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Experiments on quantum chaos using microwave cavities: Results for the pseudo-integrable L-billiard

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Abstract. We describe microwave experiments used to study billiard geometries as model problems of non-integrability in quantum or wave mechanics. The experiments can study arbitrary 2-D geometries, including chaotic and even disordered billiards. Detailed results on an L-shaped pseudo-integrable billiard are discussed as an example. The eigenvalue statistics are well-described by empirical formulae incorporating the fraction of phase space that is non-integrable. The eigenfunctions are directly measured, and their statistical properties are shown to be influenced by non-isolated periodic orbits, similar to that for the chaotic Sinai billiard. These periodic orbits are directly observed in the Fourier transform of the eigenvalue spectrum.

Keywords. Quantum chaos; billiards; eigenvalues; eigenfunctions; random matrix theory; pseudo-integrable.

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1. Introduction

The issue of non-integrability in quantum mechanics is relevant in a variety of physical systems. Indeed initial developments were in nuclear physics, where random matrix theory (RMT) was developed to account for the statistical properties of the eigenvalue spectra of heavy nuclei [1, 2]. Similar and hence universal behavior has been shown numerically in low dimensional billiard systems [3]. This universal behavior is not just confined to quantum mechanical systems such as atoms, electrons and nuclei, but is also seen in wave mechanical systems like acoustic and electromagnetic systems. The common feature in all these apparently disparate systems is that the corresponding ray or particle dynamics is chaotic. Significant developments in semiclassical theory have shown the importance of classical structures, such as periodic orbits, in organizing quantum behavior [4–7]. Although many experimental systems exhibit quantum or wave chaos, few of these yield properties which are directly comparable to analytical results. Hence theoretical developments have relied on numerical simulations to support their claims. Although computational tools are rapidly increasing in their applicability, there are nevertheless many important problems which are still not amenable to reliable numerical solutions. In this paper we describe an experimental approach which leads to important insights to the role of classical chaos and even disorder in quantum or wave mechanics. This approach, which utilizes microwave cavities, is best suited to the study of "billiard" problems that examine the quantum properties of a particle in model 2-D geometries. In the past five years these experiments have resulted in several important contributions to the field of quantum chaos. The experiments have led to the first direct observation of scarred eigenfunctions [8]. The statistical properties of the eigenfunctions for the chaotic Sinai-stadium geometry were seen to be in good agreement with the universal Gaussian orthogonal ensemble (GOE) of RMT [9]. It was further found that chaotic geometries with non-isolated periodic orbits like the Sinai billiard show deviation from universal behavior. The eigenfunctions of disordered geometries (cavities with many finite sized scatterers) have shown the effects of localization on their statistical properties [9]. This has stimulated interesting theoretical work on the spatial distribution of eigenfunctions using supersymmetric non-linear σ-models [10–12] and to date remain the only experiments in which these theories can be tested. The eigenvalue statistics have also been analyzed and shown to be in direct agreement with GOE for the Sinai-stadium [13]. The experiments have also led to the experimental verification [14] of the theorem of isospectral domains [15] and resolved an important issue of degeneracies. Periodic orbits were observed in both closed and open billiard systems by Fourier transforming the eigenvalue and transmission spectrum respectively [16, 17]. A consistent theme of this work has been the gleaning of universal behavior and deviations from such universal behavior, by exploiting the ability to study controlled variations in geometry afforded by the microwave experiments. In this paper, we discuss in detail the eigenvalue and eigenfunction properties of an important class of billiards which are pseudo-integrable. This example is important because it is an intermediate system that is neither integrable nor fully chaotic. The theoretical description of intermediate systems is challenging and it is hoped that these quantitative experimental results will lead to rigorous tests for theory. We begin with an introduction of the experimental techniques.

2. Experimental techniques

The experiments utilize "thin" microwave cavities, in which below a cutoff frequency, the 2-D scalar limit of the Maxwell–Helmholtz wave equation applies, and the correspondence to the Schrödinger equation is exact. For a cavity of arbitrary cross-section in the $x-y$ plane but uniform along the z- direction, the z-component of the wave vector is quantized as $k_z = p\pi/d$, where p is an integer and d is the cavity thickness along the z- axis. Maxwell's equations then reduce to

$$[\nabla^2 + (k^2 - (p\pi/d)^2]\{E_z, B_z\} = 0, \tag{1}$$

where ∇ is the 2-D Laplacian operator. Of course the EM field is a vector field, however a special case occurs for the transverse magnetic (TM) modes for which $B_z = 0$, and further when $p = 0$. In this limit, (1) reduces to the Schrödinger equation $(\nabla^2 + k^2)$ $\Psi = 0$, where $\Psi \longleftrightarrow E_z$. This limit can be experimentally achieved by confining measurements to the frequency range $f < f_c = c/d$. For $d = 6$ mm, the cutoff frequency $f_c = 25$ GHz, and for all lower frequencies, the correspondence between the Schrödinger and Maxwell wave equations is exact in the thin 2-D cavities. Cavities are fabricated with the 2-dimensional cross-section cut out from 6 mm thick copper sheets, and placed between two copper plates. Coupling to the cavities is accomplished by loops terminating coaxial cables – the loops couple to the microwave magnetic fields at the perimeter. The cavity response is examined using an HP 8510 Network Analyzer (ANA). The eigenvalue spectrum is obtained from the resonances observed in the transmission spectrum, with the energy eigenvalues E_n obtained from the resonant eigenfrequencies f_n using $E_n = f_n^2$. By using appropriate care, in particular coupling at several locations, it is possible to observe the first 1000 or so levels of the spectrum, without any missed levels. Thus although the finite conductivity of the copper walls leads to broadened resonances, it is not necessary to use superconducting cavities in order to carry out reliable experiments on eigenvalue statistics. One of the important and unique features of the experiments is the ability to directly measure eigenfunctions. This method, first devised by one of us [8, 18], utilizes the perturbation of the cavity resonance by a small metallic ball. Placing the ball at coordinates (x, y) in the cavity leads to a shift of the resonance frequency $\Delta f_n(x, y) = -\beta\Psi_n^2(x, y)$, from which the eigenfunction Ψ_n can be directly determined by measuring the frequency shift $\Delta f_n(x, y)$ using the ANA. Further details about the method can be found in [18].

3. Pseudo-integrable geometries

An important class of billiard systems are the pseudo-integrable geometries as they are the closest step away from integrable systems. Most polygonal billiards, with the exception of rectangles, 30°–60°–90° angle triangles, and the equilateral triangle are pseudo-integrable. The non-integrability arises at the vertex, when it is not 90°, 60°, 45°, 30°. Trajectories diverge at this point and lead the classical phase to be a surface of genus greater than one. Since the classical phase space has to be a torus for the motion to be integrable, a higher genus surface ensures non-integrability. A formula for relating the genus, g, to the angles of the vertex has been derived and is given as [19, 20]:

$$g = 1 + \frac{N}{2}\sum\left[\frac{m_i + 1}{n_i}\right], \tag{2}$$

where n_i and m_i are such that $\alpha_i = m_i\pi/n_i$ are the internal vertex angles, and N is the smallest of the integer set n_i. A surface of genus 2 which corresponds to a pseudo-integrable system is the L-shaped billliard [21, 22]. However there are similarities to integrable systems, as the periodic orbits in these geometries also occur in families, which bifurcate at the vertex which is not-integrable. They also appear to have the same asymptotic proliferation of periodic orbits [20]. However numerical work has shown that the eigenvalue statistics of these systems show non-Poisson statistics. Below we shall discuss results on a well-known example, the L-shaped billiard.

3.1 *Eigenvalue statistics*

The eigenvalue statistics of a pseudo-integrable (PI) L-shaped cavity was studied with a 44 cm by 21.8 cm copper cavity with a 9.55 cm by 9.65 cm copper piece at one of the corners. In the frequency range 0.045 GHz to 18.497 GHz, 1000 energy levels were obtained. This data also shows good agreement with the Weyl formula [23]. The nearest neighbor energy spacing $P(s)$, and spectral rigidity $\Delta_3(L)$ were studied to understand the

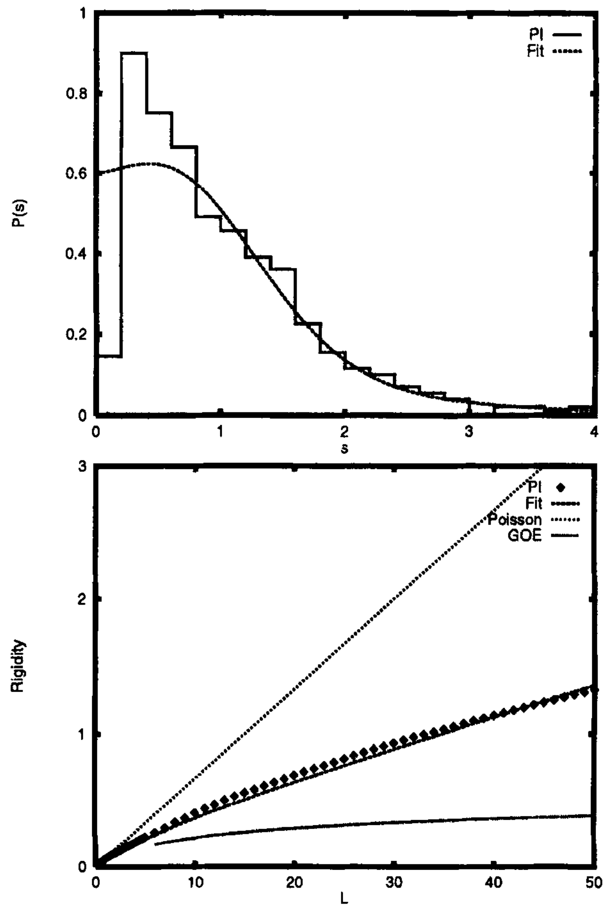


Figure 1. Eigenvalue statistics for the pseudo-integrable (PI) geometry. (Top) $P(s)$ for the pseudo-integrable geometry. The interpolation to fit the data is done using eq. (3) with $\beta = 0.62$. (Bottom) The rigidity, $\Delta_3(L)$ versus L for the pseudo-integrable geometry. The interpolation is done using (4) with $\beta = 0.65$.

nature of the eigenvalue spectrum. Even though the classical phase space structure is closer to that of an integrable geometry, level repulsion is quite evident in figure 1. The spectral rigidity $\Delta_3(L)$ was also analyzed and is also shown in figure 1. The resulting data lie intermediate between the universal integrable and chaotic curves for both the statistical measures in figure 1. We have shown elsewhere [13] that the auto-correlation function [24] has a correlation hole, as it goes below the asymptotic value in the time scales comparable to the inverse of the mean energy spacing. However, the effect is not as strong as in the chaotic case, and hence in this respect also, pseudo-integrable geometries display intermediate behavior.

For intermediate cases, Berry and Robnik [25] proposed a theory that the intermediate case can be characterized by the volume of phase space which is chaotic. While this argument was for quasi-integrable cases that have a mixed phase space, we have attempted to use it to describe the pseudo-integrable data.

For $P(s)$ this argument gives [25]:

$$P(s) = \beta \exp(-s) + (1-\beta)\frac{\pi}{2} s \exp\left(-\frac{\pi}{4}s^2\right), \tag{3}$$

where, β is the parameter which characterize the ratio of phase space which is integrable. The fit corresponding to eq. (3) is displayed in figure 1, where the data is well described by the fit for s greater than the first few bins. The β value obtained is 0.62. While there are a few missing levels, the discrepancy from (3) may be due to poor statistics, but is more likely to be real, indicating that a more refined theory is required. The experimental spectral rigidity in figure 1 is compared with theory [25, 26]:

$$\Delta_3(L) = (L/15)\beta + (1-\beta) \times \left(\frac{1}{\pi^2}\ln(L) - 0.007\right) \tag{4}$$

The β value which approximates the experimental curve is 0.65, in reasonable agreement with that obtained using (1). The auto correlation function for the intermediate case was also derived by Alhassid and Whelan [24] using the same argument of Berry and Robnik. The data for the L-billiard again fits the expression derived by them, expression for $\beta = 0.65$ [13]. Therefore a β of approximately 0.65 roughly describes the intermediate nature of the data for the three different statistical measures. But such matching should be taken with caution since the phase space structure of pseudo-integrable systems is different from that of a quasi-integrable system.

3.2 *Eigenfunction properties*

As noted earlier, one of the unique features of the experiments is the ability to obtain directly the eigenfunctions. While there are at least a few studies of the eigenvalues of these systems [19, 27, 28], the eigenfunctions have not been studied in much detail. Below we discuss several unique features of the eigenfunctions of the L-billiard. Eigenfunction data were obtained in the frequency range 0.5 GHz and 4.5 GHz, and can be easily taken into a higher frequency range. Sample eigenfunctions are displayed in figure 2, and appear to have similar structure as the eigenfunctions of the Sinai billiard in the same frequency range. The averaged density distribution is shown in figure 3 for

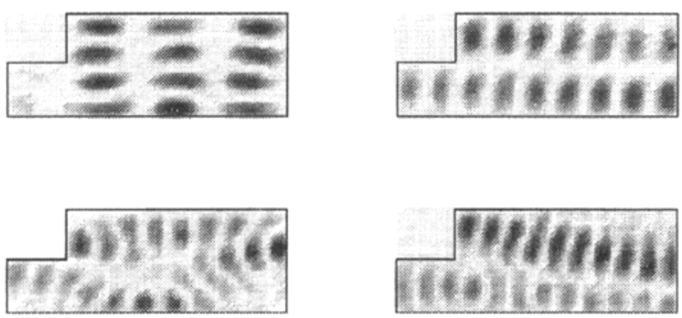


Figure 2. Sample experimental eigenfunctions of the pseudo-integrable system. The corresponding frequencies are 3.032 GHz, 3.346 GHz, 4.079 GHz, and 4.462 GHz. Dark regions correspond to high amplitudes.

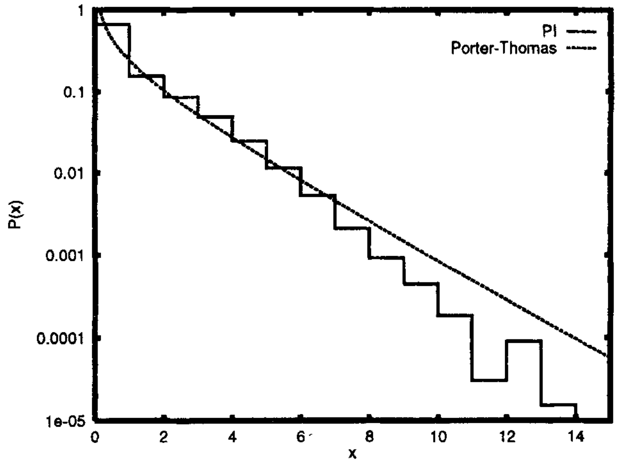
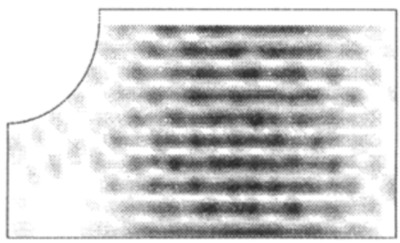


Figure 3. Density distribution of the eigenfunctions of the pseudo-integrable system ($x = |\psi|^2$).



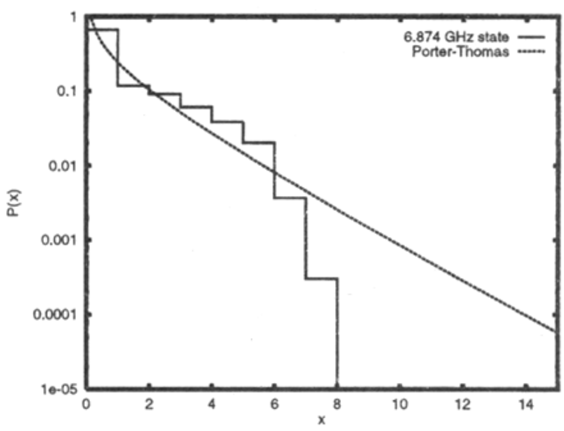


Figure 4. (Top) A bouncing ball state of the quarter Sinai billiard, and (Bottom) its density distribution.

eigenfunctions between 3.0 GHz and 4.5 GHz. A useful statistical measure to characterize eigenfunctions is the density distribution, $P(|\psi|^2)$, which gives the probability of finding a particular $|\psi|^2$ in the cavity. The distribution corresponding to the L-billiard is distinctively different from an integrable rectangle, and in fact is closer to that of

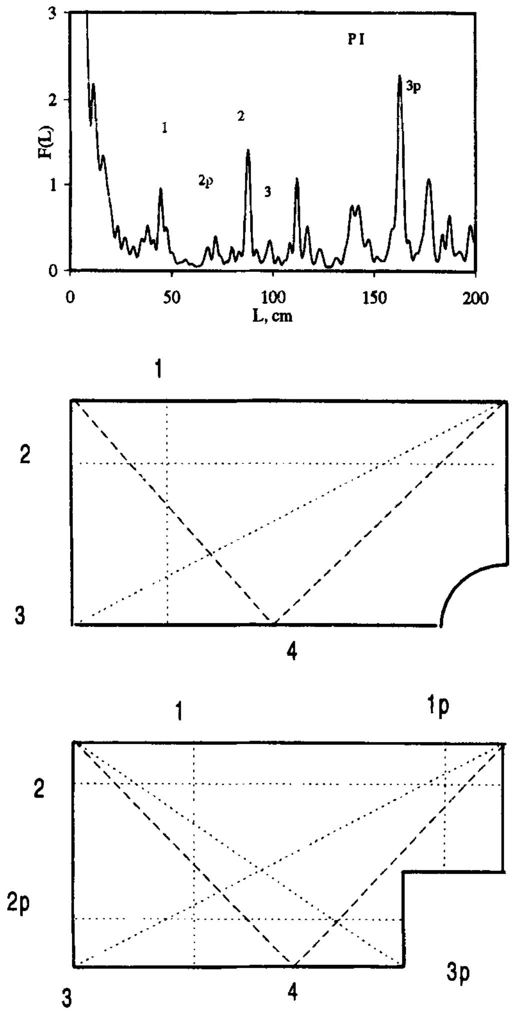


Figure 5. (Top) Fourier transform of the eigenvalue spectrum of the L-billiard. (Bottom) The peaks which can be associated with periodic orbits are labelled and shown. Also shown (middle) are corresponding orbits for the Sinai billiard.

chaotic geometries. For a rectangle, $P(|\Psi|^2)$ is truncated at $|\Psi|^2 = 4$. Thus large intensities are forbidden in the rectangle, in contrast to chaotic systems which obey the Porter–Thomas distribution [2] (see figure 3) that shows a finite though exponentially vanishing probability of finding large amplitudes. We have demonstrated this for the chaotic Sinai-stadium geometry. Closer examination shows that the data for the L-billiard actually disagrees with P-T at high densities, and instead are in closer agreement with the Sinai billiard than with the Sinai-stadium. The similarity of the behavior of the pseudo-integrable system with the Sinai billiard should not be surprising, as both have a similar structure of non-isolated periodic orbits. Very little work on the eigenfunction properties has been done for pseudo-integrable systems [29], and new theoretical developments will have to take place to describe experimental data.

3.3 *Deviations due to non-isolated periodic orbits in the Sinai billiard and the L-billiard*

As discussed in the previous section, some states of geometries like the Sinai and stadium billiards and the L-billiard, have structures which are far from irregular. In fact some show eigenfunctions that have rectangular structure, similar to the eigenfunctions of a rectangle. These eigenfunctions have an amplitude distribution with far less probability at large amplitudes. Figure 4 shows an example of a bouncing ball state of the Sinai billiard with $|\psi|^2$ almost evenly distributed between the parallel sides. The corresponding $P(|\psi|^2)$ is also displayed in the same figure that shows strongly truncated distribution. So when eigenfunctions are averaged over all states, some that are the bouncing ball, one gets an over all departure from Porter–Thomas. Observation of these small deviations clearly requires very sensitive experiments such as presented here. Recent experiments [30] with stadium cavities which probed the distribution of widths could not see this deviation from universality due to lack of accuracy and statistics [31].

3.4 *Fourier transforms and direct observation of periodic orbits*

A very useful means to directly observe and even measure the shortest periodic orbits (PO) is by taking the Fourier transform (FT) of the measured eigenvalue spectrum. The FT is shown in figure 5, and has several peaks, which correspond very well with the PO shown in figure 5. Note that only the non-isolated PO, i.e. bouncing ball orbits, are clearly present. The strength of a given PO in the FT is likely to be related to its area of stability [32, 22]. It appears that in order to manifest themselves in the FT of the eigenvalue spectrum, the relevant PO must have sufficient phase space areas of stability. Experiments on the Sinai billiard also show similar peaks in the FT since the non-isolated PO are similar in both cases (see figure 5).

4. Conclusion

The experiments described in this paper represent a unique and novel approach to studying model problems in quantum or wave chaos. We have examined the behavior of a system that is intermediate between integrable and chaotic. The statistical properties of the eigenvalues and eigenfunctions show intermediate behavior between Poisson and the Gaussian orthonormal ensemble of random matrix theory. Comparisons with existing models which examine the quasi-integrable systems shows reasonable agreement but also point to a need for a more refined theory for pseudo-integrable systems. Our results demonstrate that powerful insights can be obtained by a judicious combination of experiment and comparison with theory.

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