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The randomly stirred fluid – Turbulence as a problem in statistical mechanics

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Abstract. Properties of the randomly stirred fluid and the relevance to the problem of homogeneous isotropic turbulence are discussed.

Keywords. Homogeneous isotropic turbulence; randomly stirred fluids.

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1. Preliminaries

This article deals with applications of the methods of stastistical mechanics to the problem of homogeneous, isotropic turbulence. Turbulence [1–7] deals with a wide class of flows which seem to possess complex and seemingly random structure at some macroscopic scale of dynamic importance. The chief physical characteristic is a tremendous enhancement in transport properties—transport of momentum, energy and particles is orders of magnitude higher than possible by molecular process. Equally important is the sensitive dependence on initial conditions [8] of a turbulent flow. Two turbulent flows which are nearly identical at a given time do not remain so on timescales of dynamical interest. It is the sensitivity to initial conditions which makes the methods of statistical mechanics applicable to the problem of turbulence. The key to this is the fact that while details of fully developed turbulence are sensitive to triggering disturbances, average properties are not. Hence statistically averaged quantities are the dynamical variables of interest.

The governing dynamical equation for the study of turbulence is the Navier–Stokes equation for the velocity field $\mathbf{v}(\mathbf{r}, \mathbf{t})$,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{v} \tag{1.1}$$

where P is the pressure, ρ the density and ν the kinetic viscosity. For an incompressible fluid, there is the additional condition that

$$\nabla\cdot\mathbf{v} = 0. \tag{1.2}$$

All the information regarding the turbulence should be contained in (1.1), when it is supplemented with the boundary conditions and initial conditions. We now demonstrate that maintained turbulent motion in a closed system cannot be described by (1.1). To do so, we take the scalar product of (1.1) with **v** and integrate over all space. It is straightforward to demonstrate that if the velocity vanishes on all the enclosing boundaries, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int v^2 \mathrm{d}^3 r = -2\nu \int \mathrm{d}^3 r \left[\frac{\partial v_\alpha}{\partial x_\beta}\frac{\partial v_\alpha}{\partial x_\beta}\right] \tag{1.3}$$

and since the rate of change is always negative, motion has to cease at $t = \infty$. Thus, it is necessary to augment the right hand side of (1.1), by a force **f**, which feeds energy into the system. We will endow this force **f**, in what follows, with statistical property.

Even if the flow is not bounded, it is natural to think of an additional force, if we are worrying about the dynamics of the fluctuating part of the velocity field. To see this, we decompose the total velocity field as

$$\mathbf{v}(\mathbf{r},t) = \mathbf{V}(\mathbf{r}) + \mathbf{u}(\mathbf{r},t) \tag{1.4}$$

where **V** is a mean flow and **u** is a fluctuating field of zero mean. In other words, $\langle \mathbf{v}(\mathbf{r},t)\rangle = \mathbf{V}(\mathbf{r})$, where the prescription for taking averages will have to be provided. If we insert the above decomposition into the Navier–Stokes equation for incompressible flow, then we arrive at

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i + \nu\nabla^2 V - \frac{\partial \bar{P}}{\partial x_i} - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(V_i V_j + u_i u_j + V_i u_j + V_j u_i) \tag{1.5}$$

(note that pressure has been decomposed as $\bar{P} + p$).

Statistical average yields

$$\nu\nabla^2 V - \frac{\partial \bar{P}}{\partial x_i} = \frac{\partial}{\partial x_j}(V_i V_j + \langle u_i u_j\rangle) \tag{1.6}$$

and thus

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + f_i, \tag{1.7}$$

where

$$f_i = -\frac{\partial}{\partial x_j}(V_i u_j + V_j u_i + u_i u_j - \langle u_i u_j\rangle). \tag{1.8}$$

The fluctuating velocity field thus experiences an extra force coming mainly from the mean flow. It is this interaction of the fluctuating part with the mean flow that sustains the turbulence.

Now, about the averaging process. The available averages are space averages, time averages and ensemble averages. Spatial averages make sense only if scales over which the turbulence is homogeneous or approximately so are much larger than the scale of turbulent fluctuations. Similarly, the time averaging is useful if the time over which the turbulence is statistically stationary is longer than the time for turbulent fluctuations. Experiments usually deal with time averages obtained by using a local probe. The third average is the ensemble average where the average is taken over different realizations of the turbulent field. This ensemble averaging makes sense and can be equal to the time averaging because of the sensitive dependence on the initial conditions which makes the turbulent flow strongly mixing and ergodic. However, this would imply the existence of a stable asymptotic stationary state. It is likely that such a state exists but its existence has not been proven.

We now return to (1.7) and try to specify the properties of f. To ignore the questions of boundary conditions, mean flow, initial conditions etc., we make f a random force. We then need to specify the correlations of f. To do this, we need to keep in mind that the role of f is to supply energy at large length scales—preferably near the boundaries. If $\bar{\epsilon}$ is the rate per unit time, per unit mass at which energy is injected into the system and k_0 is a small wavenumber ($k_0 \sim O(L^{-1})$), L being the system size), then a possible prescription for the correlation is (note, dimension of $\bar{\epsilon}$ is L^2/T^3, L and T being scales for length and time)

$$\langle f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega')\rangle = \bar{\epsilon}\frac{P_{ij}(k)}{k^{D-2}}\delta(k^2 - k_0^2)\delta^D(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'). \tag{1.9}$$

In the above $P_{ij}(k)$ is the projection operator which ensures that $\nabla \cdot \mathbf{f} = 0\,[P_{ij}(k) = \delta_{ij} - (k_i k_j/k^2)]$, D is the dimensionality of space and the factor k^{D-2} is required to maintain the correct dimensions. We now use the approximation that instead of being strictly zero for $k > k_0$ as required by the δ-function in (1.9), the correlation will fall off as a power law $f(k)$ expressed by

$$f(k) = k^{2-y} \tag{1.10}$$

to write the correlation of (1.9) as

$$\langle f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega')\rangle = \frac{D_0}{k^{D-4+y}} P_{ij}\delta^D(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \tag{1.11}$$

with y taken to be an arbitrary parameter. We thus arrive at the randomly stirred model [9] of De Domincis and Martin [10, 11]. The question arises if this model is to describe fully developed turbulence, then is there a special value of y.

To address the issue of what value of y is relevant, we need to discuss correlation functions, energy spectrum, inertial range [12] etc. To begin with, let us consider the velocity correlation function $C_{ij}(k,\omega)$ defined as

$$\begin{aligned} C_{ij}(k,\omega) &= \frac{\langle v_i(\mathbf{k},\omega) v_j(\mathbf{k}',\omega')\rangle}{\delta^D(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')} \\ &= C(k,\omega)P_{ij}(k) \end{aligned} \tag{1.12}$$

The expectation value is taken over the probability distribution associated with the noise in the randomly stirred model introduced above. The total energy in the turbulence field is given by $\int \mathrm{d}^D k\, \mathrm{d}\omega C(k,\omega)$ and the energy spectrum $E(k)$ is defined as

$$\int E(k)\mathrm{d}k = \int \mathrm{d}^D k\, \mathrm{d}\omega\, C(k,\omega). \tag{1.13}$$

A dimensional analysis yields the scaling behaviour of $E(k)$ in what is known as the inertial range. The inertial range encompasses k-values which are neither too small, nor too big. The small k-values correspond to the length scales at which the forcing occurs and energy is injected into the system. The large k-values correspond to small length scales where the molecular viscosity is important and dissipates the injected energy. The result is a maintained stationary state where the energy flows from the large length scales to the short length scales at a constant rate, which is the $\bar{\epsilon}$ introduced before. The main assumption is that in the inertial range $E(k)$ is determined by $\bar{\epsilon}$ and k and does not depend upon the phenomena at long or short length scales. This leads to

$$E(k) = C_K \bar{\epsilon}^{2/3} k^{-5/3}, \tag{1.14}$$

where C_K is a universal number found to be somewhere between 1.5 and 2.0 in many experiments [13, 14]. If we use dimensional analysis to get the characteristic decay rate Γ in this stationary state, we find

$$\Gamma(k) = \Gamma_0 \, \bar{\epsilon}^{1/3} k^{2/3} \tag{1.15}$$

where Γ_0 is another universal constant. It is clear from (1.13) and (1.14), that in three dimensions

$$C(k) = \int C(k, \omega) \mathrm{d}\omega \sim k^{-11/3} \tag{1.16}$$

In general, in D-dimensions, $C(k) \sim k^{-(D+(2/3))}$. To probe the energy transfer rate, let us write the Navier–Stokes equation ((1.7)), in momentum space as

$$u_i(k) + \nu k^2 u_i = \sum_{\mathbf{p}} M_{ijl}(\mathbf{k}) u_j(\mathbf{p}) u_l(\mathbf{k}-\mathbf{p}) + f_i(\mathbf{k}) \tag{1.17}$$

where

$$M_{ijl}(\mathbf{k}) = k_j P_{il}(\mathbf{k}) + k_l P_{ij}(\mathbf{k})$$

and

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

From (1.17), we have

$$\frac{\partial}{\partial t} \sum_i u_i^2(\mathbf{k}) + \nu k^2 \sum u_i^2(k) - f_i u_i(-\mathbf{k}) = \sum_{\mathbf{p}} M_{ijl}(\mathbf{k}) u_i(-\mathbf{k}) u_j(\mathbf{p}) u_l(\mathbf{k}-\mathbf{p}) \tag{1.18}$$

If we integrate over all k in (1.18), then

$$\int \mathrm{d}^D k \sum_{\mathbf{p}} M_{ijl}(-\mathbf{k}) \langle u_i(\mathbf{k}) u_j(\mathbf{p}) u_l(\mathbf{k}-\mathbf{p}) \rangle = 0$$

in the stationary state. We define the quantity

$$\prod(k) = \int_0^k \mathrm{d}^D k' \sum_{\mathbf{p}} M_{ijl}(-\mathbf{k}) \langle u_i(\mathbf{k}') u_j(\mathbf{p}) u_l(\mathbf{k}-\mathbf{p}) \rangle, \tag{1.19}$$

which now stands for the rate at which energy is flowing from wavenumbers below k to wavenumbers above k. Kolmogorov picture asserts that $\Pi(k)$ is independent of k.

We now turn to the randomly forced model. To find out how the relaxation rate $\Gamma(k)$ scales with k, we use the effective viscosity argument of Heisenberg. In the inertial range, where the non-linear term in Navier–Stokes equation is dominant, it is this non-linear term which gives rise to an effective scale dependent viscosity. Noting that the velocity field can be written as

$$u_i(\mathbf{k},t)=\int \mathrm{d}^D p\int \mathrm{d}t' G_{in}(k,t-t')M_{njl}u_j(\mathbf{p},t')u_l(\mathbf{k}-\mathbf{p},t') \tag{1.20}$$

the averaged effect of the non-linear term is clearly expressible as

$$\int \mathrm{d}^d p \mathrm{d}t' M_{ijl}(\mathbf{k})M_{mrs}(\mathbf{p})u_s(\mathbf{k})G_{jm}(\mathbf{p},t-t')\langle u_r(\mathbf{k}-\mathbf{p},t)u_l(-\mathbf{k}-\mathbf{p},t')\rangle.$$

The Greens function $G_{jn}(\mathbf{p},t-t')$ can be written as

$$P_{jn}\,\mathrm{e}^{-\sum(\mathbf{p})(t-t')},$$

where $\sum(p)$ is the effective relaxation rate or the self energy in technical terms. The above expression then acquires the form

$$\int \mathrm{d}^D p\,\mathrm{d}t' M_{ijl}(\mathbf{k})M_{mrs}(\mathbf{p})P_{jm}(\mathbf{p})P_{rl}(\mathbf{k}-\mathbf{p}) \\ \times \mathrm{e}^{-\sum(p)(t-t')}C(\mathbf{k}-\mathbf{p},t-t')u_s(\mathbf{k},t')$$

where $C(k-p,t-t')$ is the time Fourier transform of $C(\mathbf{k}-\mathbf{p},\omega)$. Now $C(q,\omega)$, has the form $q^{-D+4-y}[\omega^2+\sum^2(q)]^{-1}$ and thus the last expression reduces to

$$\int T_{is}(\mathbf{k},t-t')u_s(\mathbf{k},t')\mathrm{d}t'$$

where the normalized trace of $T_{is}(\mathbf{k},t-t')$ is the effective relaxation rate, whose Fourier transform yields the frequency dependent rate $\sum(k,\omega)$ as,

$$\sum(k,\omega)=\int \mathrm{d}^D p \mathrm{d}\omega' M_{ijl}(\mathbf{k})M_{mri}(\mathbf{p})P_{jm}(\mathbf{p})P_{rl}(\mathbf{k}-\mathbf{p}) \\ \times G(\mathbf{p},\omega')C(\mathbf{k}-\mathbf{p},\omega-\omega') \tag{1.21}$$

Noting that $\int C(k,\omega)\mathrm{d}\omega \sim [\sum(k)]^{-1}k^{-D+4-y}$, it is easy to see that power counting consistency of (1.21) gives

$$\sum(k)\propto k^{2-(y/3)}. \tag{1.22}$$

Turning to (1.19) using (1.20) and following an identical line of argument that led to (1.22), we find that $\Pi(k)\sim k^{4-y}$. Kolmogorov asserts that $\Pi(k)=\bar{\epsilon}$ and hence one requires $y=4$ for the Kolmogorov picture of turbulence [15–18]. That raises the central question is the theory well behaved at $y=4$?

2. Technicalities

Kolmogorov scaling and the randomly stirred Navier–Stokes equation would be completely compatible if (1.21) would yield in addition to the power counting consistency as demonstrated towards the end of the last section, a finite number for universal amplitude ratios. With vertex corrections not yielding any new power laws because of Galilean invariance, the problem of turbulence would reduce to calculating the universal numbers in a loop ordered perturbation scheme. We now demonstrate that there are technical problems with this. To exhibit this, we explore the integrand on the rhs of (1.21). In particular, we probe the situation where the external momentum k is carried almost entirely by the propagator G and the momentum associated with the correlator C is very small. The zero frequency self energy can then be written as (suppressing projection operators and indices).

$$\sum(k) \simeq \int \mathrm{d}^D p \mathrm{d}\omega' M(\mathbf{k}) M(\mathbf{k}) G(k, \omega') C(p, -\omega'). \tag{2.1}$$

Now, $G^{-1}(k, \omega') = -i\omega' + \sum(k, \omega') \simeq \sum(k)$ as we cover the frequency range where ω' matches $p^{2/3}$ and hence is small compared to $\sum(k)$. Equation (3.1) thus becomes (for $y = 4$)

$$\begin{aligned} \sum(k) &\simeq \frac{M^2(k)}{\sum(k)} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}\omega}{2\pi} C(p, \omega) \\ &\simeq \frac{M^2(k)}{\sum(k)} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^{D+2/3}}. \end{aligned} \tag{2.2}$$

This integral diverges due to the zero-p range contribution and needs to be cut off at some low momentum k_0. The scaling solution is consequently no longer valid.

The above divergence comes from the very strong dynamical coupling between the eddies (fourier modes) with short wavelength and the eddies with long wavelength. This effect is spurious. The expected role of the large eddies is to simply transport the much smaller ones and not to have a strong dynamical coupling. This spurious effect was attributed to the use of the Eulerian picture and Kraichnan tried to remedy it by going into the Lagrange description. This removes the problem in principle but actual calculations are virtually impossible. A different approach was tried by Yakhot and Orszag who used the standard renormalization group technique to circumvent the problem. The idea was that one splits the velocity field into two parts—one with high momentum Fourier modes and the other with low momentum Fourier modes, integrates out the high momentum components and studies the effect of that on the low momentum ones. This leads to a new viscosity and after the usual rescalings of space, time and the velocity field a flow equation for the viscosity emerges. The fixed point of the flow corresponds to a scale dependent viscosity. By construction, this yields long wavelength, low frequency properties as stressed by Forster *et al.* In the limit of extremely high Reynolds number, one is not particularly worried by this limitation. The process of integrating over the high momentum modes bypasses the difficulty that we encountered with the low-p divergence. We sketch the steps in the following:

(i) Split the velocity field as

$$u_i(\mathbf{x}) = u_i^<(\mathbf{x}) + u_i^>(\mathbf{x})$$

where

$$u_i^<(\mathbf{x}) = \sum_{p<\frac{\Lambda}{b}} u_i(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$$

and

$$u_i^>(x) = \sum_{\frac{\Lambda}{b}<p<\Lambda} u_i(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$$

(ii) Find the equations satisfied by $u_i^<$ and $u_i^>$.
(iii) Integrate perturbatively the equation for $u_i^>$
(iv) Insert the solution for $u_i^>$ into the equation for $u_i^<$ and find the equation of motion for $u_i^<$. The linear part gives $\Delta\nu$, the change in kinematic viscosity.
(v) Rescale space and length and the velocity field, so that the new viscosity is given by

$$\nu' = \nu b^{z-2} - 8b^{4-y-2z}F_0 k^{-2}\int_{p=\frac{\Lambda}{b}}^{p=\Lambda}\frac{\mathrm{d}^D p}{(2\pi)}D\int\frac{\mathrm{d}\omega}{2\pi}M_{\rho\beta\gamma}(\mathbf{k})$$
$$\times M_{\sigma\delta\gamma}(\mathbf{k}-\mathbf{p})P_{\beta\delta}(\mathbf{k}-\mathbf{p})P_{\sigma\rho}(\mathbf{p})G(\mathbf{k}-\mathbf{p}-\omega)|G(\mathbf{p},\omega)|^2 p^{4-D-y}. \quad (2.3)$$

Requiring that the two terms on the rhs have the same dimensions ensures

$$z = 2 - \frac{y}{3} \quad (2.4)$$

and hence

$$\nu(k) = \nu k^{-y/3} \quad (2.5)$$

as in (1.22). Evaluating the integral in (2.3) in the limit of $k \to 0$ gives

$$\nu' = \nu\left[1 + \ln b\left(\frac{D_0}{2\nu^3}\frac{D-1}{D+2}\frac{S_D}{(2\pi)^D} - \frac{y}{3}\right) + \cdots\right. \quad (2.6)$$

where S_D is the surface area of a D-dimensional sphere. The fixed point condition yields

$$\frac{\nu^3}{D_0} = \frac{3}{2y}\frac{S_D}{(2\pi)^D}\frac{D-1}{D+2} \quad (2.7)$$

From (1.19), one can determine the value of D_0^2/ν^3 and thus D_0/ν which sets the scale for the Kolmogorov spectrum is obtained. For $y = 4$ and $D = 3$, one does find a value very close to the experimental results and similar success in calculating other universal numbers certainly indicates that this is a successful program.

The primary difficulties of this approach were pointed out by De Dominicis and Martin [10] at the time they introduced the model at the Kolmogorov limit i.e. $y = 4$, there are an infinite number of marginal operators and the self consistent perturbation theory has an infrared divergence at $y = 3$. In the last few years, both these effects have been investigated in a different context—the problem of growth by deposition of atoms on a substrate [19–22]. The existence of an infinite number of marginal operators and infrared divergences seem to change the roughening and the dynamical exponents of the problem

[23–26]. Consequently, in spite of impressive numerical agreement with experiments, the renormalization group approach is somewhat suspect.

We now note that the infrared divergence that started the problem in the first place is caused by the strong dynamical coupling between small scale (large k-Fourier components) energy dissipating eddies and the large scale (small k-Fourier components) energy containing eddies. For the validity of Kolmogorov scaling, the small eddies should simply be advected by the large eddies without any dynamical coupling. Hence Kolmogorov scaling requires a screening [27, 28] of the interaction between the large and small Fourier coefficients. This is provided by viscoelasticity [29]. Turning to (1.21), we note that the effective self energy $\sum(k,\omega)$ is strongly frequency-dependent for frequencies greater than $\sum(k,0)$. If we write the k-dependence of the relaxation rate as

$$\Gamma(k) = \Gamma_0 k^n \tag{2.8}$$

and that of the equal time correlation function as

$$C(k) = c_0 k^{-m} \tag{2.9}$$

then dynamic scaling yields

$$\sum(k,0) = \Gamma_0 k^n \tag{2.10a}$$

and

$$\sum(k,\omega) = k^2(-i\omega)^{(2-n)/n} \quad \text{for } \omega \gg \Gamma(k). \tag{2.10b}$$

The corresponding correlation function is

$$C(k,\omega) = \frac{C(k)}{\sum(k)}\left[\frac{1}{1 - i\omega/\sum(k,\omega)} + \frac{1}{1 + i\omega/\sum^+(k,\omega)}\right] \tag{2.11}$$

for frequencies ω such that $\omega\tau_s \gg 1$, where τ_s is the sweeping time. Light scattering from the randomly stirred fluid under the right conditions [28] should be able to establish the validity of (2.11).

We now introduce the form of $C(k,\omega)$ given above and using a Lorentzian $G(k,\omega)$ i.e., $G(k,\omega) = -i\omega + \sum(k)$, arrive at (see eq. (2.1))

$$\sum(k) \sim \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{C(p)}{\sum(p)}\left[1 + \frac{\sum(\mathbf{k}-\mathbf{p})}{\sum(p,\omega = -i\sum(\mathbf{k}-\mathbf{p}))}\right]^{-1}. \tag{2.12}$$

If we now explore the region of the integrand where $p \to 0$, then

$$\begin{aligned}\sum(k) &\sim \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{C(p)}{\sum(p)}\left[1 + \frac{\sum(k)}{p^2[\sum(k)]^{(n-2/n)}}\right]^{-1} \\ &\sim \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{C(p)}{\sum(p)}p^2 \\ &\sim \int \frac{\mathrm{d}^D p}{(2\pi)^D}C(p)\,p^{2-n}\end{aligned} \tag{2.13}$$

The infrared divergence now occurs for $m+n \geq D+2$. For the Kolmogorov case of $m = 11/3$ and $n = 2/3$ in $D = 3$ one is safe.

The above manipulations bear some similarity (in the final results) to the quasi-Lagrangian [30] approach advocated by Belinicher and L'vov [31, 32]. This has recently been taken up again by L'vov and Procaccia [33] who work entirely in coordinate space to show that the Kolmogorov spectrum does lead to a finite theory in the perturbative analysis of the forced Navier–Stokes equation.

In all our considerations so for, dimensionality plays a very minor role. The exponent z is independent of dimension while the exponent n carries a trivial dependence on dimensionality. This implies that one may possibly consider a one dimensional randomly stirred model and pick up the features that we have talked about so far. In one dimension it is convenient to consider Burger's equation which is our Navier–Stokes equation without the pressure term. For the incompressible flows that we were dealing with, the pressure term simpy modified the coupling constant associated with the nonlinear term and hence did not cause any qualitative change. Dropping it and considering the randomly stirred Burgens equation is one dimension is not going to cause any qualitative changes in the central problem. Thus

$$\dot{u} + u\frac{\mathrm{d}}{\mathrm{d}x}u = \nu\nabla^2 u + f \tag{2.14}$$

with

$$\langle f(x,t)f(x',t')\rangle = D_0 k^{-1+y-4}\delta(x-x')\delta(t-t') \tag{2.15}$$

is expected to capture the essential features of fully developed Kolmogorov turbulence for $y = 4$, there is an additional complication. Without the drive f, Burgers equation is known to have coherent structures in the form of shock waves [34]. To see Kolmogorov scaling, the random drive has to overcome the shocks. This idea that (2.14) contains the basic ingredients for understanding fully developed turbulence has recently been supported very stronglly by Cheklov and Yakhot [35], Polyakov [36] and Gurarie and Migdal [37].

The thrust of our discussion so far has been that Kolmogorov scaling is exact. Our calculations have been in the inertial range where the effective relaxation rate $\sum(k)$ dominates the molecular relaxation rate νk^2. This means that we are working with $k \ll k_D$ where k_D is the wavenumber formed from the rate of dissipation ϵ and the kinematic viscosity ν. Clearly

$$k_D = \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \tag{2.16}$$

and in the inertial range $k_D L \gg 1$, where L is the system size. With $\epsilon \sim v^3/L$ (v is a mean velocity)

$$k_D L \sim (Re)^{3/4} \tag{2.17}$$

Where $Re = vL/\nu$ is the Reynold's number and the self consistent approach that we have adopted (and which works!) requires $Re \to \infty$. Thus, in the infinite Reynold's number limit, the results for the relexation rate and the two point function $\langle v(k,\omega)v(k',\omega')\rangle$ (alternatively $\langle[v(x+r,t)-v(\mathbf{x},t)]^2\rangle$) seem to be exact.

We now have to face the fact that a whole class of experiments yield results for higher order structure factors [38–41] that are at variance with the Kolmogorov expectations. If one investigates the structure factor $\langle |v(x+r,t) - v(x,t)|^P \rangle$, then the exponent specifying the r-dependence does not depend linearly on p. On the other hand assuming that once the two point function has been shown to be finite, there would be no new divergences in the higher order correlations and hence the exponent for the pth-order structure factor should be a linear function of p. The correlation function $\langle \epsilon(x+r)\epsilon(x) \rangle$ in particular is of great interest ($\epsilon(x)$ is the dissipation rate at the point x, averaged over a local area) since in the Kolmogorov picture it has no r-dependance. Various experiments reveal $\langle \epsilon(x+r)\epsilon(x) \rangle \sim r^{-\mu}$ with μ a small number (about 0.20). This phenomenon is known as intermittency and μ is often called the intermittency exponent. We will return to this issue in § 3.

The Kolmogorov picture in two dimensions is special because there are two cascades to contend with—energy and enstrophy. Enstrophy is defined as $(\nabla \times \mathbf{v})^2$. The direction of the cascades can be determined by an argument due to Kraichnan [42]. With two conserved quantities,

$$E = \frac{1}{2}\int v^2(x)\mathrm{d}^2x = \frac{1}{2}\sum_k |v(k)|^2 = \int_0^\infty E(k)\mathrm{d}k$$

and

$$N = \frac{1}{2}\int (\nabla \times \mathbf{v})^2 \mathrm{d}^2x = \frac{1}{2}\sum_k k^2|v(k)|^2 = \int W(k)\mathrm{d}k$$

where $E(k)$ and $W(k)$ are the energy and enstrophy spectrum respectively. The canonical probability distribution for fluctuating E and N is clearly

$$P \sim \mathrm{e}^{-\beta E - \mu N} = \mathrm{e}^{-\frac{1}{2}\sum(\beta + \alpha k^2)|v_k|^2}. \tag{2.18}$$

This leads to

$$\langle |v_k|^2 \rangle = (\beta + \alpha k^2)^{-1} \tag{2.19}$$

and consequently

$$E(k) = \frac{k}{\beta + \alpha k^2}$$

and

$$W(k) = \frac{k^3}{\beta + \alpha k^2}$$

The enstrophy spectrum $W(k) \sim k$ for high wave number and hence is clearly concentrated towards high wave number side. A spectrum with $W(k) \sim k^x$, $x < 1$, will be out of equilibrium and will proceed towards equilibrium by cascading enstrophy from low to high wave numbers. Energy conservation would then demand an inverse cascade of energy. The enstrophy cascade is dissipated by molecular viscosity at high wave vectors, while the inverse energy cascade causes a condensation phenomenon at low wave numbers. The Kolmogorov argument holds for the inverse energy cascade and one ends up with $E(k) \sim k^{-5/3}$, while for the enstrophy cascade one has to set up the dimensional argument once more, postulating that in the inertial range $E(K)$ is determined by the rate of injection of enstrophy (ϵ_s say) and the local wave number k. This leads to $E(k) \sim k^{-3}$. The infrared difficulties that we talked about in the beginning of the section will appear here as well and a screening approximation will produce a self energy of the form given in (2.13). For the energy cascade in $D = 2$, $m = 8/3$ and $n = 2/3$ and the inequality $m + n < D + 2$ is satisfied which makes the theory finite. For the enstrophy cascade on the other hand, $m = 4$ and $n = 0$ and we have $m + n = D + 2$. That means a genuine logarithimic divergence and hence for the enstrophy cascade [44–46]

$$E(k) = c_0(\epsilon_s)^{2/3} k^{-3} (\ln k/k_0)^{-1/3}. \tag{2.20}$$

Yet another problem where there exists multiple cascades is the magnetohydrodynamic turbulence [47–50]. The conserved quantities in the inviscid limit are the kinetic energy and the magnetic energy. Consequently, there is a magnetic energy flux and a kinetic energy flux and one can determine [51] the Kolmogorov constants etc., for this flow in a manner analogous to that for the pure fluid. The binary liquid is another example of a situation with different fluxes [52]—an energy flux and a concentration flux and a renormalization group program for that has been carried out [53, 54] recently.

3. Subtleties

This short section has to do with the question whether Kolmogorov scaling is exact for the forced Navier–Stokes equation or not. To set the stage, we begin with the earliest discussion [54, 55] of this topic. Soon after the publication of Kolmogorov's work, it was pointed out by Landau that the theory could be internally inconsistent. The dissipation rate ϵ is found from Navier–Stokes equation to be expressible as

$$\epsilon = \nu \int \mathrm{d}^D r (\mathbf{v} \cdot \nabla^2 \mathbf{v})^2 = \sum_{\alpha,\beta} \frac{\nu}{2} \int \mathrm{d}^D r \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right)^2. \tag{3.1}$$

Being an integral over a fluctuating quantity, namely the velocity field, the dissipation rate ϵ is expected to show fluctuations which are ignored in the Kolmogorov picture. It was not till two decades had passed that Kolmogorov and Obukhov returned to this issue and took Landau's objection into account by assuming that the local dissipation rate ϵ_r at the point r (averaged over a small ball around r) had a log normal distribution i.e., $\ln\epsilon_r$ had a Gaussian distribution.

Kolmogorov assumed that the width σ of the distribution had the form

$$\sigma^2 = \langle [\ln\epsilon_r - m]^2 \rangle = \begin{cases} A + 9\delta \ln L/r, & k_d r \gg 1 \\ A' + 9\delta \ln k_D L, & k_D r \ll 1 \end{cases} \tag{3.2}$$

Where $m = \langle \ln\epsilon_r \rangle$ and δ is an universal number. The constants A and A' are determined by the large scale structure of the flow. The log normal distribution of ϵ_r requires that the probability distribution $P_r(a)$ for $\epsilon_r = a$ is given by

$$P_r(a) = \frac{1}{(2\pi\sigma^2 a^2)^{1/2}} \mathrm{e}^{-(\ln a - m)^2/2\sigma^2}. \tag{3.3}$$

This distribution leads to

$$\langle \epsilon_r^p \rangle = e^{pm+\frac{1}{2}p^2\sigma^2} \tag{3.4}$$

and subsequently

$$\langle |\mathbf{v}(\mathbf{x}+\mathbf{r},t) - \mathbf{v}(\mathbf{x},t)|^p \rangle \sim (r\bar{\epsilon})^{p/3} \left(\frac{L}{r}\right)^{(\delta/2)p(p-3)} \tag{3.5}$$

The intermittency exponent defined in the previous section is obtained from $p = 6$ and found to be $\mu = 9\delta$.

Experimental results on the high order structure factors do not quite agree with the p-dependence of the exponent shown in (3.5), but they certainly follow the qualitative feature that the exponents are not a linear function of p. The deviation from linearity prompts a multifractal description, which in turn inspires a dynamical system description of turbulence in the form of shell models. We shall not discuss these issues here they will be addressed in detail by Pandit *et al* [56] in this volume. We will confine our discussion to the randomly stirred model.

The discussion of the previous section leads us to believe that the scaling of the two point function is exact in the high Reynold's number limit (finite Reynolds number corrections [57,58] have been found which provide correction-to-scaling and hence an appearance of deviation from Kolmogorov scaling). However, the case of higher order correlation functions had not been discussed till almost the mid-nineties. The natural quantity to focus on first is the exponent μ and a lowest order calculation of the corresponding correlation function showed [59] a logarithmic behaviour i.e., $\langle \epsilon(\mathbf{x}+\mathbf{r})\epsilon(\mathbf{x}) \rangle \sim \bar{\delta} \ln(L/r)$. For small μ, the $r^{-\mu}$ behaviour can be expanded to give $\mu \ln(1/r)$ and $\bar{\delta}$ can be identified with μ. This yields a reasonable value of μ. The work of Lebedev and L'vov [60] and subsequent calculations [61, 62] examine the structure of the relevant correlation function to all orders and conclude that a resummation can lead to a power law. However, they do not calculate the exponent explicitly.

Recent developments [35–37] very strongly support the idea that the higher order structure factors in the randomly stirred fluid do show deviations from pure Kolmogorov scaling. These have to do with the one dimensional Burgers equation-turbulence discussed before. Numerical work [35] does clearly show a non-zero μ and analytical work from two different directions—operator products expansion [36] and WKB type analysis—reveals a non-Gaussian probability distribution [63].

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