PRAMANA
— journal of
physics

© Printed in India

Vol. 47, No. 5,
November 1996
pp. 393–400

Approximate analytical lower bounds on 'double asymptotic scaling' variables and physics at HERA

A SAIKIA
Department of Physics, Arya Vidyapeeth College, Guwahati 781 016, India

MS received 1 May 1996; revised 29 July 1996

Abstract. We analytically examine the asymptotic solution of gluon evolution equation in terms of the double scaling variables' ρ and σ of perturbative QCD and find the approximate lower bounds on these, above which the solution is considered to be valid. Comparison of this asymptotic solution is made with the fit obtained from data and the estimated lower bound on ρ is nearly equal to our analytical finding. To analyze the data below the lower bound on ρ, other analytical solutions of gluon evolution equation are to be used which depend highly on the input x-distributions of gluon to study the physics at low-x of HERA range.

Keywords. Gluon evolution equation; double scaling variable; small-x.

PACS No. 12·35

It has been predicted [1] that, in the HERA range, we will find double asymptotic scaling (DAS) in terms of scaling variables $\sigma(=\sqrt{\ln(x_0/x)\ln[\ln(Q^2/\Lambda^2)/\ln(Q_0^2/\Lambda^2)]})$ and $\rho(=\sqrt{\ln(x_0/x)/\ln[\ln(Q^2/\Lambda^2)/\ln(Q_0^2/\Lambda^2)]})$ if the initial gluon distribution is 'soft', from the solution of DGLAP gluon evolution equation [3, 8] for the gluon evolution at small-x. Phenomenological analysis of DAS has been performed in [1, 2]. Our purpose here is to find the approximate lower bounds on the scaling variables analytically. Comparing the solution obtained in [1] with the fit of [10], we find an approximate lower bound on ρ numerically, which is comparable to the analytical prediction justifying DAS above this lower bound. To identify small-x. physics (that is, whether the gluon is 'soft' or 'hard'), we study the region below this lower bound on ρ covered by HERA data. This can be performed with the help of analytical solutions of DGLAP equation which are sensitive to the initial x-distribution of gluon.

In [1] it has been shown that the DGLAP equation for gluon evolution at small-x can be written as

$$\left[\frac{\partial^2}{\partial\xi\partial\zeta}+\delta\frac{\partial}{\partial\xi}-\gamma^2\right]G(\xi,\zeta)=0, \tag{1}$$

where $G(\xi,\zeta)=xg(x,Q^2)$ is the gluon momentum distribution function,

$$\xi=\ln\frac{x_0}{x},$$

$$\zeta = \ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right],$$

$$\delta = \left(11 + \frac{2n_f}{27}\right)\Big/\left(11 - \frac{2n_f}{3}\right),$$

$$\gamma = \sqrt{12\Big/\left(11 - \frac{2n_f}{3}\right)},$$

n_f = the number of active flavours participating in the process and x_0, Q_0^2 are the starting scales for the perturbative evolution.

In the asymptotic limit $\sigma(\equiv\sqrt{\xi\zeta})\to\infty$ along any curve such that $\rho(\equiv\sqrt{\xi/\zeta})\to\infty$, the solution of (1) for soft boundary condition is

$$G(\sigma,\rho)\underset{\sigma\to\infty}{\simeq} Nf_g(y)\frac{1}{\sqrt{4\pi\gamma\sigma}}\left\{\exp\left[2\gamma\sigma-\delta\left(\frac{\sigma}{\rho}\right)\right]\right\}\left(1+O\left(\frac{1}{\sigma}\right)\right), \tag{2}$$

where $y=\gamma/\rho$, $f_g(0)=1$ and N is a normalization factor.

When $\xi\zeta$ is large, (2) can be written in terms of ξ and ζ (in the limit $f_g(y)\to 1$ as $y\to 0$) as

$$G(\xi,\zeta)\simeq\frac{N}{\sqrt{4\pi\gamma}}\cdot\frac{1}{(\xi\zeta)^{1/4}}\cdot\exp(2\gamma\sqrt{\xi\zeta}-\delta\zeta). \tag{3}$$

Now, we try to find the region of validity of (3) in the first quadrant of the ξ–ζ plane. When ζ is fixed (say ζ_1), $G(\xi,\zeta_1)$ has a minimum (figure 1(b)) at

$$\xi=\xi_1=\left(\frac{1}{4\gamma\sqrt{\zeta_1}}\right)^2 \tag{4}$$

since

$$\left.\frac{dG(\xi,\zeta_1)}{d\xi}\right|_{\xi=\xi_1}=0 \quad\text{and}\quad \left.\frac{d^2G(\xi,\zeta_1)}{d\xi^2}\right|_{\xi=\xi_1}>0.$$

This implies that, when ξ is increased from 0 up to ξ_1, $G(\xi,\zeta_1)$ decreases. This behaviour is contradictory to QCD predictions [6, 7]. Hence (3) does not depict the nature of gluon evolution with respect to ξ below ξ_1. So, at fixed ζ_1, from (4), the lower bound (σ_1) on σ can be predicted, which is given by

$$\sigma_1\simeq\frac{1}{4\gamma}. \tag{5}$$

Next, let us study the variation of $G(\xi,\zeta)$ with respect to ζ, when ξ is fixed (say ξ_2) (figure 1(a)). It has a maximum at ζ_2, when $\gamma^2\xi_2>\delta$, where

$$\zeta_2=\frac{1}{[2(\gamma\sqrt{\xi_2}-\sqrt{\gamma^2\xi_2-\delta})]^2} \tag{6}$$

since

$$\left.\frac{dG(\xi_2,\zeta)}{d\zeta}\right|_{\zeta=\zeta_2}=0 \quad\text{and}\quad \left.\frac{d^2G(\xi_2,\zeta)}{d\xi^2}\right|_{\zeta=\zeta_2}<0.$$

When $\gamma^2\xi_2<\delta$, ζ_2 is complex and hence within this range of analysis there is no extremum. This evolution pattern contradicts the QCD prediction at small-x and large

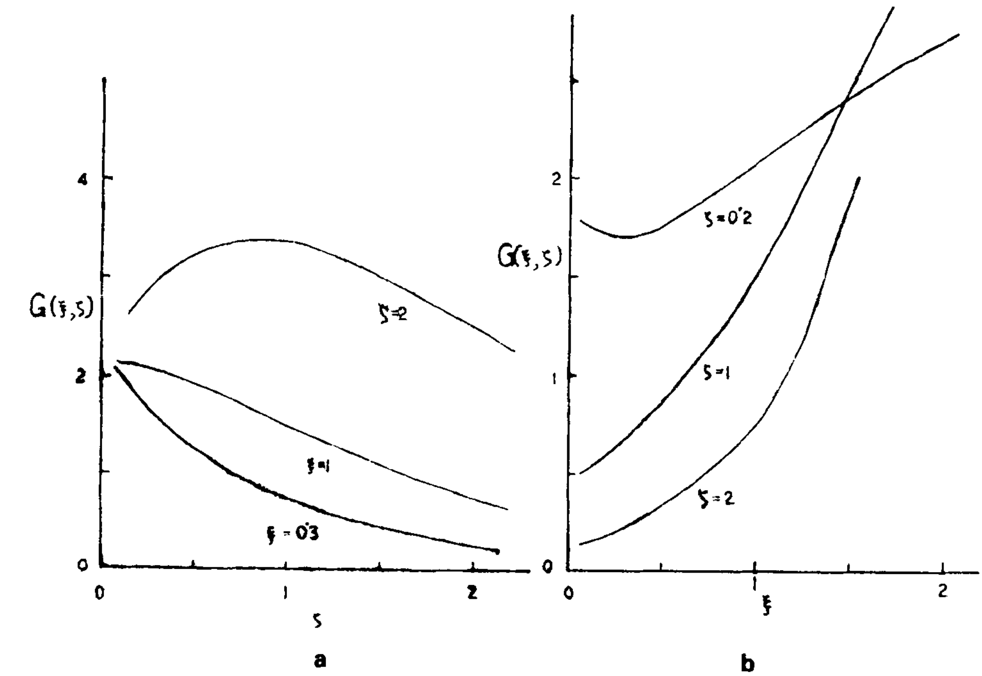

Figure 1. (**a**) Plots of ζ, $G(\xi,\zeta)$ at fixed ξ, and (**b**) plots of ξ, $G(\xi,\zeta)$ at fixed ζ, using (3) considering $N = 1$.

Q^2 [7]. So, the solution (eq. (3)) is valid only when $\xi_2 > \xi_c$, where

$$\xi_c = \delta/\gamma^2. \tag{7}$$

For $\xi_2 \gg \xi_c$, we have

$$\sqrt{\frac{\xi_2}{\zeta_2}} = \delta/\gamma = \rho_2. \tag{8}$$

Therefore, we find that $G(\xi_2,\zeta)$ attains maximum at $\rho_2 = \delta/\gamma$. For $\zeta > \zeta_2$ (i.e. $\rho < \rho_2$), $G(\xi_2,\zeta)$ decreases with increase in ζ which is opposite to the findings of [6, 7]. So, the approximate lower bound on ρ is $\rho_2(=\delta/\gamma)$. (The lower bounds on ρ and σ evaluated above are approximate, since we are not considering the exact form of $f_g(y)$ in our analysis.) Above these lower bounds, DAS is valid if the gluon is 'soft'. Departures from these lower bounds on ρ and σ may indicate, whether the small-x growth is 'hard' or 'soft', because in the 'hard' gluon case, DAS is violated [1].

For quantitative analysis, we compare DAS prediction (eq. (3)) with the leading order fit of ref. [10], since DAS is a consequence of leading order DGLAP gluon evolution equation. The fit [10] for gluon momentum distribution is

$$G(x,Q^2) = xg(x,Q^2) = \left[x^a(A + Bx + Cx^2)\left(\ln\frac{1}{x}\right)^b + S^\alpha \exp\left(-E + \sqrt{E's^\beta \ln\frac{1}{x}}\right)\right](1-x)^D, \tag{9}$$

where

$$\mu_{LO}^2 = 0{\cdot}25\,\text{GeV}^2, \quad \Lambda = 0{\cdot}232\,\text{GeV}, \quad a = 1{\cdot}00 - 0{\cdot}175,$$

$$\alpha = 0{\cdot}558, \quad \beta = 1{\cdot}218, \quad b = 0, \quad E' = 4{\cdot}066,$$

$$A = 4{\cdot}879s - 1{\cdot}383s^2,$$

$$B = 25{\cdot}92 - 28{\cdot}97s + 5{\cdot}596s^2,$$

$$C = -25{\cdot}69 + 23{\cdot}68s - 1{\cdot}975s^2,$$

$$D = 2{\cdot}537 + 1{\cdot}718s + 0{\cdot}353s^2,$$

$$E = 0{\cdot}595 + 2{\cdot}138s,$$

$$S = \ln[\ln(Q^2/\Lambda^2)/\ln(\mu_{LO}^2/\Lambda^2)].$$

(We use this fit because of its validity over a wide range of x and Q^2; $10^{-5} < x < 1$ and $1 < Q^2 < 10^8$).

For the prediction of DAS (eq. (3)), we use

$$Q_0^2 = 1\,\text{GeV}^2, \quad x_0 = 0{\cdot}1 = 0{\cdot}23\,\text{GeV}$$

because, in ref. [2] it is found that low σ data is well explained by DAS with these values. For our calculations, we take n_f to be four. So, the value of γ is 1·2 and that of δ is 1·3556.

To obtain the predictions of DAS we are to determine the value of N. The method of determination of the normalization constant N is different from [1]. From (3), the prediction from it is accurate when $\xi\zeta \to \infty$. Hence, the prediction from DAS should be nearly equal to the prediction from fit at small-x and large Q^2, that is,

$$G(\xi,\zeta)|_{\text{from DAS}} \simeq \lim_{\substack{x\to 0\\ Q^2\to\infty}} G(x,Q^2)|_{\text{from fit}}.$$

From such analysis, we found the value of N to be 3·3586 at $x = 10^{-5}$ and $Q^2 = 10^6\,\text{GeV}^2$. Putting this value in (3) we found the predictions of DAS for different values of x and Q^2 which are plotted in figure 2 as dashed curves. The continuous curves in figure 2 represent the prediction from fit using (9). Equation (3) is found to be valid for a wide range of x and Q^2. For $x \in [10^{-5}, 10^{-3}]$ and $Q^2 \in [10^3, 10^6)$ the difference between predictions from fit and DAS is below 7% of the fit; for $x = 10^{-2}$ and $Q^2 \in [10, 10^4]$ the difference is below 10%. But for $x \in [10^{-1}, 10^{-2}]$ and $Q^2 \in [10, 10^6]$ the difference is above 10%. (If we put the exact form of $f_g(y)$ at $x = 10^{-2}$, it will further reduce the prediction of DAS by 8% and at $x = 10^{-3}$ by 6%.) In figure 3 we plot the predictions from (3) and (9) for the values of $x = 10^{-5}, 10^{-2}, 1{\cdot}5 \times 10^{-2}, 2 \times 10^{-2}$ and $Q^2 \in [10, 10^6]$ to represent the difference between the two predictions. (Since $x_0 = 0{\cdot}1$, we are unable to study (3) above 0·1.) In table 1, we show the difference between the two predictions in per cent for different values of x and Q^2 and corresponding values of ρ and σ. (We have omitted from the table the prediction for $x < 10^{-2}$ because for these values of x and $\forall Q^2 \in [10, 10^6]$, $\rho > \rho_2 (\equiv (\gamma/\delta)^{-1} = 1{\cdot}13)$.) From the table we find that the disagreement is more that 10% when ρ is around 1·15 which agrees well with our analytical analysis (the difference is nearly 2%). Hence, it can be inferred that in the region $10^{-2} > x > 10^{-5}$ and $10 < Q^2 < 10^6$, DAS is applicable and 'soft' gluon process is prominent.

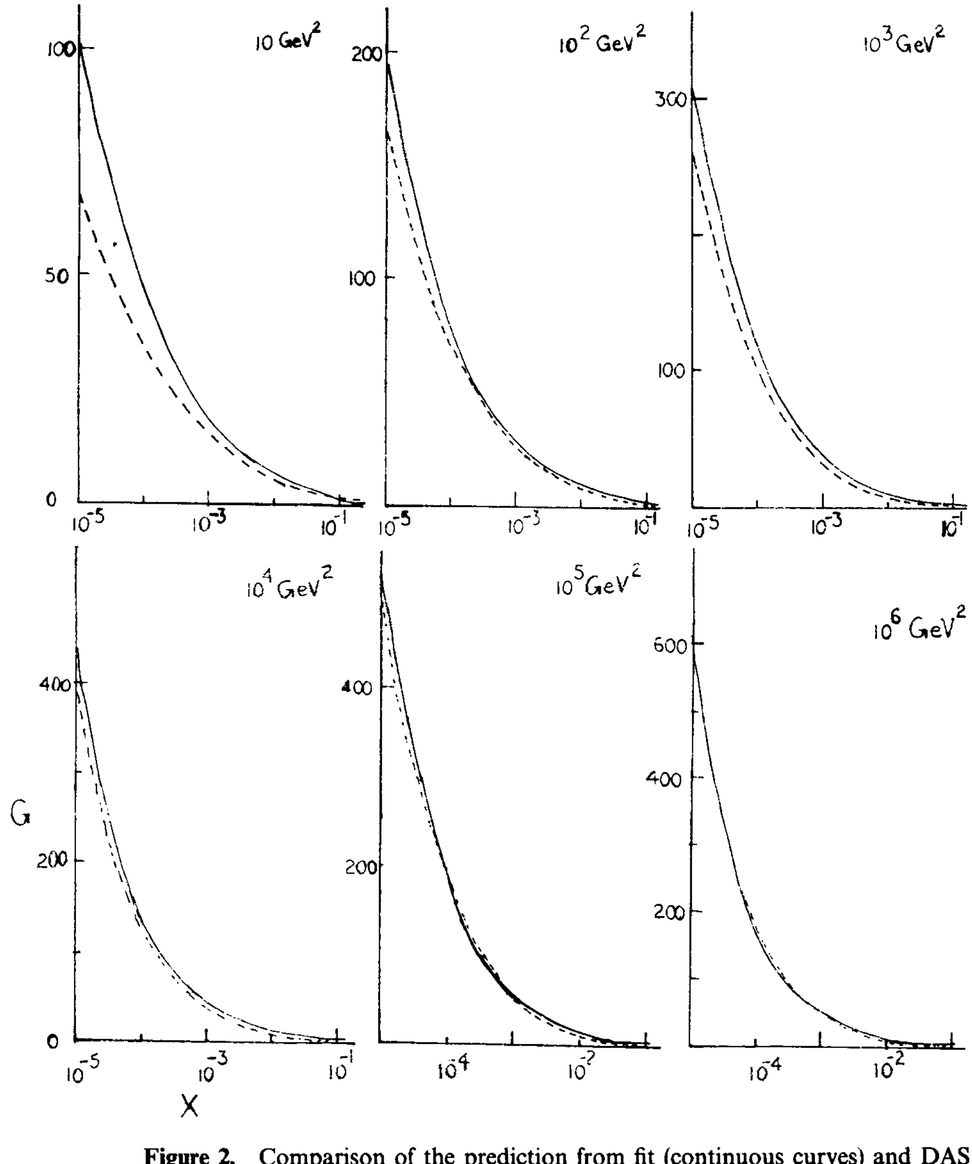

Figure 2. Comparison of the prediction from fit (continuous curves) and DAS (dashed curves) for $Q^2 = 10$, 10^2, 10^3, 10^4, 10^5 and $10^6\,\mathrm{GeV}^2$ for x in the region between $10^{-5} < x < 10^{-1}$.

Almost all of the HERA data lie in the region $\sigma \in [1, 2]$ and $\rho \in [1, 3]$ [1]. Since lower bound on σ is 0·2083 ($\equiv (1/4\gamma)$) which is outside the region probed by HERA, no comparison between fit and DAS is needed for the determination of σ_1.

To complete the exploration of physics at HERA, we study the region $1 \leqslant \rho < 1{\cdot}13$, which is out of reach for DAS. Hence other analytical solutions of DGLAP gluon evolution equation is used which are not only sensitive to the input gluon distribution functions (we need input sensitive one to distinguish between the 'hard' and 'soft'

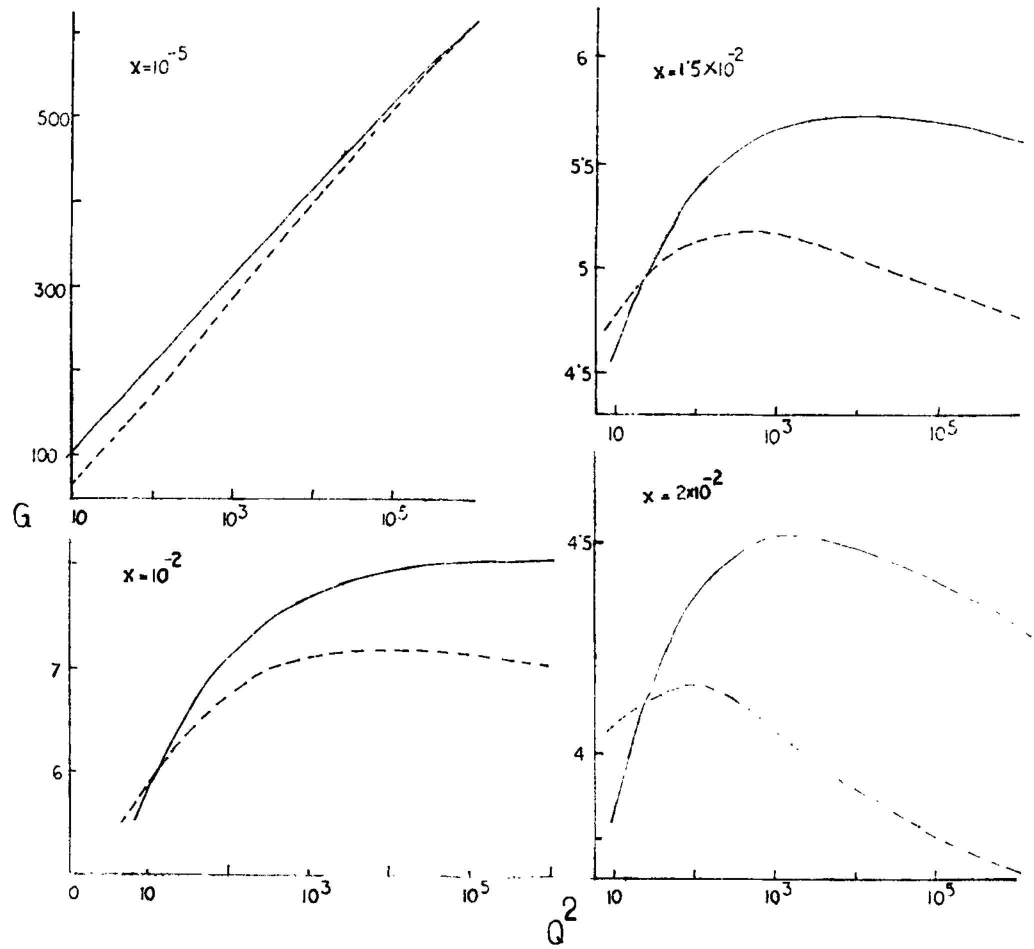

Figure 3. Comparison of the predictions from fit (continuous curves) and DAS (dashed curves) for $x = 10^{-5}$, 10^{-2}, $1{\cdot}5 \times 10^{-2}$ and 2×10^{-2} for $10\,\text{GeV}^2 < Q^2 < 10^6\,\text{GeV}^2$.

processes) but also of the pattern $\exp\sqrt{\xi\zeta}$ which is a consequence of DLA [6, 8, 9]. For the solutions of DGLAP evolution equations for structure functions, we have adopted solutions which are products of some functions of x and Q^2 [4]. This type of analysis is extended to the leading order of gluon sector [5] and to higher order in ref. [11]. This is applicable for the analysis of data around $\rho = 1$, because $\exp\sqrt{\xi\zeta}$ can be written in the factored form of functions of x and Q^2, since

$$\sqrt{\xi\zeta} = \frac{\xi + \zeta}{2}$$

when

$$\xi \simeq \zeta \text{ (i.e. } \rho \simeq 1).$$

When $G(x, Q^2)$ is written in the factored form as

$$G(x, Q^2) = g(x)h(Q^2),$$

Table 1. Difference in per cent between predictions from fit (eq. (9)) and DAS (eq. (3)) in per cent with respect to fit for different values of x and Q^2 and respective values of ρ and σ. Bold figures indicate the start of difference above 10% for respective ρ values.

	$Q = 10\,\mathrm{GeV}^2$	$Q^2 = 10^2$	$Q^2 = 10^3$	$Q^2 = 10^4\,\mathrm{GeV}^2$	$Q^2 = 10^5\,\mathrm{GeV}^2$	$Q^2 = 10^6\,\mathrm{GeV}^2$
$x = 0{\cdot}01$	−0·9%	4·1%	7·3%	8·4%	9·8%	**11·8%**
$(\rho, \sigma) =$	(1·9951, 1·1541)	(1·5629, 1·4733)	(1·3801, 1·6685)	(1·2738, 1·8077)	(1·2024, 1·915)	(1·1502, 2·0019)
$x = 0{\cdot}015$	−3·7%	4·3%	9%	**11·9%**	13·9%	15·2%
$(\rho, \sigma) =$	(1·811, 1·0476)	(1·4187, 1·3373)	(1·2526, 1·5145)	(1·1562, 1·6408)	(1·0914, 1·7382)	(1·044, 1·817)
$x = 0{\cdot}02$	−5·7%	4·9%	**10·7%**	14·3%	16·7%	18·3%
$(\rho, \sigma) =$	(1·668, 0·9649)	(1·3067, 1·2317)	(1·1538, 1·3949)	(1·0649, 1·5113)	(1·0053, 1·601)	(0·9616, 1·6737)
$x = 0{\cdot}025$	−7·1%	5·6%	12·3%	16·52%	19·3%	21·2%
$(\rho, \sigma) =$	(1·548, 0·8955)	(1·2127, 1·1432)	(1·0708, 1·2946)	(0·9884, 1·4026)	(0·933, 1·4859)	(0·8925, 1·553)

(where $g(x)$ is the input gluon distribution function and h is a function of Q^2) then.

$$\frac{\mathrm{d}G(x,Q^2)}{G(x,Q^2)} = \frac{\mathrm{d}g(x)}{g(x)} + \frac{\mathrm{d}h(Q^2)}{h(Q^2)}. \tag{10}$$

This shows that the variation of $G(x,Q^2)$ is the sum of the individual variations of the x and Q^2 dependent functions. Keeping Q^2 fixed, variation of $G(x, Q^2$ (fixed)) is equal to the variation of the input gluon distribution function, which makes the solutions to be input sensitive.

Thus, we have demonstrated that the lower limits on the scaling variables of DAS can be obtained analytically in terms of QCD parameters. The lower bound on ρ agrees well with the numerical analysis based on fit obtained from data. Above this lower bound on ρ, 'soft' gluon dominates the physics at small-x validating DAS. For complete analysis of HERA data which are out of DAS, factored form of gluon evolution function can be used near $\rho = 1$.

Acknowledgement

The author is grateful to R Deka and D K Choudhury for useful discussions.

References

[1] R D Ball and S Forte, *Phys. Lett.* **B335**, 77 (1994); **B336**, 77 (1994)
[2] J R Forshaw and R G Roberts, *Phys. Lett.* **B351**, 308 (1995)
[3] V N Gribov and L N Lipatov, *Sov. J. Nucl. Phys.* **15**, 438 (1972)
L N Lipatov, *Sov. J. Nucl. Phys.* **20**, 94 (1974)
G Attarelli and G Parisi, *Nucl. Phys.* **B126**, 298 (1977)
[4] D K Choudhury and A Saikia, *Pramana – J. Phys.* **33**, 359 (1989); **34**, 85 (1990); **38**, 313 (1992)
[5] R Deka and D K Choudhury, *Z. Phys.* (submitted)
[6] L V Gribov, E M Levin and M G Ryskin, *Phys. Rep.* **C100**, 1 (1983)
[7] R G Robert, in *The structure of proton* (Cambridge University Press, Cambridge, 1990)
[8] Y L Dokshitzer, *Sov. Phys. JETP* **46**, 641 (1977)
[9] A De Rujula, S L Glashow, H D Politzer, S B Trieman, F Wilczek and A Zee, *Phys. Rev.* **D10**, 1649 (1974)
T De Grand, *Nucl. Phys.* **B151**, 485 (1979)
J P Ralston and D W Mc Kay, in *Physics simulations at high energies* edited by V Barger (World Scientific, Singapore, 1977)
J P Ralston, *Phys. Lett.* **B172**, 430 (1986)
[10] M Gluck, E Reya and A Vogt, *Z. Phys.* **C53**, 127 (1992)
[11] D K Choudhury, R Deka and A Saikia, *Z. Phys.* (submitted)