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Geometry and nonlinear evolution equations

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Abstract. We briefly review the nonlinear dynamics of diverse physical systems which can be described in terms of moving curves and surfaces. The interesting connections that exist between the underlying differential geometry of these systems and the corresponding nonlinear partial differential equations are highlighted by considering classic examples such as the motion of a vortex filament in a fluid and the dynamics of a spin chain.The association of the dynamics of a non-stretching curve with a hierarchy of completely integrable soliton-supporting equations is discussed. The application of the surface embeddability approach is shown to be useful in obtaining such connections as well as exact solutions of some nonlinear systems such as the Belavin–Polyakov equation and the inhomogeneous Heisenberg chain.

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1. Introduction

The connection between differential geometry and certain nonlinear partial differential equations (NLPDE) has attracted the attention of both physicists and mathematicians for over a century now. Nineteenth century geometers had shown such a relationship between the structure of minimal surfaces (i.e., surfaces whose mean curvature is zero) and the hyperbolic Liouville equation. Another classic example is the connection [1] between the geometry of pseudospherical surfaces (i.e., surfaces with negative constant Gaussian curvature) and the sine-Gordon equation. This result achieved a special significance many years later, when it was realized that these equations were integrable by the powerful inverse scattering transform (IST) method discovered in 1967 by Gardner *et al* [2]. This observation inspired a lot of interesting work on the relationship [3] between the differential geometry of special surfaces and integrable soliton-supporting equations [4]. Around the same time, the differential geometry of moving curves [3] and their relation to integrable equations was also studied. Interestingly, this latter connection first arose in the context of a *physical* application in the field of hydrodynamics [5]. Pioneering work by Hasimoto [5] showed that the equation of motion of a vortex filament regarded as a space curve was equivalent to the well-known, integrable nonlinear Schrödinger equation (NLSE) [4]. Using the Hasimoto transformation that relates space curves and complex curvature functions, Lamb [6] generalized the above result by demonstrating the link between the motion of certain space curves and soliton-bearing equations. Soon afterwards, Lakshmanan [7] showed that the Landau–Lifshitz equation which describes the continuum dynamics of the classical ferromagnetic Heisenberg chain was gauge-equivalent to the NLSE .

In recent years, there has been renewed interest in looking for such connections. It has been shown that the dynamics of a non-stretching planar curve can be connected with the integrable, modified Korteweg–de Vries (MKdV) hierarchy [8]. Another example is the Belavin–Polyakov equation [9] satisfied by a unit vector field, which appears in a variety of physical applications such as the nonlinear sigma model, the two-dimensional Heisenberg ferromagnet, etc. Its equivalence to the exactly solvable elliptic Liouville equation has been established [10]. A surface embeddability approach has been demonstrated to be useful in finding explicit solutions to the spin evolution equation of an inhomogeneous Heisenberg chain [11]. In this paper, we briefly review some of the above-mentioned examples of nonlinear dynamical systems which can be described geometrically. The purpose is to illustrate how the application of the classical differential geometry of curves and surfaces can be used to find such surprising connections as well as exact dynamical solutions.

2. Space curves and surfaces

We begin by summarizing some of the salient features of the space-curve formalism and surface theory, which will be useful in the applications to be considered in subsequent sections.

2.1 *Time evolution of twisted space curves*

A *static* space curve in E^3 is described either by its parametric equations or by its natural equations: $K = K(s)$ and $\tau = \tau(s)$, where K, τ and s are the curvature, the torsion and the arc length of the space curve. Consider a curve which in its parametric form is described by the position vector $\mathbf{x} = \mathbf{x}(s)$. Let $\mathbf{t}(s) = \mathrm{d}\mathbf{x}/\mathrm{d}s$ be the unit tangent to this curve, and $\mathbf{n}$ and $\mathbf{b}$ the unit normal and binormal, so that $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ form an orthogonal triad on the curve. They satisfy the well known Frenet–Serret equations [1]

$$\mathbf{t}_s = K\mathbf{n}, \quad \mathbf{n}_s = -K\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}_s = -\tau\,\mathbf{n} \tag{2.1}$$

where the subscript denotes $(\mathrm{d}/\mathrm{d}s)$ and K and τ are given by

$$K^2 = \mathbf{t}_s \cdot \mathbf{t}_s, \quad \tau = \mathbf{t} \cdot (\mathbf{t}_s \times \mathbf{t}_{ss})/K^2. \tag{2.2}$$

Let this curve *evolve* in time t in such a way that the local 'velocity' at s is specified by

$$\mathbf{V} = (\partial\mathbf{x}(s,t)/\partial t) = \mathbf{x}_t = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}. \tag{2.3}$$

The scalar functions α, β and γ represent the tangential, normal and binormal components of $\mathbf{V}$. As we shall see, their functional form will be determined by the physics of the model under consideration.

There is another aspect of curve motion. As the tangent to a static curve is $\mathbf{x}_s = \mathbf{t}$ by definition, it is clear that in any physical application involving moving curves, it is possible to write down the time evolution of $\mathbf{t}$ from (2.3) by using

$$\mathbf{t}_t = \mathbf{x}_{st}. \tag{2.4}$$

In some other models (like spin chains), $\mathbf{t}_t$ can be written down directly. In general, since $\mathbf{t}$ is a unit vector,

$$\mathbf{t}_t = g\mathbf{n} + h\mathbf{b}, \tag{2.5}$$

where g and h will be local functions of the curvature K and the torsion τ. Further, when the compatibility conditions

$$\mathbf{t}_{ts} = \mathbf{t}_{st}, \quad \mathbf{n}_{ts} = \mathbf{n}_{st}, \quad \mathbf{b}_{ts} = \mathbf{b}_{st} \tag{2.6}$$

are imposed, one typically obtains *coupled nonlinear partial differential equations for K and τ*. In many applications, these turn out to be well-known exactly solvable equations. In such cases, the analysis shows that the above property arises essentially due to certain special geometrical features with which the curve evolution is endowed.

2.2 *Surface theory*

Consider a surface generated by a position vector $\mathbf{r}(u, v)$ which is a function of two parameters (local coordinates) u and v. The metric, i.e., the first fundamental form, is given in the usual notation [1] as $I = (\mathrm{d}\mathbf{r})^2 = E(\mathrm{d}u)^2 + 2F\mathrm{d}u\,\mathrm{d}v + G(\mathrm{d}v)^2$, where $E = (\mathbf{r}_u)^2$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$ and $G = (\mathbf{r}_v)^2$. The unit normal to the surface is defined by $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v)/|\mathbf{r}_u \times \mathbf{r}_v|$. The extrinsic curvature tensor, i.e., the second fundamental form is defined as $II = -\mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{n} = L(\mathrm{d}u)^2 + 2M\mathrm{d}u\,\mathrm{d}v + N(\mathrm{d}v)^2$, where $L = \mathbf{n}\cdot\mathbf{r}_{uu}, M = \mathbf{n}\cdot\mathbf{r}_{uv}$ and $N = \mathbf{n}\cdot\mathbf{r}_{vv}$. At every point on the surface, we can introduce a moving trihedron made up of the three linearly independent vectors $\mathbf{r}_\mathbf{u}$, $\mathbf{r}_\mathbf{v}$ and $\mathbf{n}$. Since every vector can be linearly expressed in terms of these three basis vectors, one obtains the following Gauss–Weingarten (GW) equations [1]:

$$\mathbf{r}_{uu} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n} \tag{2.7a}$$

$$\mathbf{r}_{uv} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n} \tag{2.7b}$$

$$\mathbf{r}_{vv} = \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n} \tag{2.7c}$$

$$\mathbf{n}_u = p_1\mathbf{r}_u + p_2\mathbf{r}_v \tag{2.7d}$$

$$\mathbf{n}_v = q_1\mathbf{r}_u + q_2\mathbf{r}_v \tag{2.7e}$$

where the expressions for the Christoffel symbols Γ^k_{ij} and the coefficients p_i, q_i, $i = 1, 2$ are given below for ready reference

$$\begin{aligned}
&\Gamma^1_{11} = [GE_u - 2FF_u + FE_v]/2\Lambda; \quad \Gamma^1_{12} = [GE_v - FG_u]/2\Lambda \\
&\Gamma^1_{22} = [2GF_v - GG_u - FG_v]/2\Lambda; \quad \Gamma^2_{11} = [2EF_u - EE_v - FE_u]/2\Lambda \\
&\Gamma^2_{12} = [EG_u - FG_v]/2\Lambda; \quad \Gamma^2_{22} = [EG_v - 2FF_v + FG_u]/2\Lambda \\
&p_1 = [MF - LG]/\Lambda; \quad p_2 = [LF - ME]/\Lambda. \\
&q_1 = [NF - MG]/\Lambda; \quad q_2 = [MF - NE]/\Lambda,
\end{aligned} \tag{2.8}$$

where $\Lambda = (EG - F^2)$. The Gauss–Mainardi–Codazzi (GMC) equations [1] are obtained from the compatibility conditions $\mathbf{r}_{uuv} = \mathbf{r}_{uvu}$ and $\mathbf{r}_{uvv} = \mathbf{r}_{vvu}$. We will write down the relevant equations when we discuss specific applications. The fundamental theorem of surfaces states that if the surface parameters E, F, G, L, M and N satisfy the GMC equations, then the corresponding surface $\mathbf{r}(u, v)$ is uniquely determined except for its position in space. We will see how this can be used in applications where the GMC equations turn out to be exactly solvable equations, with the surface parameters determined exactly as explicit functions of u and v.

3. Vortex filament motion and the Hasimoto transformation

It has been observed that vortex filaments in a fluid very often preserve their identity when they move. In this section, we summarize Hasimoto's work [5] which shows that on using a local induction approximation (to be explained below), the propagation of a soliton along a filament can be obtained.

A thin vortex filament in a fluid can be regarded as a space curve. The fluid is incompressible and inviscid and has a non-vanishing vorticity

$$\mathbf{w}(\mathbf{x}) = \nabla \times \mathbf{V}(\mathbf{x}), \tag{3.1}$$

where $\mathbf{V}(\mathbf{x})$ is the fluid velocity at a point with position vector $\mathbf{x}$ on the filament. Equation (3.1) implies the following Biot–Savart law:

$$\mathbf{V}(\mathbf{x}) = \int \mathrm{d}^3x' \mathbf{w}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3. \tag{3.2}$$

Thus the filament is transported by the velocity *induced* by its own vorticity. For simplicity, it is assumed that $\mathbf{w}(\mathbf{x})$ is tangential to the filament at every point. Since the vortex has a thin core, the volume integral in (3.2) can be replaced by a line integral as follows:

$$\mathbf{V}_I(\mathbf{x}) \sim \int \mathbf{t}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3 \mathrm{d}s', \tag{3.3}$$

where $\mathbf{V}_I$ is the induced velocity and $\mathbf{x} = \mathbf{x}(s, t)$ is the position vector of a point on the filament curve at time t. Because of the thinness of the core, the contribution of the local portion of the filament dominates the Biot–Savart integral. Expanding $\mathbf{x}'$ in a Taylor series around $\mathbf{x}(s, t)$,

$$\mathbf{x}'(s', t) = \mathbf{x}(s, t) + (s' - s)\mathbf{x}_s + (1/2)(s' - s)^2\mathbf{x}_{ss} + O(s' - s)^3. \tag{3.4}$$

Since $\mathbf{x}_s = \mathbf{t}$ and $\mathbf{x}_{ss} = \mathbf{t}_s = K\mathbf{n}$ from (2.1), substituting (3.4) in (3.3) yields

$$\mathbf{V}_I \sim K(\mathbf{t} \times \mathbf{n}) = K\mathbf{b}, \tag{3.5}$$

Comparing (3.5) with (2.3), we see that in this 'local induction approximation', the velocity $\mathbf{V}_I = \mathbf{x}_t$ of the vortex filament has no tangential and normal components but only a component along the binormal, proportional to the curvature K. Equation (3.5) leads to

$$\mathbf{t}_t = K_s\mathbf{b} - K\tau\mathbf{n}, \tag{3.6}$$

where (2.4) and (2.1) have been used. As explained in § 2, the compatibility conditions (2.6) lead to the following coupled set of partial differential equations for K and τ:

$$K_t = -2K_s\tau - K\tau_s \tag{3.7a}$$

$$\tau_t = [(K_{ss}/K) - \tau^2]_s - KK_s. \tag{3.7b}$$

On using the Hasimoto transformation

$$\psi = K\exp\left\{i\int^s \tau\,\mathrm{d}s\right\}, \tag{3.8}$$

Equations (3.7) reduce to the nonlinear Schrödinger equation (NLSE)

$$i\psi_t + \psi_{ss} + (1/2)|\psi|^2\psi = 0 \tag{3.9}$$

whose exact solutions are known [4]. The NLSE is a completely integrable evolution equation, associated with an infinite number of integrals of motion, strict N-soliton solutions, etc. From the soliton solution, the curvature and torsion of the filament can be read off using (3.8). Thus, the local induction approximation is seen to lead to a shape- and velocity-preserving motion of a helical vortex filament in the fluid. It is interesting that such a motion of vortex filaments has been experimentally observed in certain types of fluids. Finally, this example shows that the special geometry underlying the completely integrable NLSE is encoded in the simple geometric relationship $\mathbf{x}_t = K\mathbf{b}$ [Eq. (3.5)] for the local velocity of a space curve moving in E^3. More recently, it has been shown [12] that going beyond the local induction approximation can lead to a connection with more general soliton-bearing equations.

4. Dynamics of the classical Heisenberg ferromagnetic chain

The subject of quasi-one dimensional magnets with various symmetries is interesting because such systems can be fabricated in the laboratory. In this section we review the isotropic ferromagnetic chain.

Consider a one-dimensional lattice with classical spins $\mathbf{S}_i$ of constant magnitude (i.e., $(\mathbf{S}_i)^2 = S^2 = 1$) at every site i, with the nearest neighbour interaction Hamiltonian

$$H = -J\sum_i \mathbf{S}_i\cdot\mathbf{S}_{i+1}, \quad J > 0. \tag{4.1}$$

The spin evolution equation derived from (4.1) is $(\mathrm{d}\mathbf{S}_i/\mathrm{d}t) = J\mathbf{S}_i \times (\mathbf{S}_{i+1} + \mathbf{S}_{i-1})$. On taking the continuum limit $\mathbf{S}_i \to \mathbf{S}(x,t)$, this reduces to the nonlinear partial differential equation (the Landau–Lifshitz equation)

$$\mathbf{S}_t = J\mathbf{S} \times \mathbf{S}_{xx}, \tag{4.2}$$

where the subscripts denote partial derivatives. The underlying geometry of (4.2) is most conveniently analysed by identifying $\mathbf{S}$ with the unit tangent to a curve [13], so that the Landau–Lifshitz equation describes a *moving space curve*. (Note that the instantaneous spin configuration on the chain can be mapped to a corresponding space curve at that instant). Thus (4.2) can be cast in the form (after rescaling time t to Jt)

$$\mathbf{t}_t = \mathbf{t} \times \mathbf{t}_{ss}. \tag{4.3}$$

Pioneering work by Lakshmanan [7] showed that Eq. (4.3) is gauge-equivalent to the NLSE. The following interesting connection between the present interacting spin problem and the (seemingly unrelated) vortex filament motion considered in § 3 is worth noting: The filament evolution $\mathbf{x}_t = K\mathbf{b}$ (Eq. (3.5)) when differentiated with respect to s leads to (4.3)! Thus (4.3) also yields the *same* coupled equations (3.7) found in the fluid mechanical context, and thence to the NLSE. The soliton solutions play the role of nonlinear excitations in the spin chain [7]. It must be noted that the solution of NLSE contains information about the energy and current densities [13]. In this example we have seen that the geometry of the NLSE is encoded in the special time evolution of the *tangent* vector as given by (4.3).

Before concluding this section, we mention that there are some extensions of the Heisenberg chain which can be analysed through the space curve formalism-for instance, the inclusion [14] of the Landau–Gilbert damping term in the Hamiltonian. Integrability in higher dimensions for the spherically symmetric case has also been recently investigated [15].

5. Dynamics of nonstretching planar curves

There are many physical applications involving curve dynamics which preserve certain global geometric quantities such as the length of an open curve or the area enclosed by a closed curve. Examples are polymers, cell membranes or closed vortex filaments. An interesting paper by Goldstein and Petrich [8] considers the dynamics of curves on a *plane*. In this case the torsion $\tau = 0$, and Eqs (2.1) become

$$\mathbf{t}_s = K\mathbf{n}, \quad \mathbf{n}_s = -K\mathbf{t}. \tag{5.1}$$

Now

$$\mathbf{x}_t = U\mathbf{n} + W\mathbf{t}, \tag{5.2}$$

where U and W are the normal and tangential components of the velocity. If these are chosen to be local functions of the curvature K and its derivatives, i.e.,

$$U = U(K, K_s, K_{ss} \ldots) \quad \text{and} \quad W = W(K, K_s, K_{ss} \ldots),$$

we get from (5.2)

$$\mathbf{x}_{st} = \mathbf{t}_t = (W_s - KU)\mathbf{t} + (U_s + KW)\mathbf{n}.$$

Since $\mathbf{t}.\mathbf{t}_t = 0$, this leads to

$$W_s = KU, \quad W = \int KU\mathrm{d}s + C, \tag{5.3}$$

where C = constant. Hence, in general,

$$\mathbf{t}_t = (U_s + KW)\mathbf{n}. \tag{5.4}$$

At this stage, one poses the following question: How should a planar curve evolve in time so as to keep its length $L = \int \mathrm{d}s$ fixed? One way to ascertain $L_t = 0$ is to ensure that $\mathrm{d}s/\mathrm{d}t = 0$ so that the curve does not 'stretch' as time evolves. This imposes the compatibility conditions $\mathbf{t}_{st} = \mathbf{t}_{ts}$ and $\mathbf{n}_{st} = \mathbf{n}_{ts}$. Using (2.1), (5.3) and (5.4) in these yields the following evolution equation for K:

$$K_t = U_{ss} + K^2 U + K_s \left\{ \int KU \mathrm{d}s + C \right\} = -\Omega U, \tag{5.5}$$

where the operator Ω is given by

$$\Omega = (-\partial_{ss} + K^2 + K_s \partial_s^{-1} K) \tag{5.6}$$

with

$$\partial_s^{-1} KU = \int KU \mathrm{d}s + C.$$

It is clear that the evolution of the curve is decided by the choice of U and W in (5.2). Since U and W are related by (5.3) in general, and W should be a local function, either U should vanish or KU must be a total derivative. The simplest choice $U = U^{(1)} = 0$ gives $W = W^{(1)} = C$ and

$$K_t = -CK_s. \tag{5.7}$$

This is a linear equation representing the trivial motion of the curve along itself with velocity C. (Note that the velocity $\mathbf{x}_t^{(1)} = C\mathbf{t}$ has only a tangential component). The next choice $U = U^{(2)} = -K_s$ gives $W = W^{(2)} = (\frac{1}{2}K^2) + C$, giving the modified Korteweg-de Vries (MKdV) equation

$$K_t = -K_{sss} + (3/2)K^2 K_s \tag{5.8}$$

for the choice $C = 0$. This is a completely integrable equation. The one-soliton solution $K(s,t) = \sqrt{u}$ sech $(\sqrt{u}(s - ut))$ represents a moving localized profile (lump) traveling along the curve with velocity $u > 0$. Note that the velocity $\mathbf{x}_t^{(2)} = -K_s\mathbf{n} - (\frac{1}{2}K^2)\mathbf{t}$ has both tangential *and* normal components.

Proceeding likewise, i.e., by choosing the right-hand side of each equation for K to be the normal velocity U of the succeeding equation, a *hierarchy* of equations is obtained, the next one being

$$K_t = -K_{sssss} - (15/8)K^4 K_s - (5/2)K_s^3 - (5/2)K^2 K_{ss} - 10KK_sK_{ss}, \tag{5.9}$$

for the choice $U = U^{(3)} = K_{sss} + \frac{3}{2}K^2 K_s$.

Note that the order of the equation increases at each stage. If the curve is closed, the area enclosed is $A = \frac{1}{2}\oint(\mathbf{x} \times \mathbf{x}_s)\mathrm{d}s$, giving $A_t = \oint U\mathrm{d}s$. Since U is a total derivative for all members of the hierarchy, the corresponding evolutions preserve not only the perimeter but also the area enclosed. In addition, due to complete integrability, the dynamics also possesses an infinite number of integrals of the motion which are polynomials in the curvature and their derivatives. (There could of course be other choices like $U = K^n K_s$, $W = -K^{n+2}/(n+2)$, which would also conserve these global quantities, but it is not known whether the resulting dynamics would be integrable). In conclusion, the recursion operator Ω defined in (5.6) has a universal geometrical significance in defining the curvature evolution under arclength-conserving dynamics.

It has also been pointed out [6, 16, 17] that the Frenet–Serret equations are equivalent to the Ablowitz–Kaup–Newell–Segur (AKNS) scattering problem in the IST with zero eigenvalue. We conclude this section by mentioning that Langer and Perline [18] have shown that the dynamics of a nonstretching filament regarded as a non-planar space curve can give rise to an analogous recursion operator for the NLSE hierarchy, on using the Hasimoto transformation.

6. The Belavin–Polyakov equation

The Belavin–Polyakov equation [9]

$$\mathbf{m}_y = \mathbf{m}_x \times \mathbf{m}, \quad (\mathbf{m})^2 = 1 \tag{6.1}$$

arises in several physical applications. In these coupled nonlinear partial differential equations, both the independent variables x and y could be spatial variables, or one spatial and the other temporal, depending on the physical context. The static solutions of the nonlinear sigma model in $(2+1)$ dimensions, the static two-dimensional Heisenberg ferromagnet, and the Heisenberg antiferromagnet in $(1+1)$ dimensions [19] are examples where the relevant configurations are classical *unit* vectors $\mathbf{m}$ satisfying (6.1).

Let us first apply the moving space curve formulation to this problem [10]. It is possible to identify $\mathbf{m}$ with the tangent vector to a space curve which 'evolves' either spatially or temporally, depending on whether y is a space variable or a time variable. Thus (6.1) can be written in the form

$$\mathbf{t}_u = \mathbf{t}_s \times \mathbf{t} = -K\mathbf{b}, \tag{6.2}$$

on using (2.1). Writing $\mathbf{n}_u = a_1\mathbf{t} + a_2\mathbf{b}$ and $\mathbf{b}_u = g_1\mathbf{t} + g_2\mathbf{n}$, the compatibility conditions (2.6) determine $a_1 = 0$, $a_2 = (-K_s/K)$, $g_1 = K$ and $g_2 = (K_s/K)$. Further, the coupled evolution equations for K and τ are

$$K_u = K\tau \tag{6.3a}$$

and

$$\tau_u = (-K_s/K)_s - K^2. \tag{6.3b}$$

These equations combine to give

$$(K_u/K)_u + (K_s/K)_s = -K^2, \tag{6.3c}$$

which becomes the elliptic Liouville equation

$$\phi_{ss} + \phi_{uu} = -\exp(2\phi), \tag{6.3d}$$

on setting $\phi = \ln K$. Since the exact general solution of (6.3d) is known [10], the exact solutions for the curvature $K = \exp\phi$ and $\tau = \phi_u$ of the evolving space curve can be determined. Thus the use of classical differential geometry of space curves has enabled us to establish [10] the exact solvability of (6.2), by connecting it to the exactly solvable elliptic Liouville equation.

This connection can also be found [10] using surface theory. By identifying the solution $\mathbf{m}(x,t)$ of (6.1) with a position vector $\mathbf{r}(u,v)$, it is possible to describe a surface. The relevant expressions for the first and second fundamental forms have been given in § 2. Using them in conjunction with eq. (6.1), we see that $F = 0$ and

$G = (\mathbf{m}_y)^2 = (\mathbf{m}_x \times \mathbf{m})^2 = E$. Thus the metric has a *conformal* ("*isothermal*") form. The unit normal to the surface $\mathbf{n}$ is defined as in § 2. From (6.1), we obtain $(\mathbf{m}_x \times \mathbf{m}_y) = -E\mathbf{m}$. Hence $|\mathbf{m}_x \times \mathbf{m}_y| = E$. A short calculation shows that the second fundamental form parameters are given by $L = N = E$ and $M = 0$. Thus the Gaussian curvature $K = (LN - M^2)/(EG - F^2)$ and the mean curvature $H = \frac{1}{2}(EN + GL - 2FM)/(EG - F^2)$ are both constant. With $\mathbf{m}(x, y)$ as the position vector creating the surface, the Gauss–Weingarten equations (Eqs (2.7)) become

$$\begin{aligned}
\mathbf{m}_{xx} &= (E_x/2E)\mathbf{m}_x - (E_y/2E)\mathbf{m}_y - E\mathbf{m} \\
\mathbf{m}_{xy} &= (E_y/2E)\mathbf{m}_x + (E_x/2E)\mathbf{m}_y \\
\mathbf{m}_{yy} &= -(E_x/2E)\mathbf{m}_x + (E_y/2E)\mathbf{m}_y - E\mathbf{m} \\
\mathbf{n}_x &= -\mathbf{m}_x \\
\mathbf{n}_y &= -\mathbf{m}_y
\end{aligned} \tag{6.4}$$

We now obtain the Gauss–Mainardi–Codazzi equations [1] from the various compatibility conditions as mentioned in § 2. Note that $\mathbf{n}_{xy} = \mathbf{n}_{yx}$ is automatically satisfied, while the conditions $\mathbf{m}_{xxy} = \mathbf{m}_{xyx}$ and $\mathbf{m}_{xyy} = \mathbf{m}_{yyx}$ both reduce, after a short calculation, to the following equation for the coefficient E of the conformal metric:

$$(E_y/2E)_y + (E_x/2E)_x = -E. \tag{6.5}$$

Defining $\phi = \frac{1}{2}\ln E$, the above equation becomes the elliptic Liouville equation (6.3d). Its *general* solution can be written down as

$$\phi(x, y) = \tfrac{1}{2}\ln\{-4A'(z^*)B'(z)/[A(z^*) + B(z)]^2\} \tag{6.6}$$

where $z = x + iy, z^* = x - iy, A'(z^*) = \mathrm{d}A/\mathrm{d}z^*$, $B'(z) = \mathrm{d}B/\mathrm{d}z$ and A, B are arbitrary functions. Hence

$$E(x, y) = -4A'(z^*)B'(z)/[A(z^*) + B(z)]^2. \tag{6.7}$$

The fundamental theorem of surfaces states that if the metric and curvature tensors satisfy the Gauss–Mainardi–Codazzi equations (which in the present case is the elliptic Liouville equation), then the Gauss–Weingarten equations can in principle be integrated to determine the surface (specified by $\mathbf{m}(x, y)$) uniquely, up to a rotation and translation in space. Therefore (6.1) is exactly solvable. Some interesting special solutions have been found in Ref. [19].

It can be shown that the basic equations (6.3a) and (6.3b) determined using the 'direct' moving curve method can be combined into the following equation for the complex Hasimoto function $q = \kappa \exp(i\int_{-\infty}^{s} \tau \mathrm{d}s)$:

$$iq_u - q_s - q\int_{-\infty}^{s} |q|^2 \mathrm{d}s = 0. \tag{6.8}$$

This equation had in fact been obtained earlier [19] in another context. A short calculation verifies that its imaginary part yields (6.3a), while its real part gives (6.3b). These equations when combined become the elliptic Liouville equation (6.3d), as we have already shown above.

7. The inhomogeneous Heisenberg chain

The examples discussed in the previous sections show how classical differential geometry can be used as an effective tool to find the connection of a nonlinear system described by a vector evolution equation with a completely integrable NLPDE or with one whose exact general solution is known. As mentioned in the introduction, while these connections are indeed useful in predicting similar properties for the corresponding vector evolution equation, finding the explicit solution of the latter is not always an easy task. In many physical situations, finding special exact solutions explicitly would be of interest (the corresponding nonlinear system need not necessarily be always completely integrable). In this section we consider the example of the inhomogeneous Heisenberg chain to show how exact solutions can be found using an appropriate geometric approach [11].

Several years ago, this author proposed a model Hamiltonian [20]

$$H = -\sum_i f_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}, (\mathbf{S}_i)^2 = S^2 = 1, \tag{7.1}$$

where $\mathbf{S}_i$ denotes the spin vector and f_i the site dependent (inhomogeneous) exchange interaction between the spins. The continuum evolution for **S** was found to be

$$\mathbf{S}_t = (f\mathbf{S}\times\mathbf{S}_x)_x, \quad \mathbf{S}^2 = 1, \tag{7.2}$$

By associating **S** with the tangent to a moving curve, it was shown that the above equation is gauge-equivalent to a generalized nonlinear Schrödinger equation (GNLSE) of the form

$$iq_t + (fq)_{xx} + 2q\left\{f|q|^2 + \int_{-\infty}^{x} f_x|q|^2 \mathrm{d}x\right\} = 0. \tag{7.3}$$

Here q is given essentially by the Hasimoto function defined in (3.8), namely,

$$q = (\kappa/2) \exp \int_{-\infty}^{x} \tau \, \mathrm{d}x, \tag{7.4}$$

with the curvature κ and the torsion τ of the curve being given by

$$\kappa^2 = (\mathbf{S}_x)^2, \quad \tau = \mathbf{S}\cdot(\mathbf{S}_x \times \mathbf{S}_{xx})/\kappa^2 \tag{7.5}$$

respectively. There are only two forms of $f(x)$ for which the conventional IST method is directly applicable to the above GNLSE, *viz.*, (i) when f is a constant (in which case it reduces to the usual NLSE), and (ii) when f is a linear function [21] of x. All other $f(x)$ fall into a special class [20]. In a interesting recent paper, Cieśliński *et al* [22] have used some ideas from Sym's surface approach [3] to find exact solutions of (7.2) and (7.3) for several other forms of the inhomogeneity function $f(x)$. Representing **S** as the spatial derivative of a vector **r**, i.e., setting

$$\mathbf{S} = \mathbf{r}_x, \tag{7.6}$$

it can be verified that the kinematic equation

$$\mathbf{r}_t = f(\mathbf{r}_x \times \mathbf{r}_{xx}), \quad \mathbf{r}_x^2 = 1. \tag{7.7}$$

implies Eq. (7.2). Assuming the surface generated by the position vector $\mathbf{r}(x,t)$ to be equipped with a metric of the geodesic form, it was shown [22] that the Gauss–Mainardi–Codazzi (GMC) equations [1] for this surface can be cast in the same form as (7.3), upon making appropriate identifications between q and f on the one hand, and the coefficients of the first and second fundamental forms (i.e., the metric and the extrinsic curvature, respectively) on the other. By specializing to surfaces of revolution, it was demonstrated that the equations for the geodesics on the surface could be integrated explicitly for certain cases, and an algorithm was found for obtaining both f and the corresponding solution $\mathbf{S}$. When the geodesic coordinates were taken to be x and t, the solutions called 'spins-on-meridians' could be found for a wide variety of time-independent, bounded, positive functions $f(x)$, i.e., inhomogeneous ferromagnetic couplings.

In what follows we use [11] a geometric formulation different from the above approach in the sense that, although we too identify $\mathbf{r}(x,t)$ with a position vector that generates a surface, we do not compare the GMC equations to (7.3), nor do we use the general equations of geodesics. Our strategy is to regard the basic kinematic equation (7.7) for $\mathbf{r}$ as a *constraint* on the surface generated. Such a constraint is expected to permit only certain special geometries for the surface [3]. Indeed, for the model under investigation, the surface metric is shown to be *necessarily* of the geodesic form, and further, certain coefficients of the two fundamental forms get related through the function f. Using these results in the GMC equations, it becomes possible to 'integrate' the latter and find the expressions for L, M and N (the coefficients of the second fundamental form) as well as the corresponding function f in terms of the metric coefficient G, its x-derivatives and two (arbitrary) integration constants, for time-independent metrics. It is then demonstrated that the solution q of (7.3) can also be written in terms of the above-mentioned quantities, simply by re-expressing the moving curve parameters κ and τ in terms of surface coefficients by using the Gauss-Weingarten (GW) equations [1] for the surface. Thus we see that given an arbitrary metric coefficient, the explicit solution for q along with the corresponding f can be written down. This solution q is a *complex* function in general, and, as already mentioned in § 4, is interesting in its own right since it contains information [11] on the energy and current densities along the chain. Furthermore, on inspecting the expressions for L, M and N obtained from the GMC equations, it is readily seen that they correspond to those of surfaces of revolution, [1] when one of the two integration constants referred to above vanishes. Also, $G^{1/2}$ plays the role of the generator of revolution, and the explicit solution of $\mathbf{r}(x,t)$ can be written down in this limiting case. In this limit, our expression for $f(x)$ reduces to the result obtained [22] in the geodesic approach, for the 'spins-on-meridians' solutions. The corresponding solutions for q are necessarily *real*. For certain common surfaces of revolution like the sphere, the torus and the catenoid, the solution $\mathbf{S}$ of Eq. (7.2) can be expressed solely as a functional of the inhomogeneity function f that appears in the equation, essentially because the surface metric can also be written down as a functional of f for these surfaces. We now present a few details of this analysis:

We start with the kinematic equation for $\mathbf{r}(x,t)$ Eq. (7.7), and identify $\mathbf{r}(x,t)$ with a position vector generating a smooth surface in E^3, with local coordinates x and t. Note that $\mathbf{S}^2 = \mathbf{r}_x^2 = 1$ in the model. From the evolution, it is readily verified that $\mathbf{r}_x.\mathbf{r}_t = 0$. We use the same notation (E.F,G,L,M,N) for the metric and the extrinsic curvature parameters, respectively, as in § 6. It is easy to verify that the metric is *necessarily* constrained to be of the geodesic form given below:

$$I = (\mathrm{d}x)^2 + G(x,t)(\mathrm{d}t)^2. \tag{7.8}$$

The unit normal $\mathbf{n}$ to the surface is given by

$$\mathbf{n} = (\mathbf{r}_x \times \mathbf{r}_t)/G^{1/2}. \tag{7.9}$$

As already mentioned in § 2, the GW equations for a surface can be written in terms of L, M, N and the usual Christoffel symbols Γ^k_{ij}. For the metric given in (7.8), the latter reduce to (see Eqs (2.8)):

$$\left.\begin{array}{l}\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0;\\ \Gamma^2_{12} = \frac{1}{2}G_x/G; \quad \Gamma^1_{22} = -\frac{1}{2}G_x \quad \text{and} \quad \Gamma^2_{22} = \frac{1}{2}G_t/G.\end{array}\right\} \tag{7.10}$$

Using these, the GW equations (equations (2.7)) read

$$\mathbf{r}_{xx} = L\mathbf{n} \tag{7.11a}$$

$$\mathbf{r}_{xt} = \tfrac{1}{2}(G_x/G)\mathbf{r}_t + M\mathbf{n} \tag{7.11b}$$

$$\mathbf{r}_{tt} = -\tfrac{1}{2}G_x\mathbf{r}_x + \tfrac{1}{2}(G_t/G)\mathbf{r}_t + N\mathbf{n} \tag{7.11c}$$

$$\mathbf{n}_x = -L\mathbf{r}_x - (M/G)\mathbf{r}_t \tag{7.11d}$$

$$\mathbf{n}_t = -M\mathbf{r}_x - (N/G)\mathbf{r}_t. \tag{7.11e}$$

Further, for this (geodesic) surface it can be shown that

$$(\mathbf{r}_x \times \mathbf{r}_{xx}) = L(\mathbf{r}_x \times \mathbf{n}) = (L/G^{1/2})\mathbf{r}_t. \tag{7.12}$$

Comparing this with the given equation (7.7), we get the constraint

$$f = -G^{1/2}/L, \tag{7.13}$$

for $L \neq 0$. In other words: for a given f, the evolution can have a solution only if the surface coefficients G and L are related through f as above.

For a surface with $E = 1$ and $F = 0$, the compatibility conditions $(\mathbf{r}_{xx})_t = (\mathbf{r}_{xt})_x$ and $(\mathbf{r}_{tt})_x = (\mathbf{r}_{xt})_t$ yield the following GMC equations:

$$\begin{aligned}-(LN - M^2)/G = {} & (\Gamma^2_{12})_x - (\Gamma^2_{11})_t + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12}\\ & + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\end{aligned} \tag{7.14a}$$

$$L_t - M_x = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \tag{7.14b}$$

$$M_t - N_x = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12} \tag{7.14c}$$

On using the expressions for Γ^k_{ij} given in (7.10), and the definition

$$K = (LN - M^2)/G, \tag{7.15}$$

of the Gaussian curvature K, Eqs (7.14) reduce to

$$-K = (G_x/2G)_x + (G_x/2G)^2 \tag{7.16a}$$

$$L_t - M_x = M(G_x/2G) \tag{7.16b}$$

$$M_t - N_x = -L(G_x/2) + M(G_t/2G) - N(G_x/2G). \tag{7.16c}$$

Now, the fundamental theorem of surfaces states that if we can identify functions G, L, M and N which satisfy the Gauss equation (7.16a) and the Mainardi–Codazzi equations (7.16b) and (7.16c), then there exists a surface $\mathbf{r}(x, t)$. In other words, an exact solution of the given evolution equation exists, provided of course if the additional constraint $f = -G^{1/2}/L$ is satisfied. We note that the Gauss equation can be written in the more convenient form,

$$(G^{1/2})_{xx} = -KG^{1/2}. \tag{7.17}$$

Details of the determination of G, L, M, N are given elsewhere [11]. The equations above can be analysed to yield

$$f = \pm\phi\ (2A_o - \phi_x^2 - C_o\phi^{-2})^{1/2}/(C_o\phi^{-3} - \phi_{xx}) \tag{7.18}$$

$$L = -\phi/f = (\phi_{xx} - C_o\phi^{-3})/(2A_o - \phi_x^2 - C_o\phi^{-2})^{1/2} \tag{7.19}$$

$$M = C_o\phi^{-1} \tag{7.20}$$

$$N = f(\phi_{xx} - C_o\phi^{-3}) = -\phi(2A_o - \phi_x^2 - C_o\phi^{-2})^{1/2}, \tag{7.21}$$

where

$$\phi = G^{1/2}. \tag{7.22}$$

These expressions satisfy the GMC equations, of course, since they have essentially been obtained by integrating the latter equations. The point to be noted is that the quantities L, M and N as well as f, depend on ϕ and its derivatives. Thus, *given* an arbitrary function ϕ (with the proviso that $(2A_o - \phi_x^2 - C_o\phi^{-2}) > 0$), we can use (7.18) to (7.21) to find these quantities explicitly. Combining this result with the fundamental theorem of surfaces, we conclude that for inhomogeneity functions f determined from (7.18) in this fashion, exact solutions $\mathbf{r}(x, t)$ of (7.7) exist. In addition, the solution q corresponding to such functions can be found explicitly. Explicit solutions for $\mathbf{r}$ can also be found for a special sub-class of f, as we shall see.

As mentioned in the beginning, the solution q of (7.3) can be expressed in terms of the curvature κ and torsion τ, the parameters of a (moving) curve, as in (7.4). Using these definitions together with the GW equations (7.11), we get

$$\kappa = (\mathbf{S}_x \cdot \mathbf{S}_x)^{1/2} = (\mathbf{r}_{xx} \cdot \mathbf{r}_{xx}) = L. \tag{7.23}$$

τ can also be expressed in terms of the surface coefficients after some algebra. We obtain

$$\tau = \mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_{xx})/\kappa^2 = \mathbf{r}_x \cdot (\mathbf{r}_{xx} \times \mathbf{r}_{xxx})/L^2 = M/G^{1/2}. \tag{7.24}$$

Substituting these expressions in the Hasimoto relation (7.4) gives the following solution for q in terms of L, M and G:

$$q = \tfrac{1}{2} L \exp i \int MG^{-1/2} \mathrm{d}x. \tag{7.25}$$

We have already shown that L and M are given by (7.19) and (7.20), for $G = G(x)$. Substituting these quantities in (7.25) and using (7.22) we get

$$q = \tfrac{1}{2}(\phi_{xx} - C_o\phi^{-3})(2A_o - \phi_x^2 - C_o\phi^{-2})^{-1/2} \exp\left[iC_o \int \phi^{-2}\mathrm{d}\right]. \tag{7.26}$$

Note that the torsion $\tau = C_o\phi^{-2}$ for this system. Summarizing, we have the following result: For any arbitrary ϕ (such that $(2A_o - \phi_x^2 - C_o\phi^{-2}) > 0$), the inhomogeneity function f can be found from Eq. (4.11), and for that f, the corresponding solution q is given by eq. (7.26).

Next, we discuss solutions for $\mathbf{r}$ in the two possible classes of surfaces corresponding, respectively, to $M = 0$ and $M \neq 0$:

(i) $M = 0$:

It is evident from (7.20) that C_o must vanish when M vanishes. Using this, (7.19) and (7.21) yield

$$L = \phi_{xx}/(2A_o - \phi_x^2)^{1/2} \quad \text{and} \quad N = -\phi(2A_o - \phi_x^2)^{1/2}. \tag{7.27}$$

It is verified easily that these extrinsic curvature coefficients correspond to the following *surface of revolution* [1]:

$$\mathbf{r} = \left(\int (2A_o - \phi_x^2)^{1/2}\mathrm{d}x,\ \phi\cos t,\ \phi\sin t\right). \tag{7.28}$$

Thus ϕ plays the role of the generator of a surface of revolution. Further, since $\mathbf{r}_x^2 = \mathbf{S} = 1$, (7.28) leads to $2A_o = 1$. By computing $\mathbf{r}_{xx}$ from it we find $L^2 = (\mathbf{r}_{xx}\cdot\mathbf{r}_{xx}) = 2A_o\phi_{xx}^2/(2A_o - \phi_x^2)$, which agrees with the expression for L given in (7.27), on setting $2A_o = 1$. (The expression for N can be verified similarly). Using $A_o = \frac{1}{2}$, $C_o = 0$, in Eq. (7.18), we get

$$f = \pm\phi(1 - \phi_x^2)^{1/2}/\phi_{xx}. \tag{7.29}$$

This expression for f was also obtained by Cieślińki *et al* [22], by integrating the equations for the geodesics on a surface of revolution. We have thus established that their result corresponds to the limit $M = C_o = 0$ of the more general expression for f given in Eq. (7.18), which we derived as the inhomogeneity function supporting an exact dynamical solution for a time-dependent metric. For $C_o = 0$, Eq. (7.26) reduces to $q = \frac{1}{2}\phi_{xx}/(2A_0 - \phi_x^2)^{1/2}$.

(ii) $M \neq 0$:

This corresponds to $C_0 \neq 0$ (see Eq. (7.20)). Since surfaces of revolutions necessarily have $M = 0$, a non-vanishing M represents some other class of surfaces. The construction of $\mathbf{r}$ in accord with (7.14)–(7.21) with $C_o \neq 0$ is non-trivial and is an open problem. However, the solution for q can be found from Eq. (7.26).

Finally, we present the solutions for $\mathbf{S}$ in terms of the Gaussian curvature K. Equation (7.17) for K can be written as

$$K = -\phi_{xx}/\phi. \tag{7.30}$$

Using this in (7.29), we obtain

$$Kf = (1 - \phi_x^2)^{1/2}. \tag{7.31}$$

Substituting (7.31) in the expression for $\mathbf{r}$ in (7.28) and computing $\mathbf{r}_x$ yields, with

$2A_o = 1$,

$$\mathbf{S} = \mathbf{r}_x = ((1-\phi_x^2)^{1/2}, \phi_x \cos t, \phi_x \sin t)$$
$$= (Kf, (1-K^2f^2)^{1/2}\cos t, (1-K^2f^2)^{1/2}\sin t). \tag{7.32}$$

Hence **S** is given as a functional of the product Kf. For the cases of a torus, a sphere and a catenoid, the corresponding solutions have been found explicitly in Ref. 11.

8. Concluding remarks

Understanding the relationship between nonlinear analysis and geometry is currently an active field of research. Many interesting results have not been considered here due to limitations of space. For instance, it has been shown [23] recently that if a non-stretching curve moves on an N-dimensional sphere in such a way that its dynamics does not depend explicitly on the radius of the sphere, the MKdV hierarchy, the NLSE hierarchy and their multi-component generalizations arise in a natural fashion, for $N = 2, N = 3$ and $N > 3$ respectively. New connections have been established using the differential geometry of constant mean curvature surfaces [24]. Another aspect which we have not discussed in this paper is that of a geometric phase associated with moving curves [19] and solitons [25]. Interesting parallels have been found between elastic dynamics of ribbons [26] and the appearance of such anholonomic phases.

Our emphasis in this paper has been on showing how physical applications can be instructive in suggesting ways of obtaining interesting mathematical connections between geometry and nonlinearity. The seminal work of Hasimoto [5] in fluid dynamics and its subsequent novel developments [27] represent a case in point.The added advantage of analysing a physical system is the possibility of observing such exact mathematical solutions (e.g., solitons) in the laboratory [28]. It is clear that the underlying nonlinearity in the system plays a crucial role also in deciding the topological character of the solutions. For example, the kink configurations of the sine-Gordon equation, which are topologically non-trivial, get replaced by (topologically) uninteresting plane wave solutions when the equation is linearized. We close with the remark that it would be of great interest to understand the connection between nonlinearity and differential geometry using physical applications in higher dimensions.

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