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Signature of chaos in power spectrum

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Abstract. We investigate the nature of the numerically computed power spectral density $P(f,N,\tau)$ of a discrete (sampling time interval, τ) and finite (length, N) scalar time series extracted from a continuous time chaotic dynamical system. We highlight how $P(f,N,\tau)$ differs from the true power spectrum and from the power spectrum of a general stochastic process. Non-zero τ leads to aliasing; $P(f,N,\tau)$ decays at high frequencies as $[\pi\tau/\sin\pi\tau f]^2$, which is an aliased form of the $1/f^2$ decay. This power law tail seems to be a characteristic feature of all continuous time dynamical systems, chaotic or otherwise. Also the tail vanishes in the limit of $N \to \infty$, implying that the true power spectral density must be band width limited. In striking contrast the power spectrum of a stochastic process is dominated by a term independent of the length of the time series at all frequencies.

Keywords. Chaos; time series analysis; Power spectrum; Aliasing.

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1. Introduction

The problem of distinguishing chaos from noise as the mechanism responsible for the irregular oscillations observed in a scalar time series is of great importance owing to its relevance to a variety of experimental contexts [1]. If the erratic oscillations are chaotic, then there exists a low dimensional deterministic dynamics responsible for it. Then, in principle, it is possibile to reconstruct the relevant dynamical equations, paving way for a complete understanding of the phenomenon. There has been an intense activity in this direction resulting in the formulation of several numerical approaches based on metric and topological considerations, see for example [2]. In this paper we shall focus attention on the analysis of the spectral measure – one of the traditional tools employed in the study of stationary stochastic processes as well as deterministic dynamical systems. For example, a periodic system with frequency f has a pure point spectral measure with a spectral density (henceforth called power spectrum) constituted by sharp(delta) peaks at f, and all its harmonics, in general. On the other hand, the spectral measure of a *generic* stationary stochastic process has an *absolutely continuous* part and a corresponding broad power spectrum. Chaotic time series also leads to a broadened power spectrum and hence it becomes difficult to distinguish chaos from noise. Intuitively, it is clear that finite changes can not take place in a deterministic system linear or nonlinear (chaotic or otherwise), in an inifinitesimal interval of time, whereas finite changes are possible in any interval of time in a generic stochastic process. This argument holds good even for cases of deterministic dynamical systems with non differentiable solutions, like the kicked harmonic oscillator, for which the number of kicks are *finite* in any finite interval of time. But in a stochastic process, the trajectory is nowhere differentiable, i.e., finite changes occur in infinitesimal interval of time and these events can occur infinitely often in any infinitesimal duration of time. Whence we expect the power spectrum of a chaotic time series to differ in some way or other from that of a stochastic time series, especially at high frequencies. Recently it has been shown numerically [3] that the power spectra of time series extracted from continuous time chaotic dynamical systems exhibit an exponential decay at high frequencies. There is no formal proof to support this observation though there are some plausibility arguments [3–5]. However, we should mention that, in the limited context of the two dimensional Navier–Stokes equation, the existence of an exponentially decaying upper bound for the power spectral density has been rigorously proven, see [6].

Right at the outset we emphasize that interpretation of numerically computed power spectra is beset with problems. The computed (observed) power spectrum often differs substantially from the 'true'. The reason for such a discrepancy is two fold – (1) the time series available for the computation of the power spectrum is sampled at discrete instants of time with an interval $\tau \neq 0$, and (2) the length of the time series N is invariably finite. Contrary to the general perception, we demonstrate in this paper that the modifications to the power spectrum induced by nonzero τ and finite N are *not* undesirable features. Rather, they provide an easy means of distinguishing a chaotic process from a generic stationary stochastic process.

In § 2, the definitions of the true and computed power spectra are given. The numerical procedure employed to calculate power spectrum and the nature of the difference between the computed power spectrum and the true one are also described in this section. The necessary formalism to understand the influence of nonzero τ and finite N on the computed power spectrum is developed in § 3. Aliasing, a consequence of non zero τ, is discussed in § 4. Exact analytical expressions for the aliased versions of exponential decay and power law decay (with an even exponent) are also obtained. These are required for a proper interpretation of the power spectra of chaotic systems. Evidence for a universal power law decay exhibited by all chaotic systems is presented. In § 5, the influence of finite N on the power spectra of chaotic systems is considered and the scaling of the power spectral density with N is presented. In § 6, we present our critical remarks about the theoretical arguments in the literature for the exponential decay of the power spectrum at high frequencies for chaotic systems. We end this paper with summary of the principal conclusions. Appendix I contains the proof for $(1/f^2N)$ decay of the power spectrum, at high frequencies of deterministic dynamical systems whose solution is smooth and bounded. Appendix II contains a proof that an N independent term dominates the power spectrum at all frequencies for a generic stationary stochastic process.

In order to ensure that our conclusions are of wide applicability, we have investigated the power spectra of a variety of chaotic dynamical systems—the Rossler [7], Raman [8], and Anantha [9] oscillators, the Lorenz model [10], the quasi-periodically kicked oscillator [11], and the Lorenz intermittency model [12]. The parameters in the respective models have been so chosen as to ensure that the occurrence of chaos is through period doubling route in the first three models, quasi-periodic route in the next two and intermittency route in the last one, thus covering the full gamut of chaotic dynamical systems. The phase space dimension and the degree of nonlinearity also varies across the

Table 1. Models studied.

Model	Equations	Parameters	Route	Reference
Rossler	$dx/dt = -y - z$ $dy/dt = x + ay$ $dz/dt = b + z(x - c)$	$a = 0.15$ $b = 0.20$ $c = 10.00$	Period doubling	[7]
Lorenz	$dx/dt = \sigma(y - x)$ $dy/dt = x(r - z) - y$ $dz/dt = xy - cz$	$\sigma = 16.00$ $r = 45.92$ $c = 4.00$	Quasiperiodic	[10]
Lorenz	$dx/dt = \sigma(y - x)$ $dy/dt = x(r - z) - y$ $dz/dt = xy - cz$	$\sigma = 10.00$ $r = 170.00$ $c = 8/3$	Intermittent	[12]
Raman	$dx/dt = -\gamma_1 x - \beta_1(x+z)y^2$ $dy/dt = -\gamma_2 y - \beta_2(x^2+z^2)y$ $dz/dt = -\gamma_3 z - \beta_3(x+z)y^2$	$\gamma_1 = -1, \beta_1 = 9$ $\gamma_2 = 1.0, \beta_2 = 5.0$ $\gamma_3 = 1.5, \beta_3 = 1.0$	Period doubling	[8]
Anantha	$dx/dt = \phi^m x - bx^2 -$ $ax + y - xy$ $dy/dt = b(kbx^2 - y - xy + az)$ $dz/dt = c(x - z)$ $d\phi/dt = d(e - x\phi^m)$	$a = 0.7, b = 0.002$ $c = 0.008, d = 10^{-4}$ $k = 1.0, m = 2.0$ $e = 184.9$	Period doubling	[9]
Quasiperiodically kicked oscillator	$dx/dt = y$ $dy/dt = a[d + c\{\cos \omega z$ $+ \cos \omega_0 z\} + \cos x - y]$ $dz/dt = 1$	$a = 0.5, d = 0.8$ $c = 0.55, \omega_0 = 1.0$ $\omega = 0.6180$	Quasiperiodic	[11]

models. The quasiperiodically kicked oscillator is non-autonomous and others are autonomous. Since all known aspects relevant for the classification of dynamical systems are encompassed, the features common to all these systems must indeed be universal. The models studied and the respective parameters are given in table 1.

2. Power spectrum

Let $x(t)$ be one of the state variables describing the chaotic dynamical system. The time autocorrelation function of $x(t)$ is defined as

$$c(t) = \lim_{T\to\infty} \frac{1}{T} \int_0^{T-|t|} dt' x(t') x(t' + |t|). \tag{1}$$

The power spectrum $C(f)$ is defined as the Fourier transform of the autocorrelation function

$$C(f) = \int_{-\infty}^{\infty} c(t) e^{i2\pi ft} dt. \tag{2}$$

Power spectrum can also be equivalently defined as the modulus square of its Fourier amplitude per unit time. This is given by

$$C(f) = \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T x(t) e^{i2\pi ft} dt \right|^2. \tag{3}$$

$C(f)$ defined in (2) and (3), henceforth, will be referred to as the true power spectrum in this paper.

In general, the computer or experimental realization of state variable $x(t)$ is in the form of a scalar discrete time series. This is equivalent to obtaining a time series by sampling $x(t)$ with a non zero sampling time τ, up to a finite length N, giving $\{x_j = x(t = j\tau);\ j = 0, N-1\}$. We now proceed to define the power spectrum $P(f,N,\tau)$ of such a discrete scalar time series such that it agrees with $C(f)$ in the limit $\tau \to 0$ and $N \to \infty$. The discrete version of the autocorrelation is defined as

$$c_j(N) = \left\langle \frac{1}{N} \sum_{l=0}^{N-1-|j|} x_l x_{l+|j|} \right\rangle, \tag{4}$$

where $\langle\cdot\rangle$ represents the average over several initial conditions and trajectories. This averaging is performed so as to ensure that the autocorrelation function of the discrete time series $c_j(N)$ is identical to that of the continuous time process $c(t)$ evaluated at $t = j\tau$ in the limit $N \to \infty$. $P(f,N,\tau)$ is then defined as

$$P(f,N,\tau) = \tau \sum_{j=-(N-1)}^{(N-1)} c_j(N) \mathrm{e}^{i2\pi f\tau j}. \tag{5}$$

$P(f,N,\tau)$ can also be represented in the form analogous to equation (3). Let $X(f,N,\tau)$ be the discrete Fourier transform of $\{x_j\}$,

$$X(f,N,\tau) = \sum_{j=0}^{N-1} x_j \mathrm{e}^{i2\pi f\tau j} \tag{6}$$

and the corresponding power spectrum is

$$P(f,N,\tau) = \frac{\tau}{N} \langle |X(f,N,\tau)|^2 \rangle. \tag{7}$$

In this paper we refer to $P(f,N,\tau)$ as the computed power spectrum or simply the power spectrum. The computed power spectrum $P(f,N,\tau)$ is equal to the true power spectrum $C(f)$ in the following limit.

$$C(f) = \lim_{\tau\to 0} \lim_{N\to\infty} P(f,N,\tau). \tag{8}$$

In our numerical investigations, we used the definitions given in equations (6) and (7) to compute the power spectrum of chaotic time series. Power spectra of different segments of length N, extracted from a time series of the dynamical system corresponding to a given initial condition, are computed. The computation is repeated for different initial conditions. An average over this ensemble of power spectra is carried out to obtain $P(f,N,\tau)$.

The power spectra of chaotic dynamical systems are found to exhibit an exponential decay followed by a much slower decay (like an algebraic decay). The power spectra of two representative models exhibiting chaos, namely, the Lorenz and Rossler models, are shown in figure 1(a) and figure 1(b), respectively. The nonexponential tail, though appears like an algebraic decay, does not fit to any power law. But, surprisingly, we find

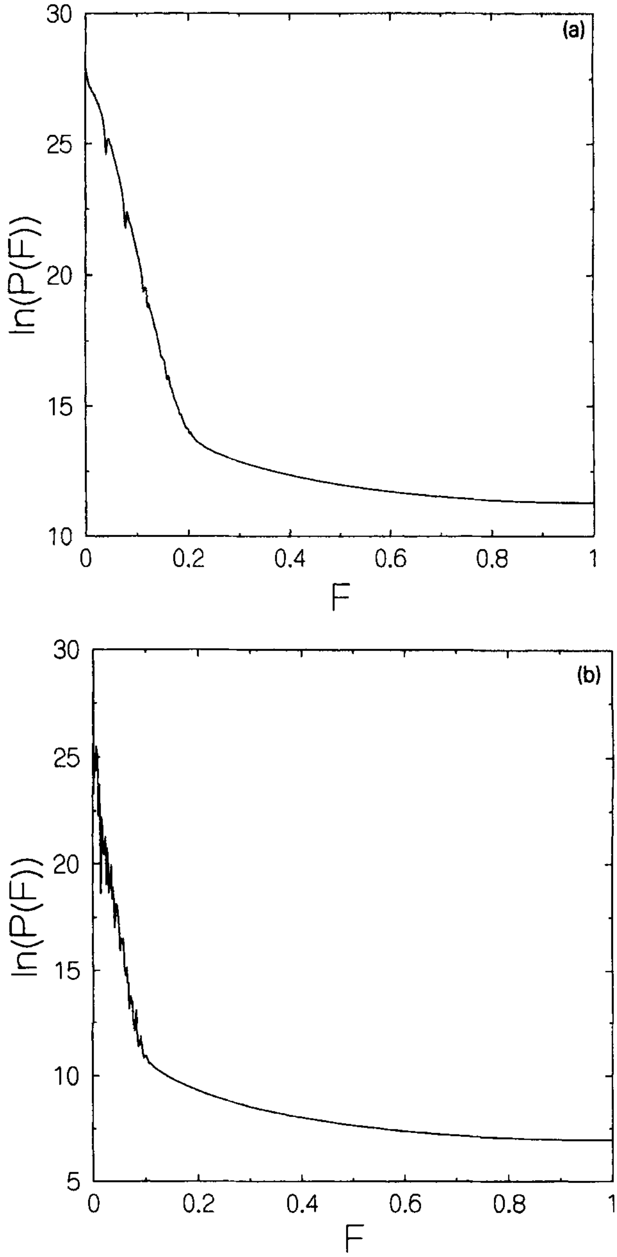

Figure 1. Computed power spectrum, $P(f, N, \tau)$ for (a) Lorenz model ($N = 2^{17}$, $\tau = 0.008$) and (b) Rossler model ($N = 2^{17}$, $\tau = 0.016$) $F = 2\tau f$.

that the tail exhibits a universal $[\pi\tau/\sin\pi\tau f]^2$ decay. Figures 2(a) – 2(f) show the power spectral density in the nonexponential region to be proportional to $[\pi\tau/\sin\pi\tau f]^2$. It is obvious from the data that, without exception, the fit is excellent. The origin of such a functional form for the computed power spectrum at high frequencies will form the

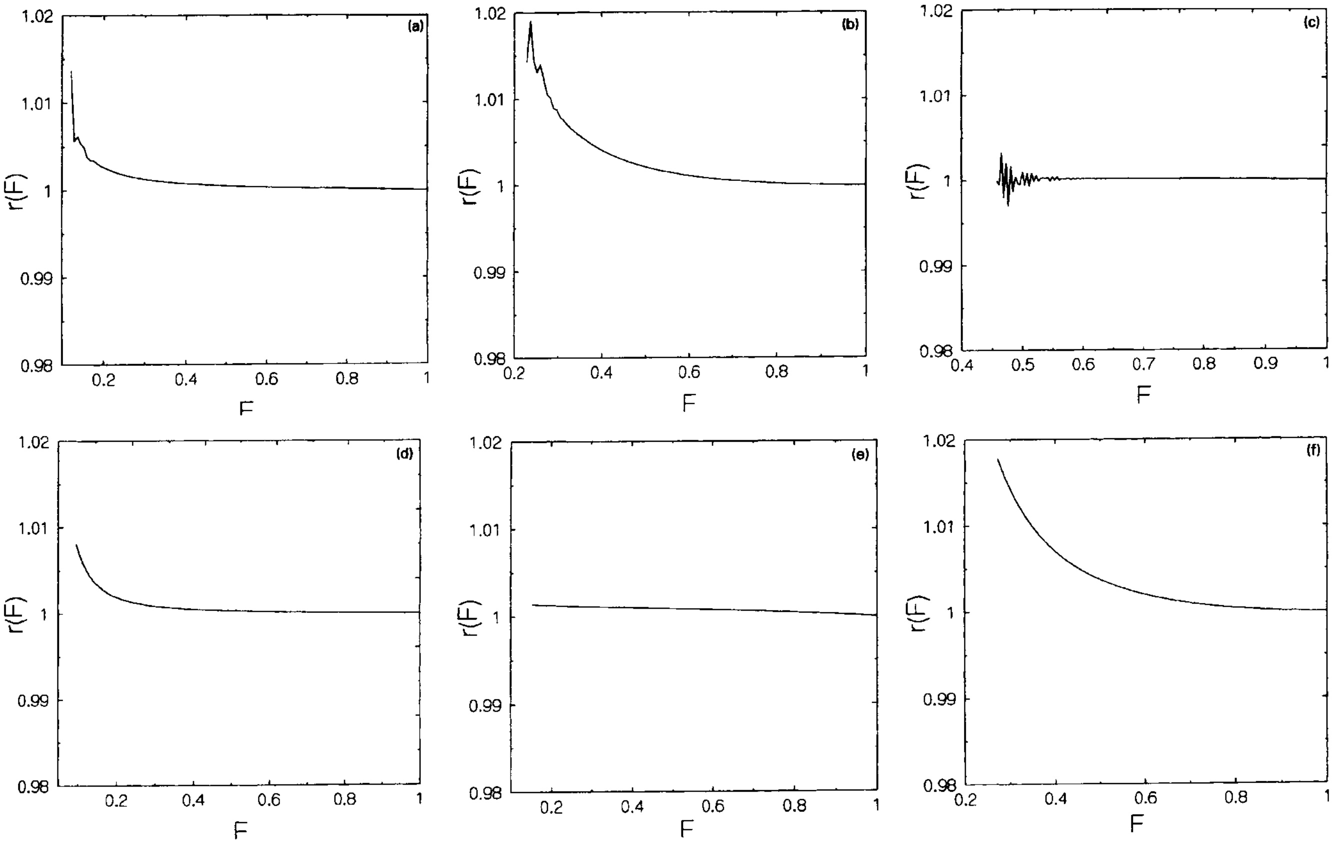

Figure 2. The ratio of the computed power spectrum and the aliased $1/f^2$ spectrum at high frequency region, $r(F) = P(f,N,\tau)/\mathrm{cosec}^2((\pi/2)F)$ where $F = f/f_m = 2\tau f$ for (**a**) Rossler model, (**b**) Lorenz model (quasi-periodic route), (**c**) Raman model, (**d**) Lorenz model (intermittency route), (**e**) Anantha oscillator and (**f**) Quasiperiodically kicked oscillator. These results are found to be independent of the choice of τ and N.

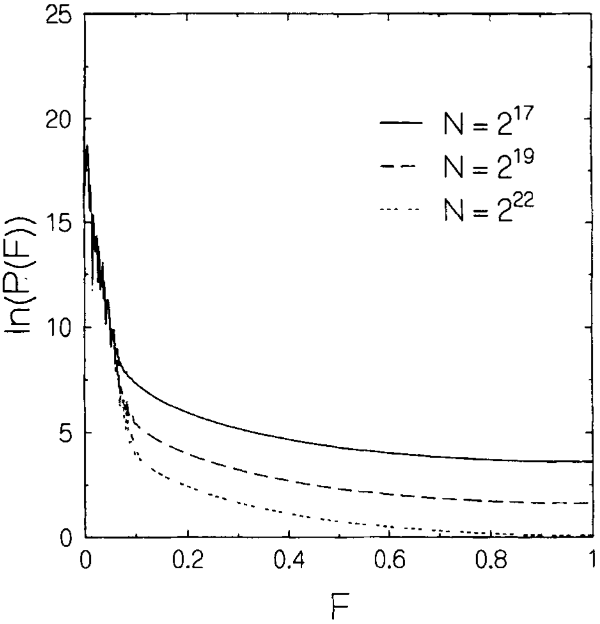

Figure 3. The scaling of the computed power spectrum with the size of the discrete time series, for Rossler oscillator.

subject matter of the next two sections. We observe that the spectral density in the nonexponential tail region is smooth and that it scales as $1/N$. The N-dependence of the power spectral density at a given frequency in the tail region is shown in figure 3. Detailed description of N-dependence is presented in § 5.

We have carried out the calculations in double and quadruple precision to see the influence of machine accuracy on our results. We find that our conclusions remain the same.

3. Relation between the true and computed power spectra

In order to understand the findings reported in the last section in a proper perspective, we derive here a formal relation between the computed and true power spectra. In doing this, we first define the following two functions, namely

$$w_N(t) = \begin{cases} 1 & \text{for} \quad -(N-1)\tau \leq t \leq (N-1)\tau \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and

$$\hat{c}(t) = \tau \sum_{j=-\infty}^{\infty} c(t)\delta(t - j\tau). \tag{10}$$

The correlation function, in general, depends on N and it can be rewritten as

$$c_j(N) = c_j + d_j(N), \tag{11}$$

where $c_j = c(t = j\tau)$ and $d_j(N)$ is a remainder which vanishes in the limit $N \to \infty$. This implies that $P(f, N, \tau)$ can be expressed as a sum of two terms

$$P(f, N, \tau) = P_1(f, N, \tau) + P_2(f, N, \tau), \tag{12}$$

where

$$P_1(f, N, \tau) = \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{c}(t) w_N(t) \exp(i2\pi ft), \tag{13}$$

and $P_2(f, N, \tau)$, related to $\{d_j(N)\}$, is the remainder that vanishes in the limit $N \to \infty$. $P_1(f, N, \tau)$ given in (13) can be rewritten in terms of the functions defined in equations (9) and (10) as

$$P_1(f, N, \tau) = \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{c}(t) w_N(t) \mathrm{e}^{i2\pi ft}. \tag{14}$$

Using the convolution theorem, we can write

$$P_1(f, N, \tau) = \int_{-\infty}^{\infty} \mathrm{d}f' \hat{C}(f - f') W_N(f'), \tag{15}$$

where $\hat{C}(f)$ and $W_N(f)$ are the Fourier transforms of $\hat{c}(t)$ and $w_N(t)$, respectively. The Fourier transform of $w_N(t)$ is given by

$$W_N(f) = \frac{\sin 2\pi(N-1)f\tau}{\pi f}. \tag{16}$$

This can be recast as

$$W_N(f) = \delta(f) + Y_N(f), \tag{17}$$

where $\lim_{N\to\infty} Y_N(f) = 0$. Substituting (17) in (15) gives

$$P_1(f, N, \tau) = \hat{C}(f) + Q_N(f, N, \tau) \tag{18}$$

such that $\lim_{N\to\infty} Q_N(f, N, \tau) = 0$. Using (12) and (18), we can write the computed power spectrum as

$$P(f, N, \tau) = \hat{C}(f) + R(f, N, \tau), \tag{19}$$

where $R(f, N, \tau)$ vanishes in the limit $N \to \infty$. Using once again convolution theorem, the Fourier transform of $\hat{c}(t)$ can be written as

$$\hat{C}(f) = \int_{-\infty}^{\infty} \mathrm{d}f' C(f - f') \sum_{m=-\infty}^{\infty} \delta\left(f' + \frac{m}{\tau}\right). \tag{20}$$

This can be simplified and we get

$$\hat{C}(f) = \sum_{m=-\infty}^{\infty} C\left(f + \frac{m}{\tau}\right). \tag{21}$$

Finally using (19) and (21), we get the desired result.

$$P(f,N,\tau) = \sum_{m=-\infty}^{\infty} C\left(f+\frac{m}{\tau}\right) + R(f,N,\tau), \tag{22}$$

where $P(f,N,\tau)$ is the computed power spectrum and $C(f)$ is the true power spectrum. This equation formally relates the computed power spectrum to the true one. It brings out clearly the role played by τ and N. The first term represents $\lim_{N\to\infty}P(f,N,\tau)$ and the second term reflects the finite size effects. We shall be analysing in the following sections, how a relation of this kind explains the results.

4. Aliasing

Power spectrum of any sampled data, with an interval of sampling τ being nonzero, is defined only in the domain $-f_m \leq f \leq f_m$, where $f_m = 1/2\tau$, is the Nyquist frequency. In sampling a given function $x(t)$, τ can be tuned. $\tau \to 0$ corresponds to sampling the entire function. But, for a fixed τ, the signal might contain frequencies higher than the Nyquist frequency. In such a case, the spectral densities corresponding to the frequencies larger than f_m will be folded into the region $|f| < f_m$. This is the phenomenon of aliasing, see for example [13].

The first term of (22) which represents $\lim_{N\to\infty} P(f,N,\tau)$ is an aliased version of true power spectrum $C(f)$. This clearly establishes the role played by nonzero τ resulting in the change in the functional form of the decay in the power spectrum due to aliasing. The remainder $R(f,N,\tau)$ in (22) depends both on τ and N. This term, owing to its dependence on τ, could itself, in principle, be an aliased version of some other function. We derive, in this section, the aliased versions of power spectra with (i) an exponential and (ii) a power law dependence on f, and discuss the implications with reference to the observed numerical results.

4.1 *Aliased exponential decay*

We consider a power spectrum with an exponential dependence on f.

$$C(f) = a\mathrm{e}^{-bf}. \tag{23}$$

The aliased version $P_a(f)$ of $C(f)$ can be written as

$$P_a(f) = \sum_{j=-\infty}^{\infty} C(f+2jf_m), \tag{24}$$

where $f_m = 1/2\tau$, the Nyquist frequency. The sum can be easily evaluated and we get

$$P_a(F) = a\frac{\cosh\,[bf_m(1-F)]}{\sinh[bf_m]}, \tag{25}$$

where $F = f/f_m$, is the scaled frequency. It is worth mentioning that, as a function of the scaled frequency F, the discrepancy between the aliased and the true power spectra gets confined to a region closer to the end point as τ decreases.

4.2 Aliasing of power law decay

Here we first derive an expression for the aliased version of $1/f^2$ decay and then general expression for any even power of $1/f$. The true power spectrum is given by

$$C(f) = \frac{A}{f^2}. \tag{26}$$

The aliased form can be written as

$$P_a(F) = \frac{A}{f_m^2} \sum_{j=-\infty}^{\infty} \frac{1}{(2j+F)^2}. \tag{27}$$

Using the integral representation

$$\frac{1}{F^2} = \int_0^\infty \mathrm{d}t\, t\, \mathrm{e}^{-Ft} \tag{28}$$

we can rewrite $P_a(f)$ as

$$\begin{aligned} P_a(F) &= \frac{A}{f_m^2} \int_0^\infty \mathrm{d}t\, t\, \frac{\cosh(1-F)t}{\sinh t} \\ &= -\frac{A}{2f_m^2} \frac{\partial}{\partial F} \int_{-\infty}^\infty \mathrm{d}t\, \frac{\sinh(1-F)t}{\sinh t} \\ &= -\frac{A}{2f_m^2} \frac{\partial}{\partial F} J(F). \end{aligned} \tag{29}$$

Using contour integration J can be evaluated as

$$J = \pi \cot \frac{\pi F}{2} \tag{30}$$

and

$$P_a(F) = \frac{A\pi^2}{4f_m^2} \operatorname{cosec}^2 \frac{\pi F}{2}. \tag{31}$$

The above can be recast in the following convenient form

$$P_a(f) = A \left[\frac{\pi\tau}{\sin \pi\tau f}\right]^2. \tag{32}$$

From (32), it is obvious that $\lim_{\tau\to 0} P_a(f) = C(f)$. However, for any finite τ, $P_a(F)$ is shape invariant and the discrepancy between the computed and the true power spectra does not vanish at any frequency. This functional form fits extremely well the power spectra of chaotic systems in the high frequency (nonexponential) region as can be seen from the figures 2(a)–2(f). This implies that the power spectra of chaotic time series have a universal $1/f^2$ tail, which is seen as a $[\pi\tau/\sin \pi\tau f]^2$ tail due to aliasing.

The aliased version of power law decay with an even exponent (say, $2l$) can also be exactly calculated.

$$P_a(F) = \frac{A}{f_m^{2l}} \sum_{j=-\infty}^{\infty} \frac{1}{(2j+F)^{2l}}. \tag{33}$$

This can be further rewritten as

$$P_a(F) = -\frac{A}{2f_m^{2l}} \frac{1}{(2l-1)!} \frac{\partial^{2l-1}}{\partial F^{2l-1}} \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{\sinh(1-F)t}{\sinh t}. \tag{34}$$

Since the integral in (34) is already evaluated (see (30)), $P_a(F)$ can easily be obtained by computing the appropriate derivative in (34).

5. Finite size effects

As has been emphasized at the beginning, the finite size of the time series also leads to a discrepancy between the computed and the true power spectra. The formal relation between the computed and the true power spectra described in (22) of § 3, has a part that depends on N, denoted by $R(f,N,\tau)$. This remainder is such that it vanishes in the limit $N \to \infty$.

Numerical investigation of the power spectral density of several chaotic dynamical systems listed in table 1, show that there is an exponential decay up to some frequency $f_\star$ followed by a $[\pi\tau/\sin\pi\tau f]^2$ tail. Our studies on the scaling of the power spectral density of the chaotic dynamical systems with respect to the system size N reveals that the spectral density corresponding to the frequencies in the exponential decay region is independent of N, whereas the ones corresponding to the frequencies in the tail region scale as $1/N$. The scaling of the power spectrum with N for Rossler is presented in figure 3. Since, by definition, see (22), only $R(f,N,\tau)$ can depend on N, it is clear from (32), together with the numerical results, that

$$R(f,N,\tau) \sim \frac{1}{N}\left[\frac{\pi\tau}{\sin\pi\tau f}\right]^2 \tag{35}$$

to the leading order in $1/N$. Since $R(f,N,\tau)$ accounts for the whole of the tail (see the fit in figures 2(a)–2(f)), within the limits of the present numerical analysis, it is evident that $\hat{C}(f)$ does not have any spectral component in the tail region. This implies that the power spectrum is band width limited.

In the experimental context, the interest is mainly to distinguish chaotic oscillations from random oscillations (or stochastic signals). The band width limited character of the power spectrum signals the presence of chaotic oscillations in the time series. In a generic noise, we do not expect the power spectrum of such a process to be band width limited. For example the power spectrum of white noise is broad and is independent of N. One can understand that in a determinisitic dynamical system, chaotic or otherwise, no drastic changes can take place in an extremely short interval of time and hence the power spectral density in the high frequency region will accordingly be zero. This results in a band width limited power spectrum. On the other hand, by definition, a generic stochastic process is one in which changes occur on any time scale. This leads to the conclusion that the power spectral density of such a process must be nonzero for all the frequencies. In Appendix II, we present a proof that for a stationary stochastic process with absolutely continuous spectral measure, the finite time power spectrum is dominated by an N independent term at all frequencies.

The $1/N$ scaling, together with an aliased $1/f^2$ decay at high frequency preceded by an exponential decay in the power spectrum single out chaotic time series from periodic, quasiperiodic and noisy signals.

6. Critical remarks on existing theoretical arguments

In [3–5] the authors present a case for the exponential decay of the power spectra of chaotic systems. The argument rests on the following definition of the power spectrum

$$C(f) = |X(f)|^2 \tag{36}$$

with

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{i2\pi ft} dt. \tag{37}$$

By extending $x(t)$ to the complex time domain, the authors argue that $X(f)$ can be represented as a sum of exponentials, and hence $C(f)$ must be a sum of exponentials. They have tacitly assumed the existence of $X(f)$.

In this section we present the problems associated with the above argument. The proper definition of $C(f)$ is, see [6]

$$C(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2 \tag{38}$$

with

$$X_T(f) = \int_0^T x(t) e^{2\pi ft} dt. \tag{39}$$

From the numerical simulations we know that the spectral measure of chaotic systems is predominantly absolute-continuous at high frequencies. We find that the associated power spectral density $C(f)$ is nonzero and bounded in the high frequency domain. But, the presence of the factor $1/T$ in (38) necessitates vanishing of $C(f)$ whenever $X(f)$ takes on a nonzero, but finite value. Since $C(f)$ of the system under investigation is nonzero, it is clear that $X(f) = \lim_{T\to\infty} X_T(f)$ can not exist. The finite time Fourier transform $X_T(f)$ should diverge as $\sqrt{T}$ as $T \to \infty$ (our $X(f, N, \tau)$ diverges as $\sqrt{N}$). Clearly these authors have neglected this part of the integral and concentrated only on the principal value, which in fact does not contribute to $C(f)$ at all.

In Appendix I, we show that the power spectra of the deterministic dynamical systems whose solution is smooth and bounded (which serves as sufficiency condition but not necessary) will have a $1/f^2N$ tail.

7. Summary and conclusion

The computed power spectrum $P(f, N, \tau)$ differs from the true one $C(f)$ due to nonzero sampling time τ and finite length of the time series N. The nonzero sampling time τ gives rise to aliasing, which changes the functional form of the decay. The computed power spectrum decays as $(\cosh bf_m(1-F))/\sinh bf_m$ up to a frequency f_c and subsequently decays as $1/N[\pi\tau/\sin\pi\tau f]^2$ for $f > f_c$. These expressions are calculated to be the aliased versions of e^{-bf} and $1/Nf^2$ respectively, which means the true power spectrum is band width limited. The feature of $1/Nf^2$ decay is observed in all the chaotic dynamical system studied. The finite size N of the time series results in contributing a spurious tail to an otherwise band width limted power spectrum. This feature turns out to be useful for distinguishing chaos from a generic stationary stochastic process as the former scales as $1/N$ while the latter is independent of N.

Appendix I

We now proceed to prove the statement that the power spectrum, defined for the signal in the interval $[0, T]$, of a bounded function belonging to C^∞ class shows a $(f^2T)^{-1}$ tail for $|f| > f_c$.

Consider a dynamical system

$$\frac{dx_i(t)}{dt} = f_i(\mathbf{x}(t)), \quad i = 1, \ldots, d \tag{A1}$$

with the initial conditions

$$x_i(0) = x_{i0}. \tag{A2}$$

The formal solution of the above equations is given by

$$x_j(t) = x_j(\mathbf{x}_0, t) = e^{tL} x_{j0}, \tag{A3}$$

where

$$L = \sum_{i=1}^{d} f_i(\mathbf{x}_0) \frac{\partial}{\partial x_{i0}}. \tag{A4}$$

The finite time Fourier transform of $x_j(t)$ is defined as

$$X_{j,T}(f) = \int_0^T dt x_j(\mathbf{x}_0, t) e^{i2\pi ft}. \tag{A5}$$

Substituting (A3) in (A5) and simplifying, we get

$$X_{j,T}(f) = \frac{1}{i2\pi f}\left[1 - \frac{iL}{2\pi f}\right]^{-1} [e^{i2\pi fT} e^{LT} x_{j0} - x_{j0}]. \tag{A6}$$

The power spectrum $P_{j,T}(f) = 1/T|X_{j,T}(f)|^2$ is given by

$$P_{j,T}(f) = \frac{1}{4\pi^2 f^2 T}\left|\left[1 - \frac{iL}{2\pi f}\right]^{-1} [e^{i2\pi fT} e^{LT} x_{j0} - x_{j0}]\right|^2 = \frac{|Q|^2}{4\pi^2 f^2 T}. \tag{A7}$$

The expression for Q can be written as

$$Q = \mathrm{e}^{i2\pi fT} \sum_{n=0}^{\infty} \left(\frac{i}{2\pi f}\right)^n L^n x_j(\mathrm{x}_0, T) - \sum_{n=0}^{\infty} \left(\frac{i}{2\pi f}\right)^n L^n x_{j0}. \tag{A8}$$

From (A8) we get

$$|Q| \leq \sum_{n=0}^{\infty} \left(\frac{1}{|2\pi f|^n}\right) (|L^n x_j(\mathrm{x}_0, T)| + |L^n x_{j0}|). \tag{A9}$$

From the formal solution (A3) it is easy to see that

$$L^n x_j(\mathrm{x}_0, T) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} x_j(\mathrm{x}_0, t)|_{t=T} \quad \text{and} \quad L^n x_{j0} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} x_j(\mathrm{x}_0, t)|_{t=0}. \tag{A10}$$

Assuming that the trajectory $x_j(\mathrm{x}_0, t)$ of the dynamical system (A1) belongs to C^∞ class, it follows that there exists an upper bound M_n for $|\mathrm{d}^n/\mathrm{d}t^n x_j(\mathrm{x}_0, t)|, n \geq 1$. Hence we obtain

$$|L^n x_j(\mathrm{x}_0, t)| \leq M_n \quad \text{for all} \quad t \in [0, \infty). \tag{A11}$$

We define a parameter $M = \sup_n (M_n)^{(1/n)}$. It can be seen that

$$\sum_{n=1}^{\infty} \frac{|L^n x_j(\mathrm{x}_0, t)|}{|2\pi f|^n} \leq \sum_{n=1}^{\infty} \left(\frac{M}{|2\pi f|}\right)^n \tag{A12}$$

for all $t \in [0, \infty)$. The above series is convergent for $|2\pi f| > M (= 2\pi f_c)$. The series can then be summed to get

$$|Q| \leq 2\left[M_0 + \frac{M/|2\pi f|}{1 - M/|2\pi f|}\right]. \tag{A13}$$

This illustrates that for $|2\pi f| > M (= 2\pi f_c)$, the power spectrum will have a tail

$$P_{j,T}(f) \sim \frac{M_0^2}{(\pi f)^2 T}. \tag{A14}$$

This will manifest as a $(\pi\tau/\sin \pi f\tau)^2$ tail for a discrete time series sampled at a time step τ (due to aliasing), consistent with our numerical results for the power spectra of chaotic dynamical systems.

Appendix II

Let $x(t)$ be a stationary stochastic process with an absolutely continuous spectral measure. Let $\{x_j = x(t = j\tau);\ j = 0, \ldots, N-1\}$ be a discretely sampled data set. $P(f, N, \tau)$ can be written as

$$P(f, N, \tau) = \frac{\tau}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle x_j x_k \rangle \mathrm{e}^{i2\pi f\tau(j-k)}. \tag{A1}$$

Since the process is stationary one can write $\langle x_j x_k \rangle = c(\tau(j-k))$. Substituting this and simplifying, we get

$$P(f,N,\tau) = \int_{-\infty}^{\infty} \mathrm{d}f' C(f') S_N, \tag{A2}$$

where

$$S_N = \tau\left[1 + 2\sum_{l=1}^{N-1}\left(1 - \frac{l}{N}\right)\cos 2\pi\tau(f-f')l\right] \tag{A3}$$

and $C(f)$ is the true (unaliased) power spectrum. $P(f, N, \tau)$ can be equivalently represented as (see eq. (22))

$$P(f,N,\tau) = \sum_{m=-\infty}^{\infty} C\left(f + \frac{m}{\tau}\right) + R(f,N,\tau). \tag{A4}$$

Thus it is only the true spectral density $C(f)$ that goes as input to the analysis. By definition, a generic stationary stochastic process is characterized by a spectral measure which is absolutely continuous, and is nonzero for all frequencies. It is clear from (A4) that if the true power spectrum is nonzero, so is the aliased power spectrum. This implies that for a given N, $P(f,N,\tau)$ of a generic stationary stochastic process consists of both an N-independent term and terms depending on N. In the following analysis we calculate the N dependent terms explicitly to understand which term dominates in the large N limit.

Let $z = 2\pi(f-f')\tau$, then

$$S_N = \tau\left[1 + 2\mathrm{Re}\,\frac{e^{iNz}-1}{e^{iz}-1} - \frac{2}{N}\frac{\partial}{\partial z}\mathrm{Im}\,\frac{e^{iNz}-1}{e^{iz}-1}\right]. \tag{A5}$$

This can be evaluated as

$$S_N = \frac{\tau}{N}\frac{\sin^2(Nz/2)}{\sin^2(z/2)}. \tag{A6}$$

The power spectrum now takes the form

$$P(f,N,\tau) = \int_{-\infty}^{\infty} \mathrm{d}f' C(f') \frac{\tau}{N}\frac{\sin^2 N\pi\tau(f-f')}{\sin^2 \pi\tau(f-f')}. \tag{A7}$$

Noting that S_N is periodic in $1/\tau$, the above integral can be written as

$$P(f,N,\tau) = \sum_{l=-\infty}^{\infty}\frac{1}{\pi}\int_{-(\pi/2)}^{\pi/2} \mathrm{d}\xi C\left(f + \frac{l}{\tau} + \frac{\xi}{\pi\tau}\right)\frac{\sin^2 N\xi}{N\sin^2\xi}. \tag{A8}$$

Since the spectral measure is absolutely continuous, the power spectrum $C(f)$ is smooth and one can expand $C(f)$ as a Taylor's series in $\xi/\pi\tau$, to give the power spectrum the following form

$$P(f,N,\tau) = \sum_{l=-\infty}^{\infty}\left[\sum_{m=0}^{\infty} C^{(m)}\left(f + \frac{l}{\tau}\right) I_m\right] \tag{A9}$$

where

$$I_m = \frac{1}{\pi m!}\int_{-(\pi/2)}^{\pi/2} d\xi \left(\frac{\xi}{\pi\tau}\right)^m \frac{\sin^2 N\xi}{N\sin^2\xi}, \quad \text{and} \quad C^{(m)}(f) = \frac{d^m}{df^m}C(f). \tag{A10}$$

It is easy to see that

$$I_0 = 1 \tag{A11}$$

$$I_{2m} \sim O\left(\frac{1}{N}\right) \quad \text{and} \quad I_{2m-1} = 0, \quad m = 1, \ldots . \tag{A12}$$

With the help of these results the power spectrum can easily be written as

$$P(f,N,\tau) = \sum_{l=-\infty}^{\infty} C\left(f+\frac{l}{\tau}\right) + \frac{1}{N}\sum_{l=-\infty}^{\infty}\left[C^{(2)}\left(f+\frac{l}{\tau}\right)\left(\frac{\ln 2}{\tau^2}\right) + C^{(4)}\left(f+\frac{l}{\tau}\right)\left(\frac{\ln 2 - \frac{9\zeta(3)}{2\pi^2}}{2\pi^2\tau^4}\right) + \cdots\right] + O\left(\frac{1}{N^2}\right), \tag{A13}$$

where $\zeta(n)$ is the Riemann zeta function of order n.

Since the first term (the N independent part) in the above equation is nonzero for any absolutely continuous nonzero spectral measure and since the coefficient of $1/N$ is finite, we can always find an N such that the first term dominates. This implies that for a generic stationary stochastic process an N independent term will dominate the power spectrum at all frequencies, if N is sufficiently large.

In the case of deterministic dynamical systems, it is known that the power spectra of regular deterministic dynamical systems are discrete, and they do not have N-independent part at high frequencies. Only chaotic systems show a semblance of a continuous power spectrum. Numerical results show that chaotic systems also do not have an N-independent part at high frequencies. Thus it is not possible that the power spectra of deterministic dynamical systems are dominated by an N-independent part at high frequency region.

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