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Control of chaos

SANGEETA D GADRE and V S VARMA
Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India

Abstract. We review the subject of control of chaotic systems paying special attention to exponential control. We also discuss the application of synchronization of chaotic systems to security in communications.

Keywords. Exponential control; synchronization; secure communications.

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1. Introduction

This started off as a general review of the subject of synchronization and control of chaotic systems. When it became clear however that this special issue on chaos would contain a number of such contributions, we decided to give it a more personal touch by also discussing at some length some of our own work in this field – the exponential control of systems exhibiting chaotic behaviour.

2. Control

The subject of controlling chaotic systems has become popular since the work of Ott *et al* [1], hereafter referred to as OGY. OGY were the first to realize that the presence of chaos instead of being a nuisance can, in fact, be of great advantage. This is because a chaotic attractor typically has embedded within it an infinite number of unstable periodic orbits. Any one of a number of such orbits can be stabilized by small time-dependent parameter perturbations of an available system parameter and a choice can be made among these orbits to achieve a desired system performance. This is to be contrasted with the situation in which the attractor is not chaotic but periodic, in which case small parameter perturbations can only produce small changes in the system orbit.

To appreciate the implications of what has been said above, consider a system which may be required to be used for different purposes and be expected to operate under different conditions at different times. If the system is chaotic, this type of multiple-use situation might be accommodated without altering the gross system configuration. In particular, depending on the use desired, the system behaviour can be changed by switching the temporal programming of the small parameter perturbations so as to stabilize different orbits. In contrast, in the absence of chaos, completely separate systems might be required for each use. Thus, when designing a system intended for multiple use, purposely building chaotic dynamics into the system can be a way of accommodating the desired flexibility.

OGY imagined a situation in which the dynamical equations describing the system are not known, but the experimental time-series of some scalar variable can be measured and using delay coordinates [2] a surface of section can be constructed. The piercings of the experimental surface of section are then studied and a particular unstable periodic orbit which gives the best system performance is selected (using embedding techniques discussed by Gunaratne *et al* [3]). The system trajectory is observed till it falls within a strip of predefined width of the selected unstable orbit, at which juncture a suitable parameter p of the system is adjusted so that the system trajectory at the next piercing of the surface of section falls on the stable manifold of the selected unstable orbit. When this occurs, the parameter perturbation can be set to zero and the orbit for subsequent time will approach the desired fixed point at a geometrical rate.

OGY demonstrated the success of their analysis numerically on the Hénon map. The method has subsequently been successfully applied to a variety of systems including magnetic ribbons [4], spin-wave systems [5], chemical systems [6], electrical diodes [7], lasers [8] and cardiac systems [9].

Some important features of the OGY technique are:

1. no model for the dynamics of the system under consideration is required,
2. the required changes in the parameter can be quite small,
3. control can be achieved even with imprecise measurements of the system eigenvalues and eigenvectors, and
4. the method is applicable to any system whose dynamics can be characterized by a nonlinear map.

Shinbrot *et al* [10, 11] adapted the OGY technique to situations in which the system can be modelled by a set of equations although these need not be exact. Their technique has been shown to be effective even in the presence of small-amplitude noise or small errors in modelling.

Dressler and Nitsche [13] showed that modifications to the original OGY procedure were necessary when the attractor is reconstructed from a time series using delay coordinates because the experimental surface of section map depends not only on the current value of the control parameter but also on its preceding value. Rollins *et al* [14] presented a recursive proportional-feedback algorithm, which is an adaptation of the method of Dressler and Nitsche for the control of highly dissipative systems.

Lai *et al* [15] extended the OGY stabilization of dissipative chaos to chaotic Hamiltonian systems. Lai and Grebogi [16] used this modification to synchronize two signals using perturbations proportional to the instantaneous difference between them. Alvarez-Ramírez [17] presented a nonlinear feedback control in which the feedback is based on linearizing the input-output dynamics of the system and control leads to large regions of stability.

Newell *et al* [18] demonstrated an experimental proportional-feedback algorithm for the synchronization of chaotic time signals generated by a pair of independent diode resonator circuits.

Bielawski *et al* [19] stabilized unstable periodic orbits by applying continuous feedback on a control parameter, the feedback signal being proportional to the difference between two values of a dynamical variable separated by a time interval equal to the unstable orbit periodicity. This method is expected to be efficient for systems near a bifurcation point and for highly dissipative systems.

Pyragas [20] proposed two methods for the control of unstable periodic orbits with small time-continuous perturbations. Feedback combined with a periodic external force of a special form is used in the first method (external force control); while the second method relies on self controlling delayed feedback (delayed feedback control). Qu *et al* [21] analyzed Pyragas' external force control and found that continuous control by feedback of a single variable is not always successful. They also showed, contrary to Pyragas' claim, that under certain conditions positive feedback can control a system where negative feedback fails to do so. Pyragas [22] subsequently extended his control to stabilize aperiodic orbits, while Kittle *et al* [23] verified the effectiveness of such control experimentally.

Matias and Güémez [24, 25] presented a new method for the stabilization of 1-D iterated maps which exhibit chaos, by acting on the system variables rather than on some available parameter. The method is implemented by applying pulses that change the system variable proportionally. The control algorithm consists of the application every Δn iterations of a feedback to the variable x having the form

$$x_n = x_n(1 + \gamma),$$

where γ represents the strength of the feedback and can be positive as well as negative. The method does not require that the particular dynamical law be known. However, the ability to control chaos depends critically on the values of γ and Δn. The latter is closely related to the periodicity of the observed cycle, this being a multiple of Δn. They illustrated their technique by an application to the logistic and exponential maps. Subsequently they applied their method to the isothermal autocatalator model [26] and the Lorenz and Rössler systems.

Singer *et al* [27] demonstrated experimentally and theoretically that low-energy, feedback control signals can be utilized efficiently to suppress (laminarize) chaotic flow in a thermal convection loop. Chen and Chou [28] showed that the continuous feedback approach is highly effective for controlling chaotic systems. Given the system's model dynamic equations, the control is designed for a well-specified system performance. The approach was applied to the Lorenz system and was able to drive the system to any steady state. The control was found to tolerate both measurement noise and modeling uncertainty as long as these were bounded.

Huberman and Lumer [12] introduced a simple adaptive control mechanism to control nonlinear systems in stable orbits. A system which is perturbed away from a stable fixed point (say X_s^*) is brought back to the fixed point by introducing changes in the parameter (μ) such that

$$\mu_{n+1} = \mu_n + \epsilon(X_n - X_s^*).$$

Sinha *et al* [29] extended this method to multi-parameter and higher dimensional nonlinear systems. They also suggested a method by which periodic motion such as limit cycles can be adaptively controlled and demonstrated the robustness of the procedure in the presence of (additive) background noise.

John and Amritkar [30] suggested a method for controlling chaotic systems which is able to synchronize the system trajectory to a desired unstable orbit. Their method uses adaptive control [12] and introduces time-dependent changes in the system parameters depending on two factors : (1) the difference between the system output variables $\{u_j\}$ and the corresponding variables of the desired orbit $\{v_j\}$, and (2) the difference between the values of the parameters (μ_i) which are controlled and their values (μ_i^*) for the desired orbit. They demonstrated the effectiveness of their method using the Lorenz and Rössler systems. They also showed that their method could be used for communication purposes.

There are other interesting and simple methods [31–38] for controlling of chaos. For example, entrainment control [31], suppression of chaos by weak periodic parametric perturbation [32] and addition of second periodic force [33] are also possible.

3. Synchronization

Although we have mentioned synchronization en passant, the possibility of synchronization in chaotic systems is not obvious. Two identical autonomous chaotic systems started at nearly identical initial points in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space. Pecora and Carroll [39] were the first to describe the linking of two chaotic systems with a common signal such that the two systems synchronized, i.e. the trajectories of the two systems converged and remained in step. Their results were obtained for flows (differential equations), but can be applied to iterated maps with only small modifications. They considered an autonomous n-dimensional dynamical system

$$\dot{u} = f(u),$$

divided the system, arbitrarily, into two subsystems $[u = (v, w)]$. They created a new system u' identical to the original u system and substituted the set of variables v for the corresponding v' in the new system. The behaviour of the new system depends on the Lyapunov exponents of the w subsystem—referred to as the sub-Lyapunov exponents (SLEs). They suggested the following theorem: the subsystems w and w' will synchronize only if the SLEs are all negative.

This theorem is necessary, but not sufficient, for synchronization. It says nothing about the set of "initial conditions" for which w' will synchronize with w. They demonstrated synchronization in the Lorenz and the Rössler systems as well as in electronic circuits and their numerical models [40]. Incidentally, Amritkar and Gupte [41] have proposed a qualitative measure of correlation between coevolving dynamical systems which is very useful in characterizing synchronization in chaotic systems.

Periodically driven nonlinear systems can exhibit multiple-period behaviour (period-2, period-3, etc.). Several such systems, when driven by the same drive, can be on identical attractors but remain out of phase with each other. This means that the basins of attraction for multiple-period systems can be divided into domains of attraction, one for each phase of the motion. A period-n attractor will have n domains of attraction in its basin. This out-of-phase situation is stable—small perturbations will not succeed in getting the systems in phase. Pecora and Carroll [42] showed that by using an almost periodic driving signal (generated from various chaotic systems) one can simultaneously (1) keep the motion of the systems nearly the same as the periodic driving signal, (2) keep the basins of attraction nearly the same, and (3) eliminate the n domains of attraction. In other words, there will be only one domain for the basin. This means that any number of such driven systems will always be in phase.

Kittle *et al* [23] pointed out that the stabilization of a desired aperiodic orbit can be interpreted in terms of Pecora and Carroll's method of synchronization. To stabilize a desired orbit we need to know a prerecorded signal $y^*(t)$, which is sufficient to control the output of a given system to a desired orbit for a given controlling method. The prerecorded signal $y^*(t)$ can be replaced by the output signal of an additional, identical chaotic system. The latter can be seen as the driving system, and the original can be interpreted as the response system. The problem of stabilizing a prerecorded desired aperiodic orbit is thus equivalent to the problem of synchronizing two identical chaotic systems.

Kapitaniak [43] applied the continuous controlling method developed by Pyragas [20] to achieve synchronization of two chaotic systems. Murali and Lakshmanan [44] also used Pyragas' method to synchronize two identical nonlinear oscillators and to transmit signals in a secure way.

Ding and Ott [45] pointed out that exact synchronism may also occur for a large class of systems that are not replicas of the original system as suggested by Pecora and Carroll. They were able to affect the following improvements over Pecora and Carroll's technique:

1. achieving synchronization when a replica subsystem does not synchronize (i.e. $\Lambda > 0$ for the replica subsystem),
2. enabling faster convergence to the synchronized state,
3. eliminating or reducing the size of spurious subsystem basins of attraction in which the subsystem does not synchronize, and
4. improving the performance of signal recovery techniques for situations where a chaotic time series is used to mask a small information bearing signal [44, 46].

Gupte and Amritkar [47] demonstrated that it is possible to stabilize unstable periodic orbits of chaotic attractors by using a suitable drive variable, the choice of drive variable being dictated by the values of the SLEs of the response system. They found that for the Lorenz and Rössler systems, the SLEs of some of the unstable fixed points appeared to govern the locking to chaotic orbits and to periodic orbits as well. On the other hand, the SLEs of the unstable period-six orbit appeared to govern the properties of the chaotic orbit for the Duffing oscillator. They also noted that the actual chaotic trajectories for these systems wound around the unstable fixed points or the periodic orbits for which the agreement between the SLEs was observed. They conjectured that the SLEs of a given chaotic orbit retained the memory of the unstable periods that were responsible for its origin.

4. Exponential control

We [48, 49] have succeeded in implementing a somewhat novel method of controlling chaos. We impose a multiplicative functional feedback control on a system parameter.

The control is of the exponential form whose argument is proportional to the feedback response of the system, i.e. the difference between the desired value and the actual value of one suitably chosen variable of the system. Pecora and Carroll's method uses a suitably chosen variable of the system as a drive for controlling chaos. The OGY method controls chaos by perturbing the system parameter. Our method is a combination of variable and parametric control as it achieves control by modulating the parameter through the variable. The control can stabilize unstable fixed points, unstable limit cycles as well as chaotic trajectories. It is effective both for maps and flows.

Consider a general N-dimensional dynamical system

$$\dot{\vec{X}} = \vec{F}(\vec{X};\mu;t) \tag{1}$$

where $\vec{X} \equiv (X_1, X_2, ..., X_N)$ are variables and $\mu \equiv (\mu_1, \mu_2, ..., \mu_K)$ are controlling parameters whose values determine the nature of the dynamics. The stabilization of a desired unstable attractor or a chaotic trajectory is possible by multiplying a suitably chosen controlling parameter say μ_r in (1) by an exponential feedback control involving only one suitably chosen variable say X_l such that the form of control is

$$\exp[\epsilon(X_l - X_l^s)] \tag{2}$$

where X_l is the actual value of one of the variables of the system after applying the control, X_l^s is the desired value of that variable and ϵ is the "stiffness" of the control which can take both positive and negative values.

The dynamics of the modulated system in the presence of control is given by

$$\dot{\vec{X}} = \vec{F}(\vec{X};\mu_1,\mu_2,...,\mu_r \exp[\epsilon(X_l - X_l^s)],..,\mu_K;t). \tag{3}$$

The form of the control is such that it becomes passive once the desired goal $\overrightarrow{X^s}$ is achieved. If fluctuations drive the system off the desired goal, the control reactivates.

The control works for those combinations of controlling parameters and variables of the system for which the real part of the largest Lyapunov exponent of the modulated system, the system after the control is applied represented by (3), is negative. The feedback function in the expression of the control involves only one selected variable X_l to convert the desired repellor – a fixed point, a limit cycle or a chaotic orbit, into an attractor, implying that knowledge of only one variable on the desired unstable orbit is sufficient to settle the system on to that orbit. This makes exponential control of particular significance for systems with several degrees of freedom. As an example, one unstable fixed point of the Lorenz system [50] can be stabilized using exponential control with feedback depending on any one of the X, Y or Z variables, whereas an unstable limit cycle or a chaotic trajectory of the system can be converted into an attractor only if we use feedback depending on the Z variable. For stabilizing most repellors, it is sufficient to apply exponential control to one selected parameter of the Lorenz system. But for stabilizing one of the two fixed points of the Hénon map [51], exponential control has to be applied to both the parameters a and b. Some of these features may also be true of other forms of control. Although in all the cases we have studied, control involving only one variable on the desired orbit is sufficient to make all the Lyapunov exponents of the system negative, this may not be true for all systems. But the fact remains that the control uses only a subset of the variables and the parameters for controlling chaos. We have successfully tested our control for stabilizing different types of orbits of the logistic map, the Hénon map and the Lorenz system.

We can extend our control to convert unstable fixed points of higher period say $(\overrightarrow{X^{*1}}, \overrightarrow{X^{*2}}, \ldots, \overrightarrow{X^{*k}})$ to stable fixed points for given values of the controlling parameters. What is required is a feedback that encodes as much information about a periodic orbit as is necessary for its unique characterization. But there is a problem here of practicality. Since in such cases the form of the control given by (2) requires the convergence of the chosen variable to one of the k values of that variable, the controlling technique diminishes in utility with increase in period. For higher period orbits of a discrete dynamical system a more effective control is one which employs a logical OR structure in the feedback function. So in order to stabilize unstable fixed points of period k, i.e. $\{\overrightarrow{X^{*i}}\}$, one of the parameters μ_r in (1) is multiplied by

$$\exp\left(\epsilon \prod_{i=1}^{k} (X_l(n) - X_l^{*i})\right) \tag{4}$$

where $X_l(n)$ is the value of the chosen variable X_l at time n of the modulated map after applying control and $(X_l{}^{*1}, X_l{}^{*2}, \ldots., X_l{}^{*k})$ is the set of k values of X_l on the desired unstable period-k orbit. We have successfully implemented such control in the case of the logistic map for stabilizing a period-2 orbit [48].

A quantity of obvious interest in the context of controlling chaos is the time required for the system to settle on to the desired orbit. This of course depends upon the stiffness of the control i.e. ϵ. For a given ϵ we study the time τ required for the system to approach within a distance ω of the desired orbit starting from some initial point. If ω_0 is the initial distance from the desired orbit, then it is clear that the length of the transient τ and ω are related by $\omega = \omega_0 \exp(\lambda\tau)$, where λ is the real part of the largest Lyapunov exponent. The slope of the plot of τ against $\ln(\omega/\omega_0)$ is nothing but $1/\lambda$. This of course has to be negative for convergence. These values of the Lyapunov exponents are found to be in good agreement with those obtained numerically [52] and, wherever possible, analytically. We also study the dependence on ϵ of the transient time τ required for settling on to different types of orbits for different systems characterized by $\ln(\omega/\omega_0)$ as well as on the accuracy ω with which it does so. We find that τ is initially a decreasing function of ϵ. But there exists an optimum stiffness of control beyond which increasing ϵ can cause τ to increase. This dependence of τ on ϵ is found to be the same as that of λ on ϵ for any orbit.

In addition to this, in the case of discrete maps we have also tried to stabilize orbits which are not the natural fixed points (stable or unstable) of the original system. We have found that exponential control succeeds in creating new stable attractors which are not the natural attractors of the unmodulated system. However we cannot stabilize any arbitrary orbit. The functional form of the map and the control criterion decide which orbits can be forced on to the system.

There is one drawback of the control – in any system, if an unstable fixed point is represented by a null vector then it cannot be stabilized using exponential control. The reason is easy to understand. The Jacobian of the modulated system given by (3) evaluated at such a point in the presence of the control is the same as of the unmodulated system (1). So the eigenvalues of the system remain unchanged in the presence of the

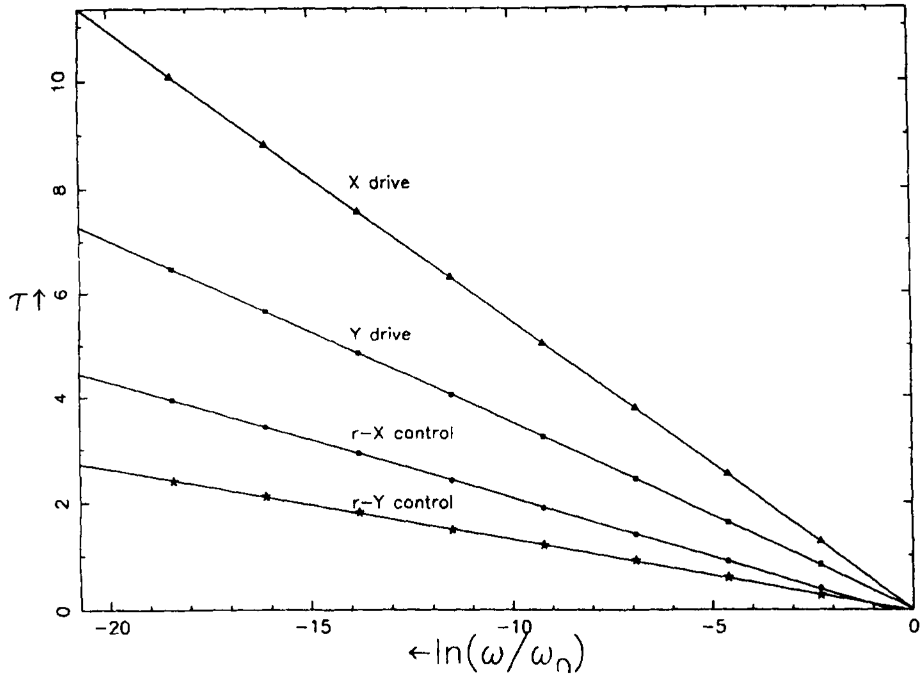

Figure 1. Plot of the transient time τ vs $\ln(\omega/\omega_0)$ for the Lorenz attractor with $\sigma = 10$, $b = 8/3$ and $r = 60$ while stabilizing the unstable fixed point $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$. Values of τ for Pecora and Carroll's method with X-drive (triangles) and Y-drive (squares) are compared with those obtained by using exponential control involving the parameter r and the variable X (circles) and also r and y (stars).

control. Hence such a fixed point remains unstable even in the presence of control. Such points may, however, get stabilized in the process of stabilizing other fixed points, resulting in the coexistence of more than one attractor.

We have compared the transient times and the basins of attraction for exponential control with other forms of control which share with it the following features:

1. the control is continuous,
2. it depends only on the variable(s) of the desired orbit,
3. is independent of the previous values of the controlling parameters μ, if the values of the parameters are changing in the presence of control, and
4. does not increase the dimensionality of the original system.

To compare our control with Pecora and Carroll's, both for X- and Y-drives in the Lorenz system, we consider the control where the parameter r is multiplied by the exponential feedback function involving the X and Y variables keeping the same set of parameter values in both cases. We find that the value of Λ with this form of the control is $-4.536\cdots$ for $\epsilon = 1.7$. We also find that Λ for the control involving the parameter r and the variable Y is $-7.593\cdots$ for $\epsilon = -0.05$ for the same set of parameter values. The corresponding values of Λ for Pecora and Carroll's method are -1.83 for the X-drive and

-2.85 for the Y-drive [39]. Our values of Λ are less than both these values and hence exponential control can be considered to take hold more rapidly. To verify this we find the transient time τ, required to stabilize the second unstable fixed point of the Lorenz system for a given set of parameter values, for different values of ω using Pecora and Carroll's method and for exponential control involving the parameter r and the variables X and Y for $\epsilon = -1.7$ and $\epsilon = -0.05$ respectively. We plot the variation of τ with $\ln(\omega/\omega_0)$ in figure 1, which confirms that our control stabilizes the unstable fixed point faster than Pecora and Carroll's. This is not altogether surprising in view of the fact that our control has an exponential form.

We have also compared our control with those suggested by Singer *et al* [27], and Chen and Chou [28]. For the first method, we find that the basin of attraction, while stabilizing nonzero fixed points of the Lorenz system, is smaller than the one obtained using exponential control. Also, the transient time is of the order $\sim 10^5$ which is larger than that with exponential control ($\sim 10^3$). The method of Chen *et al* [28] shows the same transient time for stabilizing the unstable fixed points of the Lorenz map but the basin of attraction is smaller than that obtained with exponential control. These comparisons should not be taken to imply that exponential control will always be the better choice. They only show that the performance of exponential control is different from other controls and may be more efficient for some systems.

With exponential control it is also possible to control spatiotemporal chaos. A spatiotemporal system can be described by a coupled map lattice (CML) model [53–57]. Feedback pinnings [58] are used to control chaos in the system by stabilizing a certain unstable reference state. To control such a system F, we use pinnings defined by

$$X_{n+1}(i) = F \sum_{k=1}^{L/I} \delta(i - Ik - 1) \exp[\epsilon(X_n(i) - \overline{X}_n(i)] \tag{5}$$

where I is the distance between two neighbouring pinnings, $\overline{X}_n(i)$ is the reference state to be stabilized and $\delta(j) = 1$ for $j = 1$ and $\delta(j) = 0$ otherwise. We have succeeded in stabilizing various unstable spatiotemporal patterns like an unstable homogeneous stationary state, an unstable space-period-two pattern and a time-period-two with a space-period-four running wave.

5. Application of synchronized chaos to communication

Cuomo and Oppenheim [46] used an analog circuit implementation of the chaotic Lorenz system to illustrate the use of synchronized chaotic systems in communications. The set of equations representing Lorenz based chaotic circuits are

$$\begin{aligned} \dot{u} &= \sigma(v - u) \\ \dot{v} &= ru - v - 20uw \\ \dot{w} &= 5uv - bw. \end{aligned} \tag{6}$$

Their transmitter and receiver systems were represented by identical equations, which helped to achieve perfect synchronization between them. They considered $u(t)$ as the

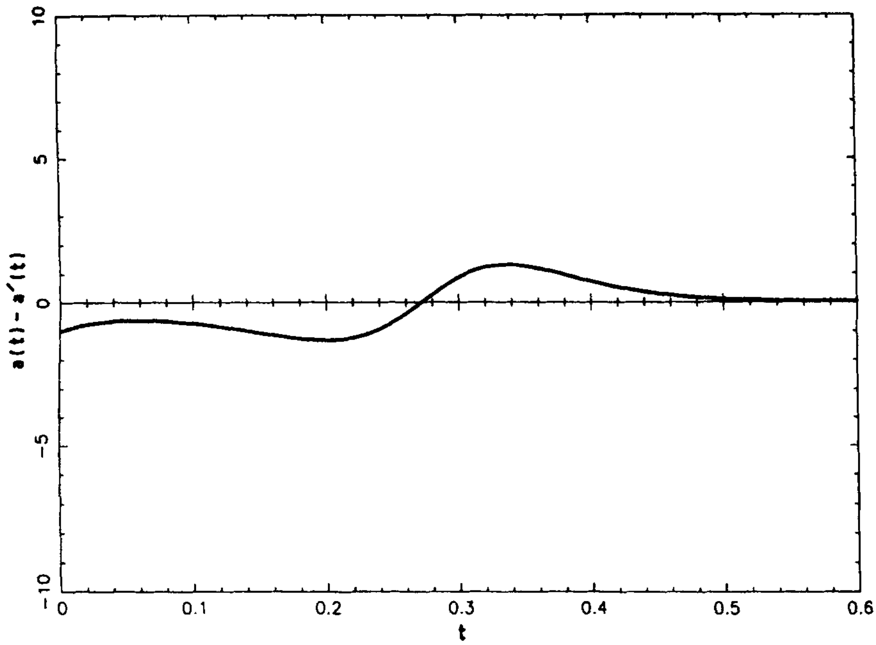

Figure 2. The error i.e. $a'(t) - a(t)$, in regeneration of the information signal at the receiver.

drive signal. The basic idea was to modulate a transmitter coefficient b with the information-bearing wave form, $m(t)$, which was taken to be a square wave. The square wave changed the value of b to a new value b_n with the one bit and no change was produced with the zero bit. A chaotic drive signal $u(t)$ was transmitted. At the receiver, the modulation produced a synchronization error between the received drive signal and the receiver's regenerated drive signal u_r with an error signal amplitude that depended on the modulation.

We have tried to use exponential control for the modulation/detection process in communications. The idea is to take the transmitter and receiver systems to be represented by the same set of equations with the same set of values for the controlling parameters. As demonstration we choose the Lorenz system to represent the transmitter and the receiver, with the parameter values $\sigma = 16.0$, $b = 4.0$, and $r = 40.0$. The largest Lyapunov exponent, for these values of the parameters is positive, implying that the system is in a chaotic regime. We chose an arbitrary chaotic trajectory starting with the initial point, say, $(10.0, 0.0, 30.0)$. The information signal $a(t)$ is taken to be a square wave defined by

$$\begin{aligned} a(t) &= 10 \qquad 0 \leq t < 5\Delta t \\ &= 0 \qquad 5\Delta t \leq t < 10\Delta t \end{aligned} \tag{7}$$

where Δt is the integration step taken to be 0.001. We superimpose the information signal $a(t)$ on one of the variables, say, $X(t)$ of the drive system such that the resultant signal $s(t) = a(t) + X(t)$. Then $s(t)$ and another variable, say, $Z(t)$ of the drive system are transmitted. At the receiver, we start with an arbitrary initial state, say, (X_r^0, Y_r^0, Z_r^0). We apply $b - Z$ control such that the parameter b gets multiplied by $\exp(\epsilon(Z_r - Z))$, where Z_r is the Z coordinate of a point on a chaotic trajectory of the receiver, evolving in the presence of exponential control, so that its dynamics can be represented by

$$\begin{aligned}\dot{X}_r &= \sigma(Y_r - X_r)\\ \dot{Y}_r &= -X_r Z_r + r X_r - Y_r\\ \dot{Z}_r &= X_r Y_r - b\exp(\epsilon(Z_r - Z))Z_r.\end{aligned} \tag{8}$$

We choose the value of ϵ to be 0.034 for which the transmitted chaotic trajectory becomes stable in the presence of exponential control. Consequently nearby trajectories converge to the desired trajectory. Thus after transients have died out, the information signal can be regenerated by subtracting the $X_r(t) \approx X(t)$ signal from the transmitted signal $s(t)$, i.e, $a(t) \approx a'(t) = s(t) - X_r(t)$. We find that the error in the regeneration of the information signal $a(t)$ for the chosen values of the constants, lies between 0.1% and 1.0% of the amplitude of the information signal. This can be seen in figure 2.

The advantage of using this approach for the modulation/detection of a signal is that there is no restriction on the frequency or the amplitude of the information signal. The quality of the regenerated signal is quite good. The only drawback is that we have and transmit two signals, viz. $s(t)$ and $Z(t)$, from the transmitter.

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