PRAMANA
—journal of
physics

Vol. 48, No. 1,
January 1997
pp. 143–161

Painlevé analysis and integrability aspects of some nonlinear partial differential equations

K PORSEZIAN
Institut für Theoretische Physik, Universität Hannover, 30167 Hannover, Germany and
Centre for Laser Technology, Department of Physics, Anna University, Madras 600 025, India

Abstract. A brief review of the Painlevé singularity structure analysis of some autonomous and nonautonomous nonlinear partial differential equations is discussed. We point out how the Painlevé analysis of solutions of these equations systematically provides the integrability properties of the equation. The Lax pair, Bäcklund transformation and bilinear forms are constructed from the analysis.

Keywords. Painlevé analysis; integrability; solitons; Lax pair.

PACS Nos 02.30; 42.65; 43.25; 47.20; 52.35

1. Introduction and historical background

The subject of completely integrable models is fascinating. Decades of research in this area has led to several new physical and mathematical developments which are quite beautiful and which unify various aspects of physical problems that appear to be desperate. Mathematically, we encounter new concepts such as complete integrability, Lax pair, solitons and so on and several new methods have been developed. Among them the Painlevé singularity structure analysis is one of the systematic and powerful method in nonlinear science to identify the integrability case(s) and the complete integrability properties of nonlinear systems. From the physics point of view, the models discussed in this article describe physical phenomena in such diverse areas as nonlinear optics, hydrodynamics, condensed matter, plasma physics, relativistic theory and so on. In this review article we will discuss the Painlevé analysis of various nonlinear models which are completely integrable. These models describe systems of nonlinear differential equations which can be solved exactly. Most of these models would be continuum models (both one and higher dimensions as well as autonomous and nonautonomous)although we will also investigate the Painlevé analysis of the Toda lattice and some nonintegrable systems. Before analysing the singularity structure analysis of these equations, first we will briefly discuss the historical background of Painlevé analysis.

Determining whether or not a given system of nonlinear ordinary or partial differential equations is integrable raises many fundamental issues. In particular there is the issue of what is meant by 'integrability' and the issue of how that integrability can best be determined without having to resort to a complete solution of the problem. For Hamiltonian systems the notion of integrability is well defined, i.e. the existence of as many involutive first integrals as there are degrees of freedom. For non-Hamiltonian systems things are less clearcut. Clearly the existence of integrals can lead to a reduction of the order of the system and hence to a solution in terms of an 'integration by quadratures'. However, this is clearly not the whole story since there are simple, apparently 'integrable', equations such as the Painlevé transcendents which do not have algebraic integrals and for which an integration by quadratures is not possible.It now seems that it is possible to identify many different classes of integrable systems on the basis of their analytic structure, i.e. the types of singularities exhibited by their solutions in the complex domain. The techniques for doing this can be applied to both the ordinary and partial differential equations and furthermore can be extended to provide, in many cases, explicit solutions to the systems in question. It also turns out that the complex domain of nonintegrable systems also contains much valuable information and overall it would seem that the study of analytic structure can provide a wide-ranging and unified treatment for a large class of nonlinear problems. Historically, the first application of these ideas is due to Kovalevskaya [1] in her famous work on the rigid body problem. This concerned the then popular problem of trying to find a general solution to the Euler–Poisson equations which describe the motion of a top spinning about a fixed point. Her approach was mainly to determine the conditions under which the only movable singularities, i.e. those singularities whose positions are initial-condition-dependent, exhibited by the solutions to the equations of motion, in the complex plane, are ordinary poles. She found that this only occured for four special combinations of the adjustable system parameters (the moments of inertia and the position of the centre of gravity). She was, able to identify the known integrable cases and one new one. Following Kovalevskaya came the extensive work of Painlevé [2] and co-workers to determine the categories of second order equations whose only movable singularities are ordinary poles. Working with the general class of equations

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = F\left[\frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right], \tag{1}$$

where F is analytic in x and rational in y and $\mathrm{d}y/\mathrm{d}x$, Painlevé found that there were 50 types which had the desired analytical property. Forty-four of these equations are solvable in terms of known functions. The reamining six equations, referred to as the Painlevé transcendents [3], have transcendental meromorphic solutions for which convergent expansions are explicitly known. Despite their mathematical interest it appeared, for many years, that the Painlevé transcendents were devoid of physical content. However, in the last two decades or so they have started to reappear in various important physical contexts such as the work of Wu *et al* [4] which showed that two-point correlation function of the Ising model satisfied the third Painlevé transcendent. In the last few years there is a growing very large number of literature on the Painlevé property as a means of identifying integrable cases of nonlinear ordinary, differential difference and partial differential equations. For example, this approach has been used successfully to predict integrable cases of a variety of systems such as the Lorentz equations [5], the Henon–Heiles system [6], the three particle Toda lattice [7], many systems in soliton theory [8–14], etc. It has also been observed that there are integrable systems with movable rational branch points. This has led to the concept of a so-called 'weak Painlevé property' [8].

For partial differential equations (PDE) 'complete integrability' is usually taken to mean the existence of an infinite number of conservation laws. As is well known such systems can sometimes exhibit N-soliton solutions and be solved by the inverse scattering transform(IST) method [9]. As with ODEs there is also important question of finding direct tests of complete integrability. Constructive approaches include using the method of differential forms, the use of Hirota's direct method [10] for constructing N-soliton solutions and certain criteria based on the properties of ODEs obtained by various reductions of the PDEs in question. This latter approach has been developed by Ablowitz, Ramani and Segur [11] and is based on the idea that all the ODEs obtained by similarity and travelling wave reductions from the PDE, should possess the Painlevé property. This conjecture has proved to be quite valuable, although there are clearly certain drawbacks such as the difficulty of identifying all possible reductions of the PDEs to ODEs and inability of the conjecture to provide further information about actual solutions to the equation in hand. In order to overcome these difficulties Weiss, Tabor and Carnevale [12] who have formulated a Painlevé (P-) type test that can be applied directly to PDEs without any need for reductions. This approach does seem to provide a valuable first test and, in addition, seems to be capable of yielding other important informations such as Bäcklund transformations, Lax pair, rational solutions etc. The major difference between the P-analysis of ODEs and PDEs is that the singularities of the latter are in general not isolated but lie on some analytic manifold, the singular manifold, determined by conditions of the form

$$\phi(x_1, x_2, \ldots, x_n) = 0, \tag{2}$$

where ϕ is analytic in the neighbourhood of that manifold. To show that a given equation, say

$$q_t = K(q), q = q(x,t), \tag{3}$$

possesses the generalized P-property requires a demonstration that the solutions may be expanded locally in a Laurent-like series about the singular manifold in the complex hyperspace, i.e. an expansion of the form

$$q(x,t) = \phi^{-\alpha}(x,t) \sum_{j=0}^{\infty} q_j(x,t)\phi^j(x,t), \tag{4}$$

where α is some(integer) leading order and the $q_j(x,t)$ a set of expansion coefficients analytic in the neighbourhood of the singular manifold $\phi(x,t) = 0$. Such an expansion will have certain values of j, termed 'resonances', at which the corresponding q_j should be arbitrary. One resonance always occurs at $j = -1$ and corresponds to the arbitrariness of ϕ itself and the positions of the other resonances are determined by the nonlinearities in $K(q)$.

By Cauchy–Kovalevskaya's theorem such an expansion of the general solution must have sufficient number of arbitrary functions equal to that of the order of the PDEs. Furthermore, depending on the form of $K(q)$, the equation may also have different leading orders, each one of which will have its own resonance structure. We usually call the leading order corresponding to the general solution the 'principle branch' and the others (with less arbitrary functions) as 'lower branches'. If, at a resonance (of any branch) the associated q_j fails to be arbitrary, terms of the form $\phi^j \ln \phi$ must be included in the expansion. This makes the solution multi-valued about the singular manifold and hence the P-property is lost.There are essentially four steps involved in the P-analysis of PDEs: (i) Determination of the leading-order behaviours; (ii) Identification of the powers at which arbitrary functions can enter into the Laurent series called resonances; (iii) Verifying that at each resonance values sufficient number of arbitrary functions exist without the introduction of movable manifolds; (iv) Establishing connections with the Lax pair, Bäcklund transformation (BT), bilinear form and other integrability properties.

2. Painlevé analysis of autonomous nonlinear partial differential equations

2.1 *Painlevé analysis of modified Korteweg de Vries equation*

A simple illustration is provided by the MKDV equation [12]

$$q_t - 6q^2 q_x + q_{xxx} = 0. \tag{5}$$

The leading order is easily determined to be -1 with $q_0 = \phi_x$. For finding the resonances, we substitute

$$q = \sum_{j=0} q_j \phi^{j-1}, \tag{6}$$

into (5) and equating the coefficients of ϕ^{j-4}, we get the resonance values in the form

$$j = -1, 3, 4. \tag{7}$$

The resonance $j = -1$ represents the arbitrariness of the singular manifold $\phi(x,t) = 0$. From the other powers of ϕ, one finds that

$$j = 0; \quad q_0 = \phi_x; \tag{8}$$

$$j = 1; \quad q_1 = -\frac{\phi_{xx}}{2\phi_x}; \tag{9}$$

$$j = 2; \quad \phi_t - 6q_2\phi_x^2 - \frac{3\phi_{xx}^2}{2\phi_x} + \phi_{xxx} = 0; \tag{10}$$

$j = 3; q_3$ arbitrary if

$$\frac{\partial}{\partial x}\left(\phi_t - 6q_2\phi_x^2 - \frac{3\phi_{xx}^2}{2\phi_x} + \phi_{xxx}\right); \tag{11}$$

$j = 4; q_4$ arbitrary. Clearly the 'compatibility condition' at $j = 3$ is always satisfied by virtue of the relations found at $j = 2$. Thus the expansion (6) is valid and we can say that the MKDV equation passes the P-test. To simplify the computations one can use a modification of this method proposed by Kruskal [13]. This consists of expanding ϕ about the singular manifold in the form $\phi(x,t) = x + \psi'(t)$, such that $\phi(\psi'(x,t)) = 0$, and setting $q_j = q_j(t)$. For the MKDV equation this reduced approach yields $q_0 = 1, q_1 = 0, q_2 = \psi_t/6, q_3$ arbitrary, q_4 arbitrary. From the point of view of the 'test' alone, this modification is rather useful. However, to construct the integrability properties from the Laurent series and to analyse the P-property of the inhomogeneous, mainly spatial dependent, we have to consider the general manifold expansion. Similarly the KDV equation

$$q_t + 12qq_x + q_{xxx} = 0 \tag{12}$$

is easily shown to pass the Painlevé test [12–15]. The leading order is -2 and the resonances occur at $j = -1, 4, 6$. The truncated expansion then takes the form

$$q = \frac{\partial^2}{\partial x^2} \log \phi + q_2 \tag{13}$$

with

$$\phi_x \phi_t + 12 q_2 \phi_x^2 + 4\phi_x \phi_{xxx} - 3\phi_{xx}^2 = 0, \quad \phi_{xt} + 12 q_2 \phi_{xx} + \phi_{xxxx} = 0, \tag{14}$$

and

$$q_{2,t} + 12 q_2 q_{2,x} + q_{2,xxx} = 0. \tag{15}$$

The set of equations (14) and (15) constitute an auto BT for the KDV equation. It should be noted that the second logarithmic derivative in (13) is reminiscent of the transformation used in Hirota's method [10]. By making the substitution $\phi_x = \psi^2$ leads to

$$\psi_{xx} + (2q_2 + \lambda)\psi = 0, \quad \psi_t + (6q_2 + \lambda)\psi_x + \psi_{xxx} = 0, \tag{16}$$

which are precisely the Lax pair for the KDV equation [9]. One of the most enduring techniques in the study of integrable nonlinear PDEs is Hirota method for constructing N-soliton solutions [10]. The method proceeds without the knowledge of the IST and in some cases was used to construct N soliton solutions before the scattering transform had been found [10]. To get the bilinear transformation from the P- analysis, we have to assume that the constant term in the expansion is equal to zero. i.e. in the case of KDV, $q_2 = 0$. Then, we get

$$f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2 + f f_{xt} - f_x f_t = 0 \tag{17}$$

which can be written in terms of Hirota's bilinear operators as

$$(D_x^4 + D_x D_t) f \cdot f = 0, \tag{18}$$

where $D_x^n(f \cdot f) = (\partial_x - \partial_{x'})^n f(x) f(x') \mid x = x'$. As is well known f can be developed in a series expansion which self truncates at each order yielding $1, 2, 3, \ldots$, N-soliton solutions. It is generally believed that if one can construct N-soliton solutions for $N \geq 3$, then they will exist for all N and the system can be deemed integrable. It should be noted that many equations can be reduced to bilinear form and one can often construct one and two soliton solutions for them by this method. However, self-truncation for all N only seems to occur only for completely integrable systems.

2.2 *Painlevé analysis of Pohlmeyer–Lund–Regge equation*

Lund *et al* [16] have shown that the dynamics of relativistic vortices (equivalently, strings) interacting through a scalar field has led to a set of two coupled, Lorentz-invariant, nonlinear equations in two independent variables. The equations are in the form

$$\Phi_{xt} - \sin\Phi\cos\Phi - \frac{sin\Phi}{\cos^3\Phi}\Psi_x\Psi_t = 0, \tag{19}$$

$$(\Psi_t \tan^2\Phi)_x + (\Psi_x \tan^2\Psi)_t = 0. \tag{20}$$

Eqs. (19–20) are conditions for embedding a 2-dimensional surface in a three dimensional sphere which itself embedded in a four dimensional Euclidean space. Eqs. (19) and (20) were also derived by Pohlmeyer [17] through a study of the nonlinear σ- models of field theory.

For convenience, we use the transformation given by Getmanov [18] in the form

$$q = \sin\Phi\exp(i\Psi). \tag{21}$$

Under this transformation, eqs. (19) and (20) are transformed into

$$q_{xt} + \frac{q_x q_t q^*}{1-\mid q\mid^2} - q(1-\mid q\mid^2) = 0. \tag{22}$$

To apply P-analysis, we define $q = a$ and $q^* = b$. Eq. (22) in terms of these change of variables takes the form [19]

$$a_{xt} + \frac{a_x a_t b}{1-ab} - a(1-ab) = 0, \quad b_{xt} + \frac{b_x b_t a}{1-ab} - b(1-ab) = 0. \tag{23}$$

Leading order: $\alpha = \beta = -1$ with $a_0 b_0 = -\psi_t$. Resonances: $j = -1, 0, 1, 2$. Collecting the coefficients of ϕ^{-4} and ϕ^{-4}, we obtain

$$a_0 b_1 + a_1 b_0 = 0. \tag{24}$$

From eq. (24) it is clear that either a_1 or b_1 is arbitrary. In a similar way proceeding further by collecting the coefficients of ϕ^{-3} and ϕ^{-3}, we get

$$(a_0 b_2 - b_0 a_2)\psi_t = a_{0t} b_1 + b_0 a_{1t}, (b_0 a_2 - a_0 b_2)\psi_t = b_{0t} a_1 + a_0 b_{1t}. \tag{25}$$

Using (24), it is easy to show that the functions a_2 or b_2 is arbitrary. Thus the general solution $(a(x,t), b(x,t))$ of eq. (22) admits the sufficient number of arbitrary functions without the introduction of any movable critical manifold, thus satisfying the P- property and hence the system is expected to be integrable. As eq. (22) satisfies the required condition to be integrable, we now proceed to obtain the associated integrability properties of the system. Now to establish the integrability properties, the series representation is truncated at the constant level term $(a_j = b_j = 0, j > 1)$ as

$$a = \frac{a_0}{\phi} + a_1, \quad b = \frac{b_0}{\phi} + b_1, \tag{26}$$

where a_0 and b_0 satisfy eqs (23). Equation (26) can also be treated as an auto BT of eq. (22). Then to find the bilinear form, we put $a_1 = b_1 = 0$ and then by defining

$$a = \frac{a_0}{\phi} = \frac{G}{F}, \quad b = \frac{b_0}{\phi} = \frac{G^*}{F} \tag{27}$$

after some manipulations, we obtain

$$D_xD_tF \cdot F = 2GG^*, F[(D_xD_t - 1)G \cdot F] = \tfrac{1}{2}G^*D_xD_tG \cdot G. \tag{28}$$

The above trilinear form was first obtained by Getmanov [18] and found 2 soliton solution to it and also conjectured N soliton solutions. To construct the bilinear form from (28), we define

$$F^2 = f^*f + g^*g. \tag{29}$$

Under this transformation eqs (28) are transformed into the bilinear equations

$$D_x(f^* \cdot f + g^* \cdot g) = 0, (D_xD_t - 1)f \cdot g^* = 0, D_xD_t(f^* \cdot f - g^* \cdot g) + 2g^*g = 0. \tag{30}$$

Once the bilinear forms are known, then one can generate the soliton solutions by expanding the dependent variables in terms of power series.

2.3 *Some nontrivial equations and Painlevé condition(s)*

It would seem that all the known integrable PDEs pass the Painlevé test. Some nontrivial examples include

I. Hirota–Satsuma equation [14]

$$q_t - \lambda(6qq_x + q_{xxx}) = -6pp_x, \quad p_t + 3qp_x + p_{xxx} = 0 \tag{31}$$

which is found to pass the P-test only when $\lambda = \frac{1}{2}$.

II. Unidirectional Zakharov equation [20]

$$iq_t + q_{xx} = qp, \quad p_t + p_x = (|\, q \,|^2)_x. \tag{32}$$

III. Nonlinear Schrödinger-Maxwell Bloch equations [21]

$$q_x = ikq_{tt} - ig \mid q \mid^2 q + 2\alpha_2\langle p\rangle, \quad p_t = i\omega p + fq\eta, \quad \eta_t = 2f(qp^* + q^*p). \tag{33}$$

Painlevé condition: $-2f^2k = g$.

IV. Higher order nonlinear Schrödinger equation [22]

$$iq_z + \alpha_1 q_{tt} + \alpha_2 \mid q \mid^2 q - i\varepsilon[\alpha_3 q_{ttt} + \alpha_4(q^2q^*)_t + \alpha_5 q(qq^*)_t] = 0. \tag{34}$$

Painlevé conditions: $\alpha_2 = 2\alpha_1, \alpha_3 : \alpha_4 : \alpha_5 = 1 : 6 : -3$.

2.4 *Non-Painlevé property of the integrable extended nonlinear Schrödinger equation*

Recently Liu *et al* [23] have shown that the extended nonlinear Schrödinger equation is a completely integrable system and constructed the N-soliton solutions through Hirota's bilinear method. The equation is of the form

$$iq_x + \frac{1}{2}q_{tt} + \frac{\mid q \mid^2 q}{2} - i\delta q_{ttt} + 2i\alpha_1 \mid q \mid^2 q_t + i\alpha_1 \mid q \mid^2 q_t^* = 0. \tag{35}$$

To apply P-analysis, we substitute $q = a$ and $q^* = b$ in (35), then the resulting equations are [24]

$$ia_t + \tfrac{1}{2}a_{xx} + \tfrac{1}{2}a^2b + i[-\delta a_{xxx} + 2\alpha_1 aba_x + \alpha_1 a^2 b_x] = 0, \tag{36}$$

$$-ib_t + \tfrac{1}{2}b_{xx} + \tfrac{1}{2}ab^2 - i[-\delta b_{xxx} + 2\alpha_1 abb_x + \alpha_1 b^2 a_x] = 0. \tag{37}$$

Leading Order: $\alpha = \beta = -1$; $a_0 b_0 = \frac{2\delta}{\alpha_1}$. Resonances: $j = -1, 0, 3, 3, 3, 4$. The resonance '-1' corresponds as usual to the arbitrariness of the singular manifold and '0' corresponds to the fact that either a_0 or b_0 is arbitrary as seen from the leading order coefficient. As the system admits more resonances at $j = 3, 3, 3$ than the required arbitrary functions, we conclude that eq. (35) fails to satisfy the Painlevé test.

In the cases considered so far reveals that if the P-property is satisfied, then the equation is completely integrable. Thus it is conjectured that this property indicates the solvability of a field equation. On the other hand we cannot conclude, in general, that a PDE which is completely integrable has the P-property. Examples are the Harry Dym equation $q_t = q^3 q_{xxx}$ and the nonlinear diffusion equation $q_t = (q^{-2}q_x)_x$ [14]. Both equations are integrable. The first can be solved by the IST and the second can be linearized to the linear diffusion equation.

2.5 *Painlevé analysis of nonintegrable equations*

Painlevé analysis is also very useful to understand the dynamics of the nonintegrable evolution equations. For example, for the real Newell–Whitehead equation [25]

$$q_t = q_{xx} + q - 2q^3 \tag{38}$$

the leading order is -1 and the resonances are -1 and 4. For simplicity, we define new variables $w = \phi_t/\phi_x$, $v = \phi_{xx}/\phi_x$ and from the arbitrary analysis, we find

$$j = 0 : q_0 = \phi_x, \tag{39}$$

$$j = 1 : q_1 = \tfrac{1}{2}(w - 3v), \tag{40}$$

$$j = 2 : q_2 = \frac{1}{6\phi_x}(v_x - w_x + 1 - \tfrac{1}{6}w^2 - \tfrac{1}{2}v^2), \tag{41}$$

$$j = 3 : q_3 = \frac{1}{24\phi_x^2}(-w_t + 2w_{xx} + \tfrac{7}{3}ww_x - 2w + \tfrac{4}{9}w^3 - v_{xx} + 3vv_x + 2v - v^3 - 2vw_x - \tfrac{1}{3}w^2 v), \tag{42}$$

$$j = 4 : 0(q_4) = \frac{1}{3\phi_x^3}(w_t - ww_x + \tfrac{3}{2}w - \tfrac{1}{3}w^3). \tag{43}$$

From eq. (43), it is clear that the Laurent expansion fails the WTC test since q_4 is found not to be arbitrary. We therefore have the following "consistency condition" (assuming $\phi_x \neq 0$):

$$w_t - ww_x + \tfrac{3}{2}w - \tfrac{1}{3}w^3 = 0. \tag{44}$$

There are two possible ways to proceed: (i) either, introduce logarithmic psi-series, i.e.

$$q = \sum_j \sum_k q_{jk} \phi^{j-1} (\phi^4 \ln \phi)^k. \tag{45}$$

(ii) or, force ϕ to satisfy the compatibility condition at $j = 4$. This takes the form of a nonlinear PDE which can be solved exactly.This approach gives a type of conditional P-property and can yield special type of solutions meromorphic in the singular manifold which is now specified as opposed to being arbitrary. In view of the above facts, we can conclude that the real Newell–Whitehead equation has the "conditional Painlevé property". It is also interesting to note that some of the special techniques developed for integrable systems are also shown to be applicable to nonintegrable equations. Let us consider the second example from one parameter continuous (Lie) group of transformations point of view [26]. For this, we consider the nonintegrable KdV equation in the form

$$q_t + q^n q_x + q_{xxx} = 0, \quad n > 2. \tag{46}$$

Using the invariant variables

$$\zeta = \frac{(x + \frac{3\beta}{\alpha})}{(\alpha + \beta)^{1/3}} \tag{47}$$

and

$$q = \frac{f(\zeta)}{(\alpha t + \delta)^{2/3n}} \tag{48}$$

in eq. (46), we obtain the invariant equation in the form

$$f''' + f^n f' - \left(\frac{2f}{n} + \zeta f'\right) = 0 \tag{49}$$

Inserting

$$f \sim \alpha(\zeta - \zeta_0)^p \tag{50}$$

in eq. (49), we find that there is only one possibility

$$p = \frac{-2}{n}, \; \alpha = \left(\frac{-(2+n)(2+2n)}{n^2}\right)^{1/n}. \tag{51}$$

Since $n > 2$, eq. (49) will have a movable branch point of order $-2/n$ provided (50) is asymptotic near ζ_0. To see this asymptotic nature, we define

$$f = v^{-2/n} \tag{52}$$

The equation for v is

$$-\frac{2}{n} v^2 v''' + \frac{6}{n}\left(\frac{2}{n} + 1\right) v v' v'' - \frac{2}{n}\left(\frac{2}{n} + 1\right)\left(\frac{2}{n} + 2\right) v'^3 - \frac{2}{n} v' - \frac{2}{n} v^3 + \frac{2\zeta}{n} v^2 v' = 0 \tag{53}$$

There is a regular solution of (53), that is, regular at ζ_0, if $v(\zeta_0)=0, v'(\zeta_0)+(2+n)(2+2n)/n^2\ (v'(\zeta_0))^3=0$. $v''(\zeta_0)$ is a finite quantity,and $v'''(\zeta_0)$ is finite. Then $v(\zeta)$ is analytic at ζ_0 and so (50) is asymptotic near ζ_0. Thus eq. (49) is not of Painlevé type. For more information about the P-analysis of nonintegrable systems, we refer the reader to refs. [25, 26].

3. Painlevé property for differential-difference equations

In this section, we will briefly discuss the Painlevé analysis of the discrete systems. The Toda lattice [27]

$$Q_{n,tt} = e^{Q_{n-1}-Q_n} - e^{Q_n-Q_{n+1}} \tag{54}$$

is probably the best known member of a class of differential-difference equations that are integrable. To establish the P-property [27], we define the variables $q_n = Q_{n,t}$, $p_n = e^{Q_n-Q_{n+1}}$, i.e.

$$p_{n,t} = p_n(q_n - q_{n+1}), \quad q_{n,t} = p_{n-1} - p_n. \tag{55}$$

A solution of these equations was found by Toda on making the substitution

$$p_n = \frac{\partial^2}{\partial t^2} \log f_n \tag{56}$$

which immediately leads to

$$q_n = \frac{\partial}{\partial t} \log\left(\frac{f_{n-1}}{f_n}\right) \tag{57}$$

and

$$\left[\frac{f_{n+1}f_{n-1}}{f_n^2} - 1\right] = \frac{\partial^2}{\partial t^2} \log f_n. \tag{58}$$

The one soliton solution corresponds to the choice f_n $\cosh(\alpha n - (\sinh\alpha)t)$. These results suggests a truncated 'Painlevé type' expansion of the form

$$p_n = \frac{p_n^{(0)}}{\phi_n^2} + \frac{p_n^{(1)}}{\phi_n} + p_n^{(2)}, \quad q_n = \frac{q_{n-1}^{(0)}}{\phi_{n-1}} - \frac{q_n^{(0)}}{\phi_n} + q_n^{(1)} \tag{59}$$

from which one can easily determine that $q_n^{(0)} = \phi_{n,t}, p_n^{(0)} = -\phi_{n,t}^2$ and $p_n^{(1)} = \phi_{n,tt}$, i.e.

$$p_n = \frac{\partial^2}{\partial t^2} \log \phi_n + p_n^{(2)}, \quad q_n = \frac{\partial}{\partial t} \log\left(\frac{\phi_{n-1}}{\phi_n}\right) + q_n^{(1)} \tag{60}$$

and that

$$p_n^{(2)}\left[\frac{\phi_{n+1}\phi_{n-1}}{\phi_n^2} - 1\right] = \frac{\partial^2}{\partial t^2} \log \phi_n. \tag{61}$$

Thus, in a sense we are expanding each dependent variable p_n about its own 'singular manifold' function ϕ_n. The same approach for the two dimensional Toda lattice,

$$p_{n,x} = p_n(q_n - q_{n+1}), \quad q_{n,t} = p_{n-1} - p_n. \tag{62}$$

yields

$$p_n = \frac{\partial^2}{\partial t \partial x} \log \phi_n + p_n^{(2)} \tag{63}$$

which immediately leads to

$$q_n = \frac{\partial}{\partial x} \log\left(\frac{\phi_{n-1}}{\phi_n}\right) + q_n^{(1)} \tag{64}$$

and

$$p_n^{(2)} \left[\frac{\phi_{n+1}\phi_{n-1}}{\phi_n^2} - 1\right] = \frac{\partial^2}{\partial t \partial x} \log \phi_n. \tag{65}$$

From the relations (63) and (64) one can easily determine Lax pairs for the corresponding systems. For example, in the case of the 2-D lattice the scattering scheme is found to be

$$\phi_{n,x} = \alpha_{n+1}\phi_n + \beta_n \phi_{n+1}, \quad \phi_{n,t} = \gamma_n \phi_{n-1} + \delta_{n-1}\phi_n \tag{66}$$

and on substituting into (65), combined with cross differentiation of (66), deduce that $\beta_n = \lambda$ (spectral parameter), $\alpha_n = -q_n^{(1)}$ and $\gamma_n = \frac{p_n^{(2)}}{\lambda}$. This type of P-analysis also yields the BT's for the Toda lattices and reveals some amusing connections with the rational solutions of the second Painlevé transcendent [27].

4. Painlevé analysis of nonautonomous nonlinear partial differential equations

The problem of nonlinear wave propagation in dispersive and inhomogeneous media has been of general interest and has had wide range of applications, e.g. radio waves in the ionosphere, waves in the ocean,nonlinear optics, magnetic systems, etc, [31–35]. For the past few years, several inhomogeneous nonlinear PDEs have been studied from the Painlevé property and soliton point of view. In a series of papers, we have systematically investigated the Painlevé analysis of many of these equations and explained the construction of soliton solutions [35–40]. In this section, we will briefly discuss the P-analysis of some nonautonomous equations. As pointed out in section I, to identify the Painlevé condition(s) of these systems, we have to use the general manifold.

4.1 *Painlevé analysis of inhomogeneous spherically symmetric nonlinear Schrödinger equation*

First, we will discuss the P-analysis of the inhomogeneous spherically symmetric nonlinear Schrödinger equation of the form [35]

$$\begin{aligned} iq_t + (fq)_{rr} + \left(f\frac{n-1}{r}q\right)_r + 2f \mid q \mid^2 q \\ + \left[2\int_0^r f_{r'} \mid q \mid^2 \mathrm{d}r' + 4(n-1)\int_0^r \frac{f}{r'} \mid q \mid^2 \mathrm{d}r'\right] q = 0. \end{aligned} \tag{67}$$

The above equation can be derived from the inhomogeneous spherically symmetric continuum Heisenberg ferromagnet in arbitrary $(n-)$ dimensions through moving helical space curve [35]. Equation (67) contains the following integrable models:

(i) $n=0, f=$ constant: nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2 \mid q \mid^2 q = 0(-\infty < x < \infty). \tag{68}$$

(ii) $n=1, f=\alpha x+\beta$: deformed nonlinear Schrödinger equation

$$iq_t + (\alpha x+\beta)(q_{xx} + 2 \mid q \mid^2 q) + 2\alpha\left(q_x + q\int_0^x \mid q \mid^2 \mathrm{d}x'\right) = 0. \tag{69}$$

(iii) $n=2, f=$ constant : deformed radial nonlinear Schrödinger equation

$$iq_t + q_{rr} + \frac{1}{r}q_r - \frac{1}{r^2}q + 2 \mid q \mid^2 q + 4q\int_0^r \frac{1}{r} \mid q \mid^2 dr' = 0. \tag{70}$$

After removing the integral through new variable R and by denoting $q=a$ and $q^*=b$, eq. (67) can be written as

$$ia_t + f\left(a_{rr} + \frac{n-1}{r}a_r - \frac{n-1}{r^2}a\right) + 2Ra + 2f_r a_r + \left[f_{rr} + \frac{n-1}{r}f_r\right]a = 0, \tag{71}$$

$$-ib_t + f\left(b_{rr} + \frac{n-1}{r}b_r - \frac{n-1}{r^2}b\right) + 2Rb + 2f_r b_r + \left[f_{rr} + \frac{n-1}{r}f_r\right]b = 0, \tag{72}$$

$$R_r - 2f_r ab - f\left[ab_r + a_r b + 2\frac{n-1}{r}ab\right] = 0. \tag{73}$$

Leading order: $\alpha=\beta=-1,\ \gamma=-2,\ a_0b_0=\phi_r^2,\ R_0=-f\phi_r^2$. Resonances: $j=-1$, $0,2,3,4$. From the detailed arbitrary analysis we find that eqs. (71–73) are free from movable critical manifolds only when f is of the form:

$$f = \varepsilon_1 r^{-2(n-1)} + \varepsilon_2 r^{-(n-2)}, \tag{74}$$

where ε_1 and ε_2 are integration constants.Hence, the inhomogeneous spherically symmetric nonlinear Schrödinger equation is expected to be integrable in arbitrary $(n-)$ dimensions only when the inhomogeneity is of the form (74). Working with the truncated expansions

$$a = \frac{a_0}{\phi} + a_1, \quad b = \frac{b_0}{\phi} + b_1, \quad R = \frac{R_0}{\phi^2} + \frac{R_1}{\phi} + R_2, \tag{75}$$

the following overdetermined systems of equations are obtained: $O(\phi^{-3}, \phi^{-3}, \phi^{-3})$:

$$a_0 b_0 = \phi_r^2, \quad R_0 = -f\phi_r^2. \tag{76}$$

$O(\phi^{-2}, \phi^{-2}, \phi^{-2})$:

$$-ia_0\phi_t - \left(2f_r + \frac{n-1}{r}f\right)a_0\phi_r - f(2a_{0r}\phi_r + a_0\phi_{rr}) + 2(a_0R_1 + R_0a_1) = 0, \tag{77}$$

$$ib_0\phi_t - \left(2f_r + \frac{n-1}{r}f\right)b_0\phi_r - f(2b_{0r}\phi_r + b_0\phi_{rr}) + 2(b_0R_1 + R_0b_1) = 0, \tag{78}$$

$$R_{0r} - R_1\phi_r - 2f_ra_0b_0 - f[(a_0b_0)_r - (a_0b_1 + a_1b_0)\phi_r] - 2\,\frac{n-1}{r}fa_0\,b_0 = 0. \tag{79}$$

$O(\phi^{-1}, \phi^{-1}, \phi^{-1})$:

$$ia_{0t} + (fa)_{rr} + \left(f\,\frac{n-1}{r}a\right)_r + 2(a_0R_2 + a_1R_1) = 0, \tag{80}$$

$$-ib_{0t} + (f\,b)_{rr} + \left(f\,\frac{n-1}{r}b\right)_r + 2(b_0R_2 + b_1R_1) = 0, \tag{81}$$

$$R_{1r} - 2f_r(a_0b_1 + a_1b_0) + f(a_0b_1 + a_1b_0)_r - 2\,\frac{n-1}{r}f(a_0b_1 + a_1 + b_0) = 0. \tag{82}$$

By noting that $(a_0\,b_{0t} + a_{0t}\,b_0) = -2\phi_r\phi_{rt}$ and using the expression for ϕ_t from eqs. (77–78) one may, after some simplifications, obtain

$$\left[\frac{a_{0r} + 2a_1\phi_r}{r^{n-1}a_0}\right]_r = \left[\frac{b_{0r} - 2b_1\phi_r}{r^{n-1}b_0}\right]_r. \tag{83}$$

Treating the constant of integration as the spectral parameter λ and identifying $a_0 = i\psi_1^2$, $b_0 = i\psi_2^2$, $a_1 = q$, $b_1 = q^*$, and $\phi_r = -i\psi_1\psi_2$, gives the scattering problem

$$\psi_{1r} = i\lambda r^{n-1}\psi_1 + q\psi_2, \quad \psi_{2r} = -q^*\psi_1 - i\lambda r^{n-1}\psi_2. \tag{84}$$

The time dependent part of the problem may be obtained directly from equations (80–82) by repeated use of (83) and (84), giving

$$\psi_{1t} = (iR - 2i\lambda^2 f\,r^{2(n-1)})\psi_1 + \left(-2\lambda f\,qr^{n-1} + i(fq)_r + i\frac{n-1}{r}fq\right)\psi_2, \tag{85}$$

$$\psi_{2t} = \left(2\lambda fq^*r^{n-1} + i(fq^*)_r + i\frac{n-1}{r}fq^*\right)\psi_1 + (-iR + 2i\lambda^2 fr^{2(n-1)})\psi_2. \tag{86}$$

Here f is given in eq. (74) and the consistency of the linear eigenvalue problem (84) and (85–86) is indeed the evolution equation (67) (only for the choice of f given in eq. (74)), provided λ evolves as

$$\lambda_t = (\varrho + i\xi)_t = 2n\varepsilon_2\lambda^2. \tag{87}$$

4.2 *Painlevé analysis of the generalized x-dependent MKDV equation*

We consider the GMKDV equation in the form [31]

$$q_t+\mu_1 q+(\nu_3+\mu_3 x)(q_{xxx}-6q^2q_x) + (\mu_1 x+\nu_1)q_x+\mu_3\left(3q_{xx}+4q^3-2q_x\int_x^{\infty}\mathrm{d}x' q^2\right)=0 \quad (88)$$

In this equation ν_1, ν_3, μ_1 and μ_3 are real constants and q is a real. For $\mu_1=\mu_3=0$, eq. (88) reduces to the well known MKDV equation. If instead $\mu_3\neq 0$, eq. (88) is an integro differential equation; it can be reduced to a pure differential equation by changing the dependent variable from q to $\int_x^{\infty} q^2(x',t)\mathrm{d}x'$ or differentiating with respect to x after dividing by q_x. However, we can remove the integral term by defining a new dependent variable $R=\int_x^{\infty} q^2(x',t)\mathrm{d}x'$. Now eq. (88) can be rewritten as a set of coupled equations in the form [40]

$$q_t+\mu_1 q+(\nu_3+\mu_3 x)(q_{xxx}-6R_x q_x)+(\mu_1 x+\nu_1)q_x + \mu_3(3q_{xx}+4qR_x-2q_x R)=0, \quad (89)$$

$$R_x=q^2 \quad (90)$$

to within an arbitrary function of t (which occurs when (90) is integrated),which can always be removed by a simple time dependent gauge transformation of q. Leading order: $\alpha=\beta=-1$ with $q_0^2=-\phi_x^2$, $R_0=\phi_x$.

Resonances: $j=-1,1,3,4$. From the truncated series, we get

$$q=\frac{q_0}{\phi}+q_1, \quad R=\frac{R_0}{\phi}+R_1. \quad (91)$$

With suitable transformation for q_0 and q_1 [15, 35, 40], after some simplifications, we obtain

$$\psi_{1x}=-i\lambda\psi_1+q\psi_2, \quad \psi_{2x}=q\psi_1+i\lambda\psi_2, \quad (92)$$

$$\psi_{1t}=A\psi_1+B\psi_2, \quad \psi_{2t}=C\psi_1+A\psi_2, \quad (93)$$

where $A=-4i\lambda^3(\mu_3 x+\nu_3)-2i\lambda[\mu_3(x\int q^2\mathrm{d}x')_x+\nu_3 q^2]+i(\nu_1+\mu_1 x)\lambda$, $B=4\lambda^2(\mu_3 x+\nu_3)q+2i\lambda[\mu_3(xq)_x+\nu_3 q_x]+\mu_3[-(xq)_{xx}+2q(x\int q^2\mathrm{d}x')_x]+\nu_3[-q_{xx}+2q^3]-(\nu_1+\mu_1 x)q$, $C=4\lambda^2(\mu_3 x+\nu_3)q-2i\lambda[\mu_3(xq)_x+\nu_3 q_x]+\mu_3[-(xq)_{xx}+2q(x\int q^2\mathrm{d}x')_x]+\nu_3[-q_{xx}+2q^3]-(\nu_1+\mu_1 x)q$ with $\lambda_t=-\mu_1\lambda-4\mu_3\lambda^3$.

4.3 *Painlevé analysis of inhomogeneous deformed Kaup system*

The deformed Kaup system is in the form [39]

$$q_t+\tfrac{1}{2}q^2+xqq_x+x\eta_x=0, \quad \eta_t+2\eta q+x\eta_x q+x\eta q_x+3q_{xx}+xq_{xxx}=0 \quad (94)$$

Leading order: $\alpha=-1$, $\beta=-2$, $q_0=\pm 2\phi_x$, $\eta_0=-2\phi_x^2$. Resonances: $j=-1,2,3,4$. Lax pair:

$$\psi_{xx}+U\psi=0, \quad \psi_t=A\psi_x+B\psi \quad (95)$$

with

$$U = \lambda^2 + \frac{i\lambda q}{2} + \frac{\eta}{4} - \frac{q^2}{16}, \quad A = -2ix\lambda - \frac{xq}{2}, \quad B = -\frac{A_x}{2} \tag{96}$$

The compatibility leads to eq. (94) only when λ satisfies

$$\lambda_t = -2i\lambda^2. \tag{97}$$

4.4 *Painlevé analysis of parametrically driven Sine–Gordon equation*

We consider the generalized sine-Gordon equation [41]

$$q_{tt} - q_{xx} + f(t)\sin q = 0. \tag{98}$$

It is well known that the sine-Gordon equation $q_{tt} - q_{xx} + \sin q = 0$, which is completely integrable, does not pass the Painlevé test directly, but after the transformation $V = e^{iq}$ [13, 41]. Under this transformation, eq. (98) takes the form.

$$VV_{tt} - V_t^2 - VV_{xx} + V_x^2 + \tfrac{1}{2}f(t)(V^3 - V) = 0. \tag{99}$$

Leading order: $\alpha = -2$, $V_0 = 4(\phi_x^2 - \phi_t^2)/f$

Resonances: $j = -1, 2$. At the resonance $j = 2$ we obtain the ODE

$$f\frac{d^2f}{dt^2} = \left(\frac{df}{dt}\right)^2. \tag{100}$$

As expected this ODE has the P-property. The general solution is given by

$$f(t) = C_1 e^{C_2 t}, \tag{101}$$

where C_1 and C_2 are integration constants.Similar type of equation with damping was also investigated in [42].

4.5 *Some inhomogeneous equations and Painlevé condition(s)*

I. Generalized KDV equation: [43]

$$q_t + f(t)qq_x + g(t)q_{xxx} = 0. \tag{102}$$

Painlevé condition: $g(t) = f(t)(a_0 \int^t \mathrm{d}s f(s) + b_0)$, a_0 and b_0 are arbitrary constants.

II. Generalized KDV equation [44]:

$$q_t + qq_x + q_{xxx} + a(x,t)q = b(x,t). \tag{103}$$

Painlevé conditions: $a_x = 0, a_t + 2a^2 - b_x = 0$.

III. Cylinderical KDV equation [45]

$$q_t + qq_x + q_{xxx} + \frac{aq}{t} = 0. \tag{104}$$

Painlevé condition: $a = \frac{1}{2}$.

IV. Generalized nonlinear Schrödinger equation [43]

$$iq_t + \alpha(t)q_{xx} + \beta(t) \mid q \mid^2 q = 0. \tag{105}$$

Painlevé condition: $\alpha(t) = \beta(t)(a_1 \int^t \mathrm{d}s f(s) + b_1)$,$a_1$ and b_1 are arbitrary constants.

V. Damped Nonlinear Schrödinger equation [46]

$$iq_t + q_{xx} - \mid q \mid^2 q = a(x,t)q + b(x,t). \tag{106}$$

Painlevé conditions: $a = x^2(\beta_t/2 - \beta^2) + x\alpha_1 + \alpha_0 + i\beta, b = 0, \alpha_0(t), \alpha_1(t)$ and $\beta(t)$ are arbitrary real analytic functions.

VI. Cylinderical Nonlinear Schrödinger equation [47]

$$iq_t + q_{xx} + \mid q \mid^2 q = \frac{q}{2t}. \tag{107}$$

VII. Inhomogeneous nonlinear Schrödinger equation [38]

$$iq_t + (f\,q)_{xx} + 2f \mid q \mid^2 q + \left[2\int_0^x f_{x'} \mid q \mid^2 \mathrm{d}x'\right] q = 0 \tag{108}$$

Painlevé condition: $f = \alpha x + \beta$.

VIII. Generalized inhomogeneous nonlinear Schrödinger equation [37]

$$iq_t + (\alpha x + \beta)(q_{xx} + 2 \mid q \mid^2 q) + 2\alpha\left(q_x + q\int_0^x \mid q \mid^2 \mathrm{d}x'\right) + i\mu_1(xq)_x + i\nu(q_{xxx} + 6 \mid q \mid^2 q_x) = 0 \tag{109}$$

IX. Radially symmetric nonlinear Schrödinger equation [38]:

$$iq_t + q_{rr} + \left(\frac{n-1}{r}q\right)_r + 2 \mid q \mid^2 q + 4(n-1)q\int_0^r r' \mid q \mid^2 \mathrm{d}r' = 0 \tag{110}$$

Painlevé conditions: $n = 1, 2$.

X. Inhomogeneous Nonlinear Schrödinger–Maxwell Bloch equations [36]

$$q_x = ikq_{tt} - ig \mid q \mid^2 q + \alpha_2(x,t)q + \langle p \rangle, \quad p_t = i\omega p + fq\eta, \quad \eta_t = 2f(qp^* + q^*p). \tag{111}$$

Painlevé conditions: $-2f^2k = g$, $\alpha_2 = 1/2(x + x_0)$.

5. Painlevé analysis of some higher dimensional equations

5.1 *Painlevé analysis of Kodomtsev–Petviashvili equation*

The KP equation [12, 14]

$$q_{tx} + q_x^2 + qq_{xx} + \sigma q_{xxxx} + q_{yy} = 0, \quad \sigma = \pm 1 \tag{112}$$

possesses the P-property. Leading order: -2 with $q_0 = -2\phi_x^2$ Resonances: $-1, 4, 5, 6$. Then equating the various powers of ϕ to zero we obtain sets of equations and one can easily verify that the functions q_4, q_5 and q_6 are arbitrary and so the P-property holds.

From the truncated expansion, the Lax pair is found to be in the form

$$\sigma\psi_y + \psi_{yy} + q\psi = 0,\ \psi_t + 4\psi_{xxx} + 6q\psi_x + 3\left(q_x - \sigma\int_{\infty}^{x} q_y \mathrm{d}x'\right) = 0 \quad (113)$$

the compatibility of which is the KP eq. (112). The Painlevé analysis of similar type of equations is also discussed in [48].

5.2 *Painlevé analysis of cylinderical Kadomtsev–Petviashvili equation*

The cylinderical Kadomtsev–Petviashvili equation is of the form [49]

$$q_{xt} + qq_{xx} + q_x^2 + q_{xxxx} + a(t)q_x + b(t)q_{yy} = 0 \quad (114)$$

Painlevé conditions: $da/dt + 2a^2 = 0$, $db/dt + 4ab = 0$. Solving the above conditions,we get $a(t) = 0$, $b(t) =$ constant, or $a(t) = 1/2(t - t_0)$, $b(t) = b_0/(t - t_0)^2$.

5.3 *Painlevé analysis of Davey–Stewartson equation*

The higher dimensional version of the nonlinear Schrödinger equation [50]

$$iq_t - \sigma_1 q_{xx} + q_{yy} = \sigma_2 \mid q \mid^2 + 2\sigma_1\sigma_2 Rq,\ \sigma_1 R_{xx} + R_{yy} + (\mid q \mid^2)_{xx} = 0 \quad (115)$$

where $\sigma_i = \pm 1$, $i = 1, 2$. Leading order: $\alpha = \beta = -1$, $\gamma = -2$ with $(a_0^2 + b_0^2) = 2/\sigma_2$ $(\sigma_1\phi_x^2 + \phi_y^2)$, $R_0 = -(2/\sigma_2)\phi_x^2$. Resonances: $-1, 0, 2, 3, 3, 4$. From the truncation, the bilinear form is constructed as

$$(iD_t - \sigma_1 D_x^2 + D_y^2 - \sigma_2\lambda^2)g \cdot F = 0(\sigma_1 D_x^2 + D_y^2 - \sigma_2\lambda^2)F \cdot F = -\sigma_1 G \cdot G \quad (116)$$

where λ = constant.

6. Summary and conclusions

In this short review, we have briefly discussed the Painlevé analysis and integrability properties of some autonomous and nonautonomous nonlinear partial differential equations. For a class of physically important equations we have demonstrated that this technique is one of the very useful and powerful method in nonlinear science to establish the integrability properties like Lax pair, Bäucklund transformation, bilinear form and so on. In this article, I have not included the Painlevé analysis of the coupled nonlinear Schrödinger equations. For coupled systems, this analysis is found to be cumbersome and the construction of Lax pair (from the truncation) is also an unsolved problem.

Acknowledgements

KP is grateful to Professor H J Mikeska for his very kind hospitality and constant encouragement. He expresses his thanks to DAAD for offering the fellowship, and to

DST, and CSIR, Govt. of India, for the financial support through major research projects. This paper is dedicated to Professor M Lakshmanan on the eve of his 50th birthday.

References

[1] S Kovalevskaya, *Acta Math.* **12**, 177, (1889); **14**, 81(1889)
[2] P Painlevé, *Oeuvres*, Vol. I, CNRS, Paris (1973)
[3] E L Ince, *Ordinary differential equations* (Dover, New York, 1956)
[4] T T Wu, B M McCoy, C A Tracy and E Barouch, *Phys. Rev.* **B13**, 316 (1976)
[5] M Tabor and J Weiss, *Phys. Rev.* **A24**, 2157 (1981)
[6] Y F Chang, M Tabor and J Weiss, *J. Math. Phys.* **23**, 531 (1982)
[7] T Bountis, H Segur and F Vivaldi, *Phys. Rev.* **A25**, 1257 (1982)
[8] A Ramani, B Dorizzi and B Grammaticos, *Phys. Rev. Lett.* **49**, 1539 (1982)
[9] M J Alowitz, D J Kaup, A C Newell and H Segur, *Stud. Appl. Math. Phys.* **53**, 249 (1974)
[10] R Hirota, in *Solitons* edited by R K Bullough and P J Caudrey (Springer, Berlin,1980)
[11] M J Alowitz, A Ramani and H Segur, *J. Math. Phys.* **27**, 1006 (1980)
[12] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**, 532 (1983)
[13] M D Kruskal, Private Communication (1982)
[14] J Weiss, *J. Math. Phys.* **24**, (1983), 1405, 532; **25**, (1984), 13,2226; **26**, (1985), 258, 2174 (1986)
[15] A C Newell, M Tabor and Y B Zeng, *Physica* **D29**, 1 (1987)
[16] F Lund and T Regge, *Phys. Rev.* **D14**, 1524 (1976)
[17] K Pohlymeyer, *Com. Math. Phys.* **46**, 207 (1977)
[18] B S Getmanov, JINR Preprint E2-11093, Dubna 1977
[19] K Porsezian, *Chaos, solitons and fractals* (Submitted)
[20] P Goldstein and E Infeld, *Phys. Lett.* **A103**, 8 (1984)
[21] K Porsezian and K Nakkeeran, *J. Mod. Opt.* **42**, 1953 (1995)
[22] K Porsezian and K Nakkeeran, *Phys. Rev. Lett.* **76**, 3955 (1996)
[23] S Liu and W Z Wang, *Phys. Rev.* **E49**, 5726 (1994)
[24] K Porsezian, *Pure. App. Opt.* **5**, L345 (1996)
[25] F Cariello and M Tabor, *Physica* **D39**, 77 (1989)
[26] M Lakshmanan and P Kaliappan, *J. Math. Phys.* **24**, 795 (1983)
[27] J D Gibbon and M Tabor, *J. Math. Phys.* **26**, 1956 (1985)
[28] R Conte, A P Fordy and A Pickering, *Physica* **D69**, 33 (1993)
[29] R Conte, *Phys. Lett.* **A140**, 383 (1989)
[30] A Pickering, *J. Math. Phys.* **37**, 1894 (1996)
[31] F Calogero and A Degasperis, *Lett. Nuovo Cimento* **22**, 139, 271 421 (1978)
[32] R Balakrishnan, *Phys. Rev.* **A32**, 1144 (1985)
[33] H H Chen and C S Liu, *Phys. Fluids* **21**, 377 (1978)
[34] S P Burtsev, V E Zakharov and A V Mikhailov, *Teor. Math. Fiz.* **70**, 323 (1987)
[35] M Daniel, K Porsezian and M Lakshmanan, *J. Math. Phys.* **35**, 6498 (1994)
[36] K Porsezian and K Nakkeeran, *Phys. Lett.* **A206**, 183 (1995)
[37] K Porsezian, M Daniel and R Bharathikannan, *Phys. Lett.* **A156**, 206 (1991)
[38] K Porsezian and M Lakshmanan, *J. Math. Phys.* **32**, 2923 (1992)
[39] K Porsezian, *Phys. Lett.* **A191**, 229 (1994)
[40] K Porsezian, *Phys. Lett.* **A221**, 163 (1996)
[41] W H Steeb and J A Louw, *Phys. Lett.* **A113**, 61 (1985)
[42] S Puri, *Phys. Lett.* **A107**, 359 (1985)
[43] N Joshi, *Phys. Lett.* **A125**, 456 (1987)
[44] L Halavaty, *J. Phys. Soc. Japan* **55**, 1405 (1995); *Phys. Lett.* **A128**, 335 (1988)
[45] W H Steeb, M Kloke, B M Spieker and W Oevel, *J. Phys.* **A16**, L447 (1983)
[46] N Nirmala, M J Vedan and B V Baby, *J. Math. Phys.* **27**, 2640, 2644 (1986)
P A Clarkson, *Proc. R. Soc. Edinb.* **A109**, 109 (1988)

[47] R Radha and M Lakshmanan, *J. Phys.* **A28**, 6977 (1995)
[48] K Porsezian and A Uthyakumar, *Phys. Lett.* **A183**, 371 (1993)
[49] W Oevel and W H Steeb, *Phys. Lett.* **A103**, 239 (1984)
[50] S Ganesan and M Lakshmanan, *J. Phys.* **A20**, L1147 (1988)