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A class of stationary rotating string cosmological models

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Abstract. We obtain a one parameter class of stationary rotating string cosmological models of which the well-known Gödel universe is a particular case. By suitably choosing the free parameter function, it is always possible to satisfy the energy conditions. The rotation of the model hinges on the cosmological constant which turns out to be negative. String-dust distribution in Gödel-type universes is also briefly discussed.

Keywords. General relativity; exact solutions; rotating string cosmology.

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1. Introduction

Cosmic strings have been considered in the study of early universe cosmology. They may be one of the sources of density perturbations that are required for formation of large scale structure in the universe [1, 2]. They possess stress-energy and hence couple to gravitational field. Their various features have been considered by some authors [3–5]. Cosmic strings as source of gravitational field in general relativity (GR) was discussed by Letelier [6] and Stachel [7]. Letelier [8] has further constructed string cosmological models for Bianchi I and Kantowski–Sachs spacetimes by introducing the energy-momentum tensor

$$T_{ik} = \rho u_i u_k - \lambda w_i w_k, \quad u_i u^i = 1 = -w_i w^i, \quad u_i w^i = 0 \tag{1.1}$$

as the source term in Einstein's equation

$$R_{ik} - \tfrac{1}{2} R g_{ik} = -8\pi T_{ik} - \Lambda g_{ik}, \tag{1.2}$$

where Λ is the cosmological constant. T_{ik} represents the energy momentum of a cloud of strings attached with mass particles. The density ρ is made up of particle density ρ_p and the string tension density λ, and is given by

$$\rho = \rho_p + \lambda. \tag{1.3}$$

The energy conditions will require $\rho \geqslant 0$, $\rho_p \geqslant 0$ leaving the sign of λ undetermined. However, λ has to be positive whenever $\rho_p = 0$. The matter flow and the string fibre directions are respectively specified by the unit timelike u^i and spacelike w^i vectors.

String cosmological models have been studied for Bianchi type spacetimes by several authors [9–13]. It is also shown that, cylindrically symmetric non-singular spacetimes also admit physically reasonable string cosmological models [14]. So far all the models considered are free of rotation.

In this paper we obtain stationary rotating string solutions of Einstein's equation. We have obtained a one parameter class of rotating string spacetimes and the free function can be suitably chosen to satisfy the energy conditions. The well-known rotating Gödel universe follows as a special case of this class. It turns out that the cosmological constant Λ plays a very important role in the sense that it measures rotation as well as particle density ρ_p. We also consider string dust distribution in Gödel universe.

In § 2 we give the basic equations and their solutions are considered in § 3. String dust distribution in Gödel universe is discussed in § 4 and we conclude with remarks in § 5.

2. Basic equations

We consider the stationary line-element admitting R^3 group of Killing vectors in the form

$$ds^2 = -dx^2 - \alpha^2 dy^2 - dz^2 + (dt + H dy)^2, \tag{2.1}$$

where α and H are functions of x alone.

We introduce the orthonormal tetrad $\theta^1 = dx$, $\theta^2 = \alpha dy$, $\theta^3 = dz$ and $\theta^4 = dt + H dy$ and in what follows all quantities will be referred to the tetrad frame.

The surviving R_{ab} are given as follows:

$$\begin{aligned} R_{11} &= R_{22} = \alpha''/\alpha - H'^2/2\alpha^2, \\ R_{44} &= -H'^2/2\alpha^2, \quad R_{24} = -(1/2\alpha)(H'' - H'\alpha'/\alpha), \end{aligned} \tag{2.2}$$

where overhead prime denotes differentiation with respect to x. Substituting this in (1.2) and using (1.1), we get

$$R_{24} = 0, \tag{2.3}$$

$$R_{11} = -4\pi(\rho + \lambda) - \Lambda, \tag{2.4}$$

$$R_{44} = 2\Lambda = 8\pi(\lambda - \rho), \tag{2.5}$$

where we have used $u_i = \delta_i^4$ and $w_i = \delta_i^3$ (string is along the z-axis).

From eqs (2.2)–(2.5) we readily obtain

$$H' = m\alpha, \quad 8\pi\rho = m^2 - \alpha''/\alpha, \quad 8\pi\lambda = m^2/2 - \alpha''/\alpha, \tag{2.6}$$

where m is a constant of integration. Clearly $\rho \geqslant 0$ is ensured if $\alpha'' \leqslant 0$ and the particle density

$$\rho_p = \rho - \lambda = m^2/16\pi \geqslant 0 \tag{2.7}$$

whereas the cosmological constant

$$\Lambda = -m^2/4 \leqslant 0. \tag{2.8}$$

The vorticity of the velocity field, $\Omega = w_{ab}w^{ab}$ turns out to be

$$\Omega = \sqrt{2}m = 2\sqrt{-2\Lambda} \tag{2.9}$$

which will vanish only when $\Lambda = 0$ and so does the particle density ρ_p.

The metric function α is undetermined and hence we have one parameter class of rotating string spacetimes. In the following we consider some simple interesting cases.

3. Solutions

Solution 1. The simplest case will obviously be $\alpha = x$ leading to

$$\lambda = \rho_\mathrm{p} = \rho/2 = m^2/16\pi, \quad H = mx^2/2. \tag{3.1}$$

Then the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}x^2 - x^2\mathrm{d}y^2 - \mathrm{d}z^2 + (\mathrm{d}t + \tfrac{1}{2}mx^2\mathrm{d}y)^2. \tag{3.2}$$

Solution 2. Let us put $\lambda = 0$ which will imply $\alpha = \mathrm{e}^{mx/\sqrt{2}}$, $H = \sqrt{2}\alpha$. Then we obtain

$$\mathrm{d}s^2 = -\mathrm{d}x^2 - \mathrm{e}^{\sqrt{2}mx}\mathrm{d}y^2 - \mathrm{d}z^2 + (\mathrm{d}t + \sqrt{2}\mathrm{e}^{mx/\sqrt{2}}\mathrm{d}y)^2 \tag{3.3}$$

which is the well-known Gödel universe [15] with $8\pi\rho = m^2/2 = -2\Lambda$.

Solution 3. Let us consider the equation of state of the kind $\rho = (1+k)\lambda$ where k is a positive constant. Then we have

$$\alpha'' + \frac{1-k}{2k}m^2\alpha = 0,$$

the solution of which depends upon the sign of $(1-k)/2k$.

Case (i) $a^2 = ((1-k)/2k)m^2 > 0$. In that case,

$$\alpha = \cos ax, \quad 8\pi\rho = \frac{m^2(1+3k)}{2k}, \quad H = -\frac{m}{a}\sin ax$$

and the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}x^2 - \cos^2 ax\mathrm{d}y^2 - \mathrm{d}z^2 + \left(\mathrm{d}t - \frac{m}{a}\sin ax\mathrm{d}y\right)^2. \tag{3.4}$$

Case (ii) $-b^2 = ((1-k)/2k)m^2$. We then obtain

$$\alpha = \mathrm{e}^{bx}, \quad 8\pi\rho = \frac{1+k}{2k}, \quad H = \frac{m}{b}\mathrm{e}^{bx}$$

and

$$\mathrm{d}s^2 = -\mathrm{d}x^2 - \mathrm{e}^{2bx}\mathrm{d}y^2 - \mathrm{d}z^2 + \left(\mathrm{d}t + \frac{m}{b}\mathrm{e}^{bx}\mathrm{d}y\right)^2. \tag{3.5}$$

Solution 4. Let us consider the case of vanishing Λ which means $\rho_p = \Omega = 0$ and $\rho = \lambda$. This is the case of the universe filled with the cosmic strings alone. Note that α still remains free to be chosen. We choose $\alpha''/\alpha = -n^2$, n being a constant. Then we get

$$\rho = \lambda = n^2/8\pi$$

and the metric has the simple form

$$ds^2 = -dx^2 - \cos^2 nx dy^2 - dz^2 + dt^2. \tag{3.6}$$

It is interesting to note that ρ is a constant and switching that off leads to flat spacetime.

This metric can be transformed to the solution for the interior of a string given by Hiscock and Gott [4, 16]. Raychaudhuri [17] criticized their interpretation and suggested that it would represent a homogeneous magnetic universe with Λ. This is what was exactly done by Patel and Vaidya [18] much earlier.

Solution 5. Raychaudhuri and Guha Thakurta [19] have investigated the conditions for the metric (2.1) to be homogeneous from Killing equations without reference to the field equations. These conditions are

$$\alpha''\alpha - \alpha'^2 = \text{const.}, \frac{H'}{\alpha} = \text{const.} \tag{3.7}$$

All the four solutions cited above satisfy the conditions (3.7). Hence they are homogeneous. But our metric function α is arbitrary and therefore it is not necessary to satisfy the first condition in (3.7). For example, let us take α in the form

$$\alpha = A\cosh^2\left(\frac{m}{2}x\right), \tag{3.8}$$

where A is a constant.

In this case we have

$$8\pi\rho = \tfrac{1}{2}m^2 \operatorname{sech}^2\left(\frac{m}{2}x\right), \quad 8\pi\lambda = -\frac{m^2}{2}\tanh^2\left(\frac{m}{2}x\right),$$

$$H = \tfrac{1}{2}mx + \frac{A}{2}\sinh mx. \tag{3.9}$$

Thus the choice (3.8) of α gives us an inhomogeneous string cosmological model in which the string-dust density is always negative. This solution also satisfies the energy conditions. This is just an illustration of inhomogeneous models.

4. String-dust in Gödel-type universes

We now remove the cosmological constant Λ and bring in an isotropic pressure by considering geometric strings (string-dust) in the Gödel type universes described by the metric (2.1). Here the energy momentum tensor T_{ik} is given by

$$T_{ik} = \lambda(u_i u_k - w_i w_k) + (p + \rho)u_i u_k - p g_{ik}. \tag{4.1}$$

It is easy to see that Einstein's field equation (1.2) for T_{ik} given by (4.1) reduces to the following system of equations:

$$R_{24} = 0, \tag{4.2}$$

$$8\pi p = -\tfrac{1}{2}R_{44} = 8\pi\rho, \tag{4.3}$$

$$8\pi\lambda = -R_{22}. \tag{4.4}$$

Using eqs (2.2) we get

$$\frac{H'}{\alpha} = m = \text{const.}, \; 8\pi p = 8\pi\rho = \frac{m^2}{2},$$

$$8\pi\lambda = \frac{m^2}{2} - \frac{\alpha''}{\alpha}. \tag{4.5}$$

Thus by specifying λ or α the solution gets determined. By taking $\alpha''/\alpha \leqslant 0$, we can make $\lambda > 0$. $\lambda = 0$ gives us the Gödel universe. For different choices of α, we get different string-dust distributions in Gödel universe.

5. Concluding remarks

It can be easily seen that when $H^2 - \alpha^2 < 0$ the curve with constant t, x and z is closed. For example, consider the case (ii) of solution 3. It is clear that when $k < 1$, the curve $t = \text{const.}$, $x = \text{const.}$, and $z = \text{const.}$ is closed. However of late there is a growing opinion to keep the question of causality open. There are two opposing views: (i) the laws of physics forbid the occurrence of closed time-like lines [20] and (ii) the laws of physics allow closed time-like lines and nature exhibits them [21, 22]. Thus the question of occurrence of closed timelike curves is far from resolved.

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