PRAMANA
— journal of
physics

© Printed in India

Vol. 47, No. 5,
November 1996
pp. 339–345

Dynamics of the logistic map under discrete parametric perturbation

P P SARATCHANDRAN*, V M NANDAKUMARAN and G AMBIKA[†]
International School of Photonics, Cochin University of Science and Technology, Cochin 682 022, India
[†]Department of Physics, Maharajas College, Cochin 682 011, India
*Permanent Address: Department of Physics, The Cochin College, Cochin 682 002, India

MS received 30 September 1995; revised 12 August 1996

Abstract. By introducing a periodic perturbation in the control parameter of the logistic map we have investigated the period locking properties of the map. The map then gets locked onto the periodicity of the perturbation for a wide range of values of the parameter and hence can lead to a control of the chaotic regime. This parametrically perturbed map exhibits many other interesting features like the presence of bubble structures, repeated reappearance of periodic cycles beyond the chaotic regime, dependence of the escape parameter on the seed value and also on the initial phase of the perturbation etc.

Keywords. Logistic map; periodic perturbation; bubble structures; fractal basin of escape.

PACS No. 05·45

1. Introduction

Nonlinear difference equations have proved to be efficient mathematical models in the study of many physical and biological systems [1]. The simplest and one of the most extensively studied nonlinear difference equations is the logistic map

$$X_{n+1} = 4\lambda X_n(1 - X_n) \quad 0 \leqslant X, \lambda \leqslant 1. \tag{1}$$

In actual experimental situations the control parameter represents the ambient conditions such as voltage, discharge current, intensity of the electromagnetic field, etc [2, 3] which in turn can often have a time dependence. The simplest case of a time dependence is a linear time dependence [4], where properties like bistability and hysteresis have been observed. One can consider a more general situation in which λ itself evolves as a discrete nonlinear map and most often X gets enslaved to the periodicity of the perturbation [5, 6]. Quadratic maps with additive periodic forcing leading to bistability and co-existence of multiple attractors have also been studied [7]. Time dependence can also be incorporated into the dynamics of (1) as a periodic perturbation [8–11].

In this paper we consider a situation wherein instead of changing λ continuously it is changed in a discrete sequence as a train of pulses repeated periodically, the envelope of the amplitude of the pulses forming a positive sine profile. Such a sequence of pulses may be relevant in the study of biological systems subjected to periodic stimuli [12].

With a similar perturbation, but with a cosine profile that includes both the positive and negative half cycles, it has been found that the map undergoes a transition from a fixed point and gets locked into the periodicity of the perturbation [14]. For odd periodic perturbation, the period locked system undergoes the usual period doubling cascade whereas for even periodic perturbation there are multiple period locked cycles co-existing for an interval and having different basins of attraction. These period locked cycles then undergo period doublings independently. In our work we have restricted the sine profile in the perturbation to its positive half cycle. We have already analyzed the symbolic dynamics of the period-locked cycles in the presence of this perturbation [13] and observed many sequences that do not fall under the MSS prescription. We explain the presence of bubble structures in certain bifurcation diagrams. So also the parameter value for escape is dependent both on the initial phase and on the seed value and that the basin for escape is a fractal. The dimension D_0 of this basin has been computed. The difference in the behaviour of the map for odd and even perturbations as reported in [14] is absent in the present model. The paper is organized as follows. Section 2 discusses the dynamics of the perturbed map for perturbations of various periods with the help of the bifurcation diagrams. Section 3 attempts an explanation for the presence of bubble structures in the bifurcation diagram. In §4 the phase dependent nature of the escape parameter is detailed. Our concluding remarks are given in the last section.

2. Parametrically perturbed logistic map

In this section we outline the nature of the perturbation employed in the system and its effects on the bifurcation diagrams of the map. The control parameter of the logistic map is perturbed by a periodic perturbation so that it takes the form

$$\lambda_{n+1} = \lambda^0 + \lambda^1 \sin\phi_{n+1} \bmod \pi, \tag{2}$$

where

$$\phi_{n+1} = \phi_n + \pi\omega. \tag{3}$$

Here λ^0 is the time independent part and λ^1 refers to the amplitude of the time dependent part of the control parameter. For a rational ω, say $\omega = p/q$, the control parameter forms a q cycle with cycle elements.

$$\lambda_q = \lambda^0 + \lambda^1 \sin((n-1)\pi/q) \text{ where } n = 0, 1, 2, \ldots, q-1. \tag{4}$$

The logistic map with the sinusoidal perturbation can then be written as

$$X_{n+1} = 4(\lambda^0 + \lambda^1 \sin\phi_n) X_n (1 - X_n). \tag{5}$$

If $L(X) = 4X(1-X)$ the parametrically perturbed map becomes

$$X_{n+1} = \lambda^0 L(X_n) + \lambda^1 \sin\phi_n L(X_n). \tag{6}$$

We can easily identify λ^0 $L(X)$ as the usual logistic map and $\lambda^1 \sin\phi_n L(X)$ as the perturbation component. The dynamics of the parametrically perturbed map (6) has been studied by fixing λ^0 and varying the strength of the perturbation amplitude λ^1. To obtain a wide tunability for λ^1 we fix $\lambda^0 = 0{\cdot}1$. The advantage of fixing λ^0 at $0{\cdot}1$ is that the unperturbed logistic map possesses only one stable cycle, a stable fixed point $x^* = 0$ so that the full effect of the perturbation can be studied. At $\lambda^1 = \lambda^{1*}$ determined by

$$+1 = \prod_{i=0}^{q-1} 4^q(\lambda^0 + \lambda^{1*}\sin\pi i/q) \tag{7}$$

the fixed point changes stability with a q cycle. This q cycle thereafter undergoes period doubling bifurcations. If λ^0 is such that the unperturbed logistic map has a stable k cycle then the perturbation would result in a transition from k cycle to kq cycle. The transition from a fixed point to a q cycle in such cases can still be observed by a perturbation which is out of phase i.e., $\lambda^1 \sin\phi_n \to -\lambda^1 \sin\phi_n$, the value of λ^{1*} can be obtained from (7). Figures 1 and 2 give the bifurcation diagram for the map (5) for an odd ($q = 3$) and an even ($q = 6$) periodic perturbation respectively. It is clear from table 1 that the stability zone for the various q cycles have increased considerably in comparison with the respective stability zones in the unperturbed logistic map. This perturbation also provides a prescription for period-locking the system to any desired periodicity. It is obvious from figure 1 that the q cycle becomes superstable more than once within a zone of stability. This is in contrast with the behaviour of the unperturbed logistic map (1). The symbolic dynamics of these superstable cycles have already been analyzed [13]. Also the cycles possess sequences which do not fall under the usual MSS prescription in addition to MSS ones.

3. Bubble structures, co-existence of multiple attractors in the parametrically perturbed logistic map

In this section we explain the presence of bubble structures in the bifurcation diagram of (5) for certain q values (for e.g. $q = 10$). A bubble is formed when a periodic cycle of q period doubles but later recombines to form a q cycle again. Such bubble structures

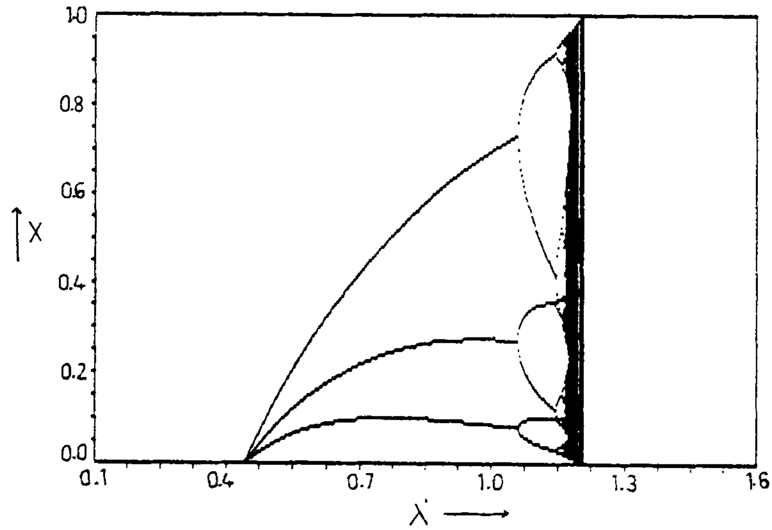

Figure 1. The bifurcation diagram of the map for $q = 3$. Note the widening of the zone of stability for the 3 cycle which appears only as a small window in the logistic map (1).

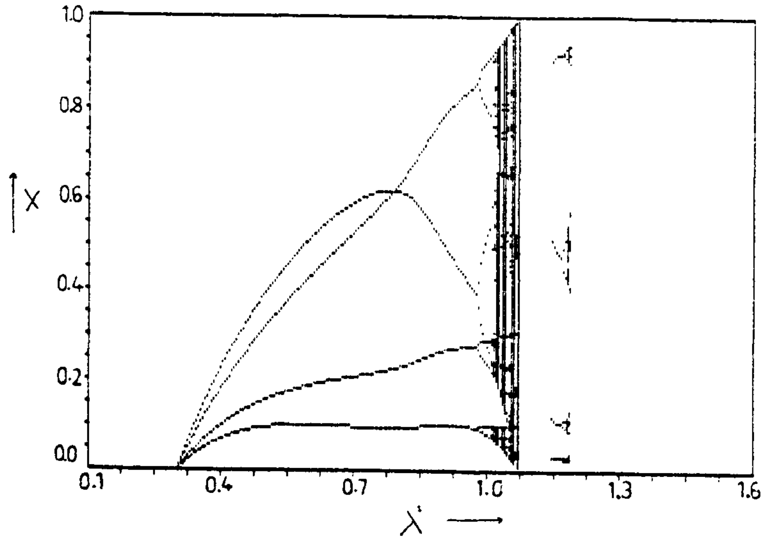

Figure 2. The bifurcation diagram of the map for $q = 4$.

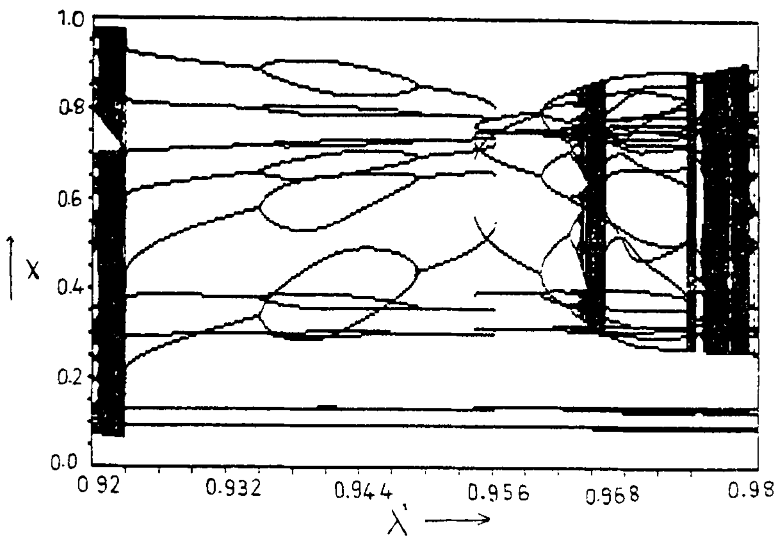

Figure 3. Section of the bifurcation diagram for $q = 10$. The diagram has been drawn using different initial values. Notice the hysteresis region in the range $0{\cdot}953 < \lambda^1 < 0{\cdot}956$.

have been reported earlier [15–17]. We find that the symmetry condition for the bubble structures as reported in [16] is not a necessary condition for the formation of bubbles. Figure 3 plots the bifurcation diagram for $q = 10$ in the parameter range $0{\cdot}9 < \lambda^1 < 1$. Here a 10 cycle reappears after the period doubling cascade and remains stable up to $\lambda^1 \approx 0{\cdot}935$. This 10 cycle then period doubles but recombines later to form a 10 cycle again thus giving rise to a bubble structure. This bubble structure is followed by

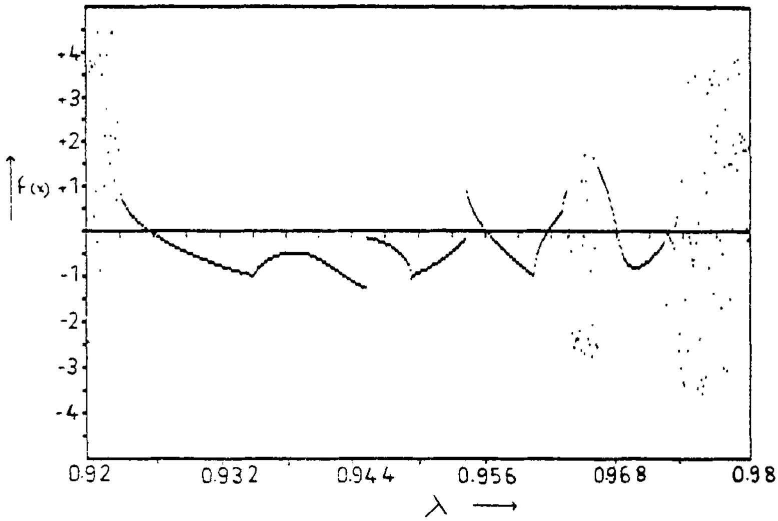

Figure 4. Variation of the $f^{10'}(X)$ with λ^1. Notice the discontinuity in slope in the hysteresis region.

a hysteresis region where a new 10 cycle coexists with this 10 cycle in the parameter range $0{\cdot}953 < \lambda^1 < 0{\cdot}956$. Beyond $\lambda^1 = 0{\cdot}956$ the original 10 cycle loses stability and the new 10 cycle later undergoes period doubling bifurcations. In the region of coexistence the two 10 cycles have their basins of attraction intertwined, which is quite different from earlier reports [7, 14, 18]. Bubble structures have also been found for $q = 5, 8$. We have found that for this map bubble structures are always being followed by a region of co-existing attractors having intertwined basins of attraction. These sequence of events can be visualized in the plot of $f^{10'}(x)$ vs λ^1 given in figure 4. The birth of this new 10 cycle is simultaneous with the discontinuity in the $f^{10'}(x)$ curve. We also find that the range of λ for which the system has bounded orbits is discontinuous, that is there are regions of escape within the tuning range of the parameter λ.

4. Fractal nature of the basin of escape

In this section we discuss the fractal nature of the basin of escape for the perturbed logistic map (5). For the usual logistic map (1) the range of the control parameter for bounded orbits is $0 \leqslant \lambda \leqslant 1$. The value of the control parameter is $\lambda = 1$ beyond which successive iterates of the map speeds away rapidly towards $-\infty$. It has also been found that beyond $\lambda = 1$ those seed values which remain in the [0, 1] after infinite iterations form a cantor set [19]. Therefore $\lambda = 1$ is considered as the parameter value for escape from the interval [0, 1]. In map (5) the parameter value for escape λ_e is found to be dependent both on the initial phase and seed value of the iteration. For a given seed value X_0 the iterates may either remain within the interval [0, 1] or go out depending on the initial phase of iteration.Thus for $q = 10$ and $\phi_0 = \pi/10$ the iterates escape towards $-\infty$. If we choose a seed value $X_0 = 0{\cdot}3$ the iterates are unbounded. In fact for $\omega = 1/4$ and $\lambda^1 = 0{\cdot}904$, a parameter value for which a superstable 4 cycle exist we find

Table 1. Stability zone for the q cycle in the parametrically perturbed logistic map

q	Stability zone
2	$0{\cdot}52500 < \lambda^1 < 2{\cdot}0211$
3	$0{\cdot}34096 < \lambda^1 < 0{\cdot}959$
4	$0{\cdot}30034 < \lambda^1 < 0{\cdot}9761$
5	$0{\cdot}28406 < \lambda^1 < 0{\cdot}847$
6	$0{\cdot}27578 < \lambda^1 < 0{\cdot}8541$
7	$0{\cdot}27094 < \lambda^1 < 0{\cdot}907$
8	$0{\cdot}26788 < \lambda^1 < 0{\cdot}883$
9	$0{\cdot}26579 < \lambda^1 < 0{\cdot}9234$
10	$0{\cdot}26432 < \lambda^1 < 0{\cdot}8966$

Table 2. The capacity dimension of the basin of escape for $\omega = 1/10$ for $\phi_0 = \pi/10$.

λ^1	D_0
0·923	0·8526099
0·94	0·9071668
0·948	0·9581154
0·956	0·9728904

that certain seed values do not converge onto the 4 cycle, instead they go out of the interval. This basin of escape is of fractal nature. The fractal dimension D_0 has been calculated and tabulated in table 2.

5. Concluding remarks

We have studied the dynamics of a parametrically perturbed logistic map. The map can be an efficient model in the study of physical systems subjected to periodic stimuli [12]. The map is shown to exhibit many interesting properties like bubble structures, hysteresis region, reappearence of periodic cycles beyond chaotic region. The dynamic region of the control parameter appears to be discontinuous. We find that the escape parameter λ_e depends on the initial phase ϕ_0 and seed value X_0. The basin for escape has a fractal structure. We have also found that the range of λ^1 for which bounded orbits exist is discontinuous.

References

[1] R M May, *Nature (London)* **261**, 459 (1976)
[2] M Ciofini, R Meucci and F T Arecchi, *Phys. Rev.* **E52**, 95 (1995)
[3] R Badii, E Brun, M Finardi, L Flepp, R Holzner, J Parisi, C Reyl and J Simonet, *Rev. Mod. Phys.* **66**, 1389 (1994)
[4] R Kapral and P Mandel, *Phys. Rev.* **A32**, 1076 (1985)

[5] K P Harikrishnan and V M Nandakumaran, *Phys. Lett.* **A125**, 465 (1987); *Pramana – J. Phys.* **29**, 533 (1987); *Phys. Lett* **A133**, 305 (1988)
[6] S Batra and V S Varma *Pramana – J. Phys.* **37**, 83 (1991)
[7] Sanju and V S Varma, *Phys. Rev.* **E48**, 1670 (1993)
[8] R Lima and M Pettini, *Phys. Rev.* **A41**, 726 (1990)
[9] Y C Lai, M Ding and C Grebogi, *Phys. Rev.* **E47**, 86 (1993)
[10] Y C Lai, C Grebogi, *Phys. Rev.* **E49**, 1094 (1994)
[11] W Yang, M Ding, A J Mandell and E Ott, *Phys. Rev.* **E51**, 102 (1995)
[12] S J Schiff, K Jerger, D H Duong, T Chang, M L Spano and W L Ditto, *Nature (London)* **370**, 615 (1994)
[13] P P Saratchandran, V M Nandakumaran and G Ambika, *Proc. of the Seminar on Computational Aspects in Chaos and Nonlinear Dynamics* edited by G Ambika and V M Nandakumaran (Wiley Eastern Ltd., 1994) p. 215
[14] M Lucke and Y Saito, *Phys. Lett.* **A91**, 205 (1982)
[15] M Bier and T C Bountis, *Phys. Lett.* **A104**, 239 (1982)
[16] T Midavaine, D Dangoisse and Glorieux, *Phys. Rev. Lett.* **55**, 1989 (1985)
[17] P R Krishnan Nair, V M Nandakumaran and G Ambika, *Proc. of the Seminar on Computational Aspects in Chaos and Nonlinear Dynamics* edited by G Ambika and V M Nandakumaran (Wiley Eastern Ltd., 1994) p. 144
[18] C Grebogi, E Ott and J A Yorke, *Physica* **D7**, 181 (1983)
[19] R L Devaney, *An introduction to chaotic dynamical systems* (The Bengamin/Cummings Publishing Co., 1989)