset of the C-L transition is coincident with the onset of the magnetic transition to the long range order (LRO) where two-point long range spin-spin correlations vanish. However, the significance of the E-C transition on the magnetic properties of the quantum chain remains a mystery.

The C phase of the model is found to be described by three different universality classes [6]. The E-C transition is characterized by a different universality class from that of the C-L transition line. The critical phase sandwiched between the E-C and C-L transitions is conjectured to form its own universality class although the existence of a RG cycle has been confirmed only in the Ising limit $J_y = 0$ (see the next section). All these three universality classes are different from that of the Harper critical point.

The fluctuations in the L phase are believed to be universally characterized by a strong coupling fixed point. This was confirmed in the Ising limit discussed in the next section.

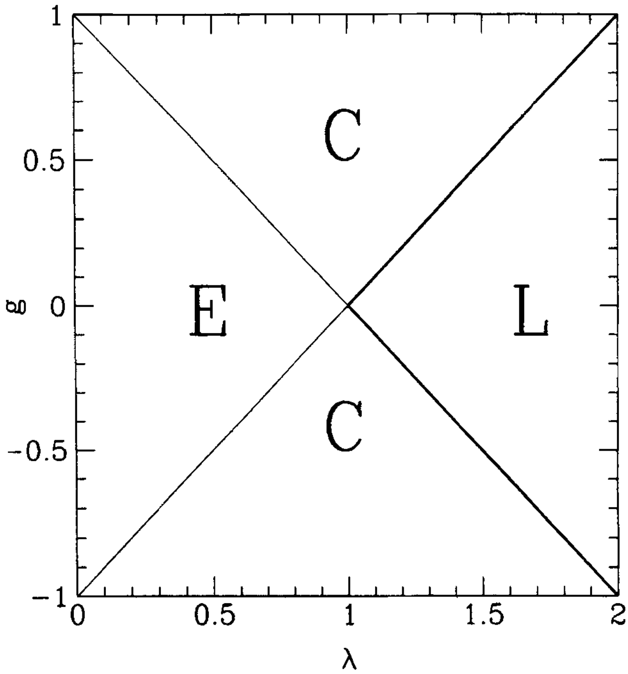

Figure 5. Phase diagram for the quasi-particle excitations of the quantum spin chain with $J_x = 1 - g$ and $J_y = 1 + g$. For $g = \pm 1$, the model reduces to Ising model with C and L phases. The dark boundary line separating C and L phases is the conformal line describing the onset to magnetic long range order.

5.2 *Relation to the bloch electrons*

It is rather interesting that the Bloch electron problem and the spin problem are related in certain limits. The TBM for the triangular lattice (see (11)) resembles the TBM for the Ising limit ($J_y = 0$) of the quantum spin chain discussed in the previous section [19, 20]. Both are modeled by a NN TBM where the diagonal as well as the off-diagonal terms are modulating. The periodicity of the diagonal term is twice the periodicity of the off-diagonal term. The only difference between these two models is in the relative phase differences between the diagonal and off-diagonal terms. As a consequence, the wave functions at the band edges have exact symmetry in the triangular model but only asymptotic symmetry in the quantum Ising model (QIM).

The phase diagram is the same for both systems with C phase for $\lambda \leq 1$ and the L phase for $\lambda > 1$ (see figure 5 at $g = -1$). There are three RG limit cycles which determine the asymptotic scaling of the wave functions at the upper or lower band edges in both systems for $\lambda < 1$, $\lambda > 1$, and $\lambda = 1$, respectively. However, an exception to this rule is formed by the lower band edge of the QIM at $\lambda = 1$ which is described by its own RG fixed point. The corresponding state is believed to be conformally invariant mapping to none of the quantum states of the triangular model [5, 6].

The fact that RG analysis singles out the conformal state in distinguishing the two models that bear striking mathematical resemblance (inspite of some important physical differences) proves the fact that our RG method provides an extremely useful description of the physics underlying the QP models discussed here.

Another interesting feature is that the conformally invariant state of the Ising model also describes the fluctuations in the exponentially localized strong coupling limit of the Harper equation for $E = 0$ [8]. In other words, the self-similar fluctuations at the band center of the supercritical Harper model describe the self-similarity of the quantum state responsible for the long range order in Ising spin. This fact can be understood by noticing that at the conformal point, the TBM described by (13) resembles equation (8) describing the fluctuations in the localized phase of the Harper equation. Therefore, the excitations in the spin model are related to the fluctuations in the localized states of the Harper equation and the conformal universality class of the spin model is related to the strong coupling universality of the Harper equation.

6. Conclusions

Our study of a variety of systems with a common feature being the competing length scales, shows the existence of interesting phase diagrams exhibiting E-L, E-C and C-L transitions. Furthermore, another intriguing possibility is the reentrant E phase. This richness in the phase diagram manifests itself in interesting variations in the transport properties as the parameters of the system are varied. We believe that our results can be experimentally realized on two-dimensional mesoscopic systems. The results on the quantum spin chain may be relevant in magnetic superlattices.

Wave functions in both the *critical* and localized phases exhibit fractal character. The universal scaling properties in these phases can be studied with very high precision using a decimation method. The decimation scheme unveils some interesting relationships between seemingly unrelated problems and limits the number of universality classes. One of our important results is that a wide class of systems can be shown to belong to the universality class of the NN square lattice (Harper universality) or the triangular lattice. In addition to determining the phase diagram and scaling characteristics, our method provides a new tool to determine eigenvalues, phase boundaries, and Lyapunov exponents. This is because unknown parameters in the TBM can be determined self-consistently with the RG limit cycle upto machine precision. We refer the readers to our earlier papers for various detailed examples.

Another important result of our studies that the L phase exhibit the same richness as the C phase with fractal characteristics, has to be viewed from a more general perspective. The localization phenomenon in QP systems is analogous to the Anderson localization of the random systems. Therefore, our findings strengthen the likelyhood of fractal characteristics in other random or aperiodic systems exhibiting localization.

Finally, we would like to point out that there are many other systems in both classical and quantum chaos [24] which can be modeled by [TBM]. One interesting example is the study of phonon modes in the classical Frenkel-Kontorova model exhibiting the pinning-depinning transition [22, 23]. Our very recent studies have shown the existence of a new type of phonon modes with eigenstates represented by an infinite series of step functions [24]. Another important problem that is under investigation is the quantum kicked rotor which has been mapped to a TBM [21]. Thus, we believe that there are many more interesting results to be discovered using the RG methods.

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References

[1] For a review, see J B Sokoloff, *Phys. Rep.* **126**, 189 (1985)
[2] V I Arnold and A Avez, *Ergodic problems of classical mechanics* (Benzamin, New York, 1968)
[3] B Simon, *Adv. Appl. Math.* **3**, 463 (1982)
[4] J A Ketoja, *Phys. Rev. Lett.* **69**, 2180 (1992)
[5] J A Ketoja and I I Satija, *Phys. Lett.* **A194**, 64 (1994)
[6] J A Ketoja and I I Satija, *Physica* **A219**, 212 (1995)
[7] J A Ketoja, I I Satija and J C Chaves, *Phys. Rev.* **B52**, 3026 (1995)
[8] J A Ketoja and I I Satija, *Phys. Rev. Lett.* **75**, 2762 (1995)
[9] L Pietronero, A P Siebesma, E Tosatti and M Zannetti, *Phys. Rev.* **B36**, 5635 (1987)
M Schreiber and H Grussbach, *Phys. Rev. Lett.* **67**, 607 (1991)
F Lema and C Wiecko, *Phys. Scr.* **47**, 129 (1993)
[10] P G Harper, *Proc. Phys. Soc. London* **A68**, 874 (1955)
[11] S Ostlund and R Pandit, *Phys. Rev.* **B29**, 1394 (1984)
S Ostlund, R Pandit, D Rand, H J Schellnhuber and E D Siggia, *Phys. Rev. Lett.* **50**, 1873 (1983)
[12] P B Wiegmann and A V Zabrodin, *Phys. Rev. Lett.* **72**, 1890 (1994)
L D Faddeev and R M Kashaev, *Commun. Math. Phys.* **169**, 181 (1995)
[13] S Aubry and G André, *Ann Israel Phys. Soc.* **3**, 133 (1980)
[14] D R Hofstadter, *Phys. Rev.* **B14**, 2239 (1976)
[15] J H Han, D J Thouless, H Hiramoto and M Kohmoto, *Phys. Rev.* **B50**, 11365 (1994)
[16] J A Ketoja and I I Satija, *Reentrant phase diagram of the generalized Harper equation*, preprint (1996); *J. Phys.* **C9**, 1123 (1997)
[17] F H Claro and G H Wannier, *Phys. Rev.* **B19**, 6068 (1979)
D J Thouless, *Phys. Rev.* **B28**, 4272 (1983)
[18] E Lieb, T Schultz and D Mattis, *Ann. Phys. (N.Y.)* **16**, 407 (1961)
[19] I I Satija *Phys. Rev.* **B48**, 3511 (1993); *Phys. Rev.* **B49**, 3391 (1994)
I I Satija and M Doria, *Phys. Rev.* **B39**, 9757 (1989)
[20] I I Satija and J C Chaves, *Phys. Rev.* **B49**, 13239 (1994)
[21] S Fishman, D R Grempel and R E Prange, *Phys. Rev. Lett.* **49**, 509 (1982)
[22] P Bak, *Rep. Prog. Phys.* **45**, 587 (1982)
[23] S Aubry, *Physica* **D7**, 240 (1983)
[24] J A Ketoja and I I Satija, *Critical phonons of the supercritical Frenkel–Kontorova model: Renormalization bifurcation diagrams*, *Physica* **D** (in press)