PRAMANA
— journal of
physics

© Printed in India

Vol. 47, No. 3,
September 1996
pp. 211–224

Stochastic motion of a charged particle in a magnetic field: II Quantum Brownian treatment

SUSHANTA DATTAGUPTA and JAGMEET SINGH*
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India
*Also at: Physics Department, S.G.T.B. Khalsa College, University of Delhi, Delhi 110007, India

MS received 2 May 1996

Abstract. We study the quantum Brownian motion of a charged particle in the presence of a magnetic field. From the explicit solution of a quantum Langevin equation we calculate quantities such as the velocity correlation function and the mean-squared displacement. Our calculated expressions contain as special cases the motion of a *classical* particle in a magnetic field and that of a *free* (but quantum) particle, in a dissipative environment.

Keywords. Magnetic field-induced dynamics; quantum Langevin equation; velocity correlation; quantum diffusion.

PACS Nos **05·40; 42·50; 41·70**

1. Introduction

In the preceding paper [1] (henceforth referred to as I), we have considered the dissipative dynamics of a charged particle in a magnetic field. Two distinct stochastic models were employed: one based on diffusion processes as appropriate to classical Brownian motion, the other based on jump processes as in the Boltzmann–Lorentz model of classical kinetic theory. Our objective here is to extend the previous classical treatment to the realm of quantum mechanics, for the present to the case of diffusion processes. The case of quantum Boltzmann–Lorentz model will be dealt with later.

The quantum dynamics of a charged particle in the presence of a magnetic field is characterized by coherent (Larmor) precession in circular orbits around the magnetic field. This occurs at a rate given by the cyclotron frequency $\omega_c = |e|B/_{mc}$, where e is the charge, B is the strength of the field, m is the mass of the particle and c is the speed of light. Dissipation leads to incoherence in the motion, brought about by a characteristic rate generally known as the friction coefficient γ. When $\gamma \gg \omega_c$, one expects to see classical-like evolution in an otherwise quantum problem. In addition to γ and ω_c, there is of course a third frequency in the system, $\nu = k_B T/_{\hbar}$, where k_B is the Boltzmann constant, T is temperature and $\hbar$ is the Planck constant. The frequency ν is ubiquitous in a quantum system at a finite temperature characterizing the interplay of thermal and quantal fluctuations. It can be as large as $10^{11}\,s^{-1}$ even at such low temperatures as 1 K which means that one has to probe within a time 10^{-11}s in order to see quantum coherence effects. From the preceding discussion it follows that we are in a situation in which we expect to see different behaviour in different regimes of the three competing frequency scales set by ω_c, γ and ν. Furthermore, it is also evident that the case at hand belongs to a large class of problems which falls in the domain of quantum dissipative systems. Such problems permeate wide ranging areas of physics including quantum optics, condensed matter physics, chemical physics and quantum measurement theory [2]. Thus, the presently studied system of quantum Brownian motion of a charged particle in the presence of a magnetic field can be viewed as a paradigm of irreversible behaviour of a quantum system which is otherwise governed by unitary evolution.

Having presented the general background to our present investigation, it is important for us to clarify what precisely is meant by "quantum Brownian motion". In answering this question we follow the method of Caldeira and Leggett (CL) in which a system-plus-reservoir approach is adopted [3]. The reservoir is constituted of an infinitely large number of quantum harmonic oscillators which are linearly coupled to the system coordinates. While the details of this model will be discussed later, it suffices for the present to state that by choosing a particular density of states for the reservoir oscillators, CL show that in the appropriate classical limit (i.e. $\nu \to \infty$), the probability distribution in phase space follows a Fokker–Planck equation. Since the latter and its cousin, the Langevin equation for position and momentum variables, are believed to provide a theoretical framework describing classical Brownian motion, CL proceed to extend the validity of the model to the quantum domain, and define the resultant process as the "quantum Brownian motion". We shall follow the same approach here, with an additional input of the magnetic field. In this sense, the present treatment may be viewed as the quantum version of the treatment in I, for diffusion processes.

The CL model is based on a Hamiltonian written down by Feynman and Vernon [4]. There is a lengthy discussion about the justification for the chosen form of the coupling between the system and reservoir oscillators, in a review article [5]. The idea is to consider the time evolution of the so called reduced density operator, obtained from the actual density operator by tracing out the reservoir variables. The Weyl mapping is used to obtain the appropriate Wigner distribution function from the reduced density operator. The Wigner function follows the Fokker–Planck equation in the classical limit, for a specific choice of the density of states for the reservoir oscillators [3]. While this approach may be viewed to be based on the Schrödinger-like picture of quantum mechanics, there is a corresponding Heisenberg-like picture adopted by authors such as Ford *et al* [6], Zwanzig [7] and Ford *et al* [8]. In this, one starts from the same Feynman–Vernon Hamiltonian equations of motion for the position and momentum variables for both the system and the reservoir, integrates out the reservoir variables, and obtains the quantum Langevin equation for the system, under identical assumption about the density of states for the reservoir oscillators, as made by CL. The derivation of the quantum Langevin equation can be easily extended to incorporate the presence of the magnetic field [9]. We find the resultant equation extremely amenable for analyzing the dissipative magneto-transport of a charged particle.

With the preceding background to the aim and scope of the paper the outline is as follows. In § 2, we introduce the basic Feynman–Vernon Hamiltonian, both in the absence and presence of magnetic field and sketch the steps leading to the quantum Langevin equation. The solution of the latter plus the results on velocity correlation function are presented in § 3. The question of mean-squared displacement and the concomitant diffusion behaviour are dealt with in § 4. Finally, § 5 contains a few concluding remarks.

2. Quantum Langevin equation

In this section, we give an outline of the derivation of the quantum version of the classical Langevin equation, following the treatment of Ford *et al* [8]. The method uses a system-plus-reservoir approach in which the reservoir coordinates are integrated out from the equations of motion resulting from an underlying Hamiltonian or Lagrangian. In fact, the method is a straight-forward generalization of the derivation of the classical Langevin equation, as in Zwanzig [7]. The starting point is the Hamiltonian, written down by Feynman and Vernon [4]:

$$H = \frac{p^2}{2m} + V(x) + \sum_j [p_j^2/_{2m_j} + \tfrac{1}{2} m_j \omega_j^2 (q_j - x)^2], \tag{1}$$

where p and x are the momentum and position coordinates for the system-particle while p_j and q_j are the corresponding variables for the reservoir. It suffices for the present to consider a one-dimensional case though in the sequel we will be required to treat a three-dimensional model when we discuss the motion of a charged particle in the presence of a magnetic field. The momentum and coordinate variables satisfy the commutation relations

$$[x, p] = i\hbar$$
$$[q_j, p_k] = i\hbar\delta_{jk}. \tag{2}$$

The equations of motion read

$$\dot{x} = [x, H]/i\hbar = p/m,$$
$$\dot{p} = [p, H]/i\hbar = -V'(x) + \sum_j m_j \omega_j^2 (q_j - x),$$
$$\dot{q}_j = [q_j, H]/i\hbar = p_j/m_j,$$
$$\dot{p}_j = [p_j, H]/i\hbar = -m_j \omega_j^2 (q_j - x). \tag{3}$$

Eliminating the momentum variables, we obtain

$$m\ddot{x} + V'(x) = \sum_j m_j \omega_j^2 (q_j - x), \tag{4}$$

$$\ddot{q}_j + \omega_j^2 q_j = \omega_j^2 x. \tag{5}$$

Barring the dependence on the system coordinate x on the right hand side, (5) simply describes harmonic motion and hence the general solution of (5) is given by

$$q_j(t) = q_j^h(t) + x(t) - \int_{-\infty}^{t} \mathrm{d}t' \dot{x}(t') \cos[\omega_j(t - t')], \tag{6}$$

where q_j^h is the solution of the homogeneous equation

$$q_j^h(t) = q_j(0)\cos(\omega_j t) + p_j(0)\frac{\sin(\omega_j t)}{m_j \omega_j}. \tag{7}$$

It is interesting to note that our starting Hamiltonian (1) is time independent and hermitian and therefore, would normally yield unitary evolution. However by choosing the *retarded* solution of the inhomogeneous equation (5) we have (tacitly) induced breaking of the time-reversal invariance. Substitute (6) into (4), to obtain the Langevin equation

$$m\ddot{x} + \int_{-\infty}^{t} \mathrm{d}\tau \mu(t-\tau)\dot{x}(\tau) + V'(x) = f(t), \tag{8}$$

where the friction or the memory function $\mu(t)$ is given by

$$\mu(t) = \sum_j m_j \omega_j^2 \cos(\omega_j t) \Theta(t), \tag{9}$$

$\Theta(t)$ being the Heaviside step function, and the 'noise' is given by

$$f(t) = \sum_j m_j \omega_j^2 [q_j cos(\omega_j t) + \frac{p_j}{m_j \omega_j} \sin(\omega_j t)], \tag{10}$$

where q_j and p_j are time independent operators which obey commutation rules given in (2).

It is evident that the noise is a quantum mechanical operator, whose auto correlation and commutator can be obtained by assuming that in the distant past, the harmonic oscillator reservoir is in thermal equilibrium, at a temperature T. Thus, we can write [8]

$$\langle \{f(t), f(t')\} \rangle = \frac{2}{\pi} \int_0^\infty \mathrm{d}\omega \, \mathrm{Re}[\tilde{\mu}(\omega + i0^+)] \hbar\omega \coth\left(\frac{\hbar\omega}{2k_\mathrm{B} T}\right) \cos[\omega(t-t')], \tag{11}$$

$$\langle [f(t), f(t')] \rangle = \frac{2}{i\pi} \int_0^\infty \mathrm{d}\omega \, \mathrm{Re}[\tilde{\mu}(\omega + i0^+)] \hbar\omega \sin[\omega(t-t')]. \tag{12}$$

Here we have defined the Laplace transform of the memory function as

$$\tilde{\mu}(s) = \int_0^\infty \mathrm{d}t \exp(ist)\mu(t), \quad \mathrm{Im}\, s > 0. \tag{13}$$

Equations (11) and (12) completely characterize quantum Langevin equation (8).

As in the case of classical Langevin equation, it is useful to consider as a special case, constant "friction". The memory kernel $\mu(t-\tau)$ is then replaced by $m\gamma\delta(t-\tau)$, so that $\mathrm{Re}[\tilde{\mu}(\omega + i0^+)]$ reduces to $m\gamma$, a constant. In that case we have the ordinary Langevin equation:

$$m\ddot{x} + m\gamma\dot{x} + V'(x) = f(t), \tag{14}$$

where now (cf. eqs (11) and (12))

$$\langle \{f(t), f(t')\} \rangle = \frac{2m\gamma}{\pi} \int_0^\infty \mathrm{d}\omega \hbar\omega \coth\left(\frac{\hbar\omega}{2k_\mathrm{B} T}\right) \cos[\omega(t-t')], \tag{15}$$

$$\langle [f(t), f(t')] \rangle = \frac{2m\gamma}{i\pi} \int_0^\infty \mathrm{d}\omega \hbar\omega \sin[\omega(t-t')]. \tag{16}$$

Note the intriguing fact that the underlying stochastic process is still non-Markovian, even though there is no memory. This feature has to do with quantum fluctuations, which can be ignored only in the limit of $\nu \to \infty$. In the latter case, of course, the right hand side of (15) reduces to a term proportional to $\delta(t-\tau)$, restoring Markovianness (cf. eq. (5) of I).

We now turn our attention to the situation in which the particle constituting the system is electrically charged and under the influence of an external magnetic field **B**. The Hamiltonian in (1) is then generalized to [9]

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{r}) + \sum_j \left[\frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(\mathbf{q}_j - \mathbf{r})^2\right], \tag{17}$$

where e is the charge of the particle and **A** is the vector potential, in terms of which

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{18}$$

As before (cf. eq. (2)), the commutation relations are

$$[\mathbf{r}_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}, \quad [q_{j\alpha}, p_{k\beta}] = i\hbar\delta_{jk}\delta_{\alpha\beta}, \tag{19}$$

where the Greek symbols stand for the cartesian indices x, y and z.

The equations of motion for the reservoir operators are simply the three-dimensional generalization of the corresponding equations listed in (3), whereas those for the particle have additional terms due to the presence of the magnetic field. In particular, the generalized version of (4) reads [9]

$$m\ddot{\mathbf{r}} + \nabla V(\mathbf{r}) = \sum_j m_j\omega_j^2(\mathbf{q}_j - \mathbf{r}) + \frac{e}{c}(\mathbf{v} \times \mathbf{B}). \tag{20}$$

We may emphasize that the only additional contribution comes from the quantum form of the Lorenz force term. Furthermore, since there is no dependence on the vector potential **A** in (20), the present treatment is completely gauge-independent [10].

The rest of the steps are exactly as earlier, leading finally to the following quantum Langevin equation for a charged particle in the presence of a magnetic field

$$m\ddot{\mathbf{r}} + \nabla V(\mathbf{r}) + \int_{-\infty}^{t} \mathrm{d}t'\,\mu(t-t')\dot{\mathbf{r}}(t') - \frac{e}{c}(\dot{\mathbf{r}} \times \mathbf{B}) = \mathbf{f}(t). \tag{21}$$

The spectral properties of $\mathbf{f}(t)$ are the same as before i.e. they are independent of the B-field as one would expect. These are quoted here (see eqs (23) and (24) below) for only the memory-less case in which (21) reduces to

$$m\ddot{\mathbf{r}} + \nabla V(\mathbf{r}) + m\gamma\dot{\mathbf{r}}(t) - \frac{e}{c}(\mathbf{v} \times \mathbf{B}) = \mathbf{f}(t). \tag{22}$$

The auto-correlation and the commutator of $\mathbf{f}(t)$ are given by (cf. eqs (15) and (16))

$$\langle\{f_\alpha(t), f_\beta(t')\}\rangle = \delta_{\alpha\beta}\frac{2m\gamma}{\pi}\int_0^\infty \mathrm{d}\omega\hbar\omega\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right)\cos[\omega(t-t')], \tag{23}$$

$$\langle[f_\alpha(t), f_\beta(t')]\rangle = \delta_{\alpha\beta}\frac{2m\gamma}{i\pi}\int_0^\infty \mathrm{d}\omega\hbar\omega\sin[\omega(t-t')]. \tag{24}$$

Equation (22) is of the same form as its classical counterpart (cf. (4) of I) except that the spectral properties of the noise (now an operator) are much richer in structure (compare

(23) and (24) with (5) of I). As commented earlier, the underlying stochastic process is now non-Markovian and therefore, unlike the classical case, there is no appropriate Fokker–Planck description of the dynamics. Hence, for calculational purposes, it is convenient to directly employ the Langevin equation (22), as we shall discuss in the next section.

3. Velocity correlation

In this section we explicitly solve the quantum Langevin equation (22) in the memory-less case, and analyze the velocity auto-correlation function. For the sake of simplicity we ignore the potential energy $V(\mathbf{r})$. Further, we assume that the magnetic field is directed along the z-axis. The motion along z is merely that of a free, quantum particle in a dissipative environment, while that in the xy-plane is described by the equations

$$\ddot{x} + \gamma\dot{x} - \omega_c\dot{y} = \frac{1}{m}f_x(t)$$

$$\ddot{y} + \gamma\dot{y} + \omega_c\dot{x} = \frac{1}{m}f_y(t), \tag{25}$$

where ω_c is the cyclotron frequency (cf. § I)

$$\omega_c = eB/mc. \tag{26}$$

Introducing

$$Z = x + iy,$$

$$F = f_x + if_y,$$

and

$$\bar{\gamma} = \gamma + i\omega_c, \tag{27}$$

(25) can be expressed in the compact form

$$\ddot{Z} + \bar{\gamma}\dot{Z} = \frac{F(t)}{m}, \tag{28}$$

whose solution reads

$$Z(t) = Z(t_0) + \frac{\dot{Z}(t_0)}{\gamma}[1 - \exp(-\bar{\gamma}(t - t_0))] + \int_{t_0}^{t} \mathrm{d}\tau \exp(-\bar{\gamma}\tau) \int_{t_0}^{\tau} \mathrm{d}t' \exp(\bar{\gamma}t')\frac{F(t')}{m}. \tag{29}$$

Without loss of generality, we may choose

$$Z(t_0) = 0. \tag{30}$$

Corresponding to (29), we have

$$\dot{Z}(t) = \dot{Z}(t_0)\exp[-\bar{\gamma}(t - t_0)] + \exp(-\bar{\gamma}t) \int_{t_0}^{t} \mathrm{d}t' \exp(\bar{\gamma}t')\frac{F(t')}{m}. \tag{31}$$

Consider now the correlation function

$$C(t,t') = \langle \dot{Z}(t)\dot{Z}^{+}(t') \rangle, \quad t > t'. \tag{32}$$

Using the definition of Z as in (27), the correlation function can be alternately expressed as

$$C(t,t') = \langle v_x(t)v_x(t') + v_y(t)v_y(t') \rangle + i\langle (\mathbf{v}(t) \times \mathbf{v}(t'))_Z \rangle. \tag{33}$$

From (31)

$$\begin{aligned} C(t,t') = {} & \langle |\dot{Z}(t_0)|^2 \rangle \exp[-\bar{\gamma}(t-t_0) - \bar{\gamma}^*(t'-t_0)] \\ & + \frac{1}{m^2}\exp(-\bar{\gamma}t - \bar{\gamma}^* t') \int_{t_0}^{t} \mathrm{d}\tau \int_{t_0}^{t'} \mathrm{d}\tau' \exp(\bar{\gamma}\tau + \bar{\gamma}^*\tau') \langle F(\tau)F(\tau') \rangle, \end{aligned} \tag{34}$$

where we have used the fact that the initial velocity $\dot{Z}(t_0)$ is uncorrelated with $F(\tau)$ (from causality) and that

$$\langle F(\tau) \rangle = 0. \tag{35}$$

Since we are ultimately going to take the limit $t_0 = -\infty$, we may ignore the first term in (34) from subsequent analysis. Furthermore, we have (cf. eq. (27))

$$\langle F(\tau)F^{+}(\tau') \rangle = \langle f_x(\tau)f_x(\tau') \rangle + \langle f_y(\tau)f_y(\tau') \rangle, \tag{36}$$

which, from (23) and (24), can be further expressed as

$$\langle F(\tau)F^{+}(\tau') \rangle = \frac{m\gamma}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \hbar\omega \exp[i\omega(\tau-\tau')] \left\{\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right\}. \tag{36'}$$

Substituting (36) in (34), and integrating τ and τ', and taking the limit $t_0 = -\infty$, we finally obtain

$$C(t,t') = \frac{1}{m\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\gamma}{\gamma^2 + (\omega+\omega_c)^2}\hbar\omega \left\{\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right\} \exp[i\omega(t-t')]. \tag{37}$$

Thus the correlation function is manifestly a function of the time difference $(t - t')$ implying stationarity. Hence, t' can be set equal to zero.

Comparing (37) with (33) we conclude that

$$\begin{aligned} & \langle v_x(t)v_x(0) + v_y(t)v_y(0) \rangle \\ & = \frac{1}{m\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\gamma}{\gamma^2 + (\omega+\omega_c)^2}\hbar\omega \left\{\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right\} \cos(\omega\tau), \end{aligned} \tag{38}$$

and

$$\begin{aligned} \langle \mathbf{v}(t) \times \mathbf{v}(0) \rangle_Z = {} & \frac{1}{m\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\gamma}{\gamma^2 + (\omega+\omega_c)^2}\hbar\omega \\ & \times \left\{\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right\} \sin(\omega t). \end{aligned} \tag{39}$$

3a Free particle case

If we set $B = 0$ (i.e. $\omega_c = 0$), we would be naturally considering a free particle and its quantum dissipative motion. This problem was treated in detail by Hakim and Ambegaokar [11]. The latter authors also had started from the Feynman–Vernon Hamiltonian (eq. (1)), but had gone about the calculation in a very different way than the present Langevin treatment. Hakim and Ambegaokar had employed the functional integral representation of the time dependent density matrix of the system, following Caldeira and Leggett [3], and evaluated the so-called influence functional by explicitly diagonalizing the Feynman–Vernon Hamiltonian, in the free-particle case. It is instructive therefore to compare our equations (38) and (39), in the limiting case of $\omega_c = 0$, with the results derived in [11].

Setting $\omega_c = 0$ in (38) and taking cognizance of the (anti-) symmetry of the integrand with respect to ω, we easily find

$$\langle v_x(t)v_x(0) + v_y(t)v_y(0)\rangle = \frac{1}{m\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2+\omega^2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos(\omega t). \tag{40}$$

In the free particle case, of course, all the three cartesian components of the velocity have the same auto-correlation and hence,

$$\langle \mathbf{v}(t)\cdot\mathbf{v}(0)\rangle = \frac{3}{2m\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2+\omega^2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos(\omega t). \tag{41}$$

In particular, the equal time correlation leads to

$$\langle \mathbf{v}^2\rangle = \frac{3}{m\pi}\int_{0}^{\infty} d\omega \frac{\gamma}{\gamma^2+\omega^2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right), \tag{42}$$

which agrees with the answer, in equilibrium, derived by Hakim and Ambegaokar (cf. eq. (82) or (83) of [11]).

On the other hand, if we retain the dependence on the initial epoch t_0, we would have obtained from (34) (after some straightforward algebra)

$$\begin{aligned}\langle \mathbf{v}^2(t)\rangle &= \langle \mathbf{v}^2(t_0)\rangle \exp[-2\gamma(t-t_0)] \\ &\quad + \frac{3\gamma}{m\pi}\int_0^{\infty} d\omega \frac{\hbar\omega}{\gamma^2+\omega^2}\coth\left(\frac{\hbar\omega}{2k_B T}\right) \\ &\quad \times \{1+\exp[-2\gamma(t-t_0)] - 2\exp[-\gamma(t-t_0)]\cos[\omega(t-t_0)]\},\end{aligned} \tag{43}$$

which for $t_0 = 0$, yields

$$\begin{aligned}\langle \mathbf{v}^2(t)\rangle &= \langle \mathbf{v}^2(0)\rangle \exp(-2\gamma t) \\ &\quad + \frac{3\gamma}{m\pi}\int_0^{\infty} d\omega \frac{\hbar\omega}{\gamma^2+\omega^2}\coth\left(\frac{\hbar\omega}{2k_B T}\right) \\ &\quad \times \{1+\exp(-2\gamma t) - 2\exp(-\gamma t)\cos(\omega t)\}.\end{aligned} \tag{43'}$$

The difference between (42) and (43′) is precisely due to the choice of the initial epoch. Equation (42) corresponds to the case in which, in the distant past, the particle is assumed to be decoupled from the reservoir, which was maintained in thermal equilibrium, at a fixed temperature. It is therefore, not surprising that (43′) is in complete agreement with the result obtained by Hakim and Ambegaokar (cf. eq. (82) of [11]), for the initial condition in which the total density matrix is assumed to be the factorized product of the density matrices for the particle and the reservoir.

3b *Classical case*

As stated earlier (22) provides a description of what we have called quantum Brownian motion, presently of a charged particle moving under the influence of a magnetic field. Since quantum mechanics subsumes classical machines we should be able to derive the results obtained in I, for diffusion processes, in the limit of $\hbar \to 0$. We demonstrate this, explicitly, for the velocity auto-correlation.

We focus our attention to (38) and (39), in which the co-tangent function under the integral can be replaced by the inverse of its argument. Therefore, as $\hbar \to 0$,

$$\langle v_x(t)v_x(0) + v_y(t)v_y(0)\rangle = \frac{2k_B T}{m\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2 + [\omega + \omega_c)^2}\cos(\omega t), \qquad (44)$$

and

$$\langle \mathbf{v}(t) \times \mathbf{v}(0)\rangle_Z = \frac{2k_B T}{m\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2 + (\omega + \omega_c)^2}\sin(\omega t). \qquad (45)$$

The integrals in (44) and (45) can be easily evaluated by contour integration in the complex ω-plane, yielding

$$\langle v_x(t)v_x(0) + v_y(t)v_y(0)\rangle = \frac{2k_B T}{m}\exp(-\gamma|t|)\cos(\omega_c t), \qquad (44')$$

and

$$\langle \mathbf{v}(t) \times \mathbf{v}(0)\rangle_Z = \frac{2k_B T}{m}\exp(-\gamma|t|)\sin(\omega_c t). \qquad (45')$$

Since the correlation for the z-component of the velocity is that of a free particle ($\omega_c = 0$), we have from (44′)

$$\langle \mathbf{v}(t)\cdot\mathbf{v}(0)\rangle = \frac{k_B T}{m}\exp(-\gamma|t|)[1 + 2\cos(\omega_c t)]. \qquad (46)$$

As expected (45′) and (46) are in agreement with (9) of I.

4. Mean-squared displacement

It is well known that for classical Brownian motion of a particle the mean-squared displacement is (asymptotically) proportional to time, with the coefficient being the diffusion constant. This behaviour also remains valid when the particle is charged and under the influence of a magnetic field. In that case the mean-squared displacement is given by [12]

$$\langle \mathbf{r}^2(t)\rangle = 6Dt, \qquad (47)$$

where the diffusion coefficient D turns out to be

$$D = \frac{k_B T}{m\gamma} \frac{1 + \omega_c^2/3\gamma^2}{1 + \omega_c^2/\gamma^2}. \tag{48}$$

It is of interest, therefore, to enquire what the mean-squared displacement is, when the charged particle is in quantum Brownian motion. This issue is of basic importance, as certain model calculations suggest that a quantum particle may have "superdiffusive" motion [13].

Before we analyze what the mean squared displacement is, it is useful to consider the following correlation function

$$X(t,t') = \langle Z(t) Z^+(t') \rangle, \quad t \geqslant t'. \tag{49}$$

We note that

$$\mathrm{Re}\, X(t,t') = \langle x(t)x(t') + y(t)y(t') \rangle, \tag{49'}$$

and hence $\lim_{t' \to t} \mathrm{Re}\, X(t,t')$ yields the mean-squared displacement, in the xy plane.

From (29) we obtain

$$\begin{aligned}\langle X(t,t') \rangle = {} & \langle |\dot{Z}(t_0)|^2 \rangle \{[1 + \exp(-\bar{\gamma} t - \bar{\gamma}^* t' + 2\gamma t_0)] \\ & - \exp(-\bar{\gamma}(t - t_0)) - \exp(-\bar{\gamma}^*(t' - t_0))\} \\ & + \frac{1}{m^2} \int_{t_0}^{t} \mathrm{d}\tau \exp(-\bar{\gamma}\tau) \int_{t_0}^{\tau} \mathrm{d}t'' \exp(\bar{\gamma} t'') \int_{t_0}^{t'} \mathrm{d}\tau' \exp(-\bar{\gamma}^* \tau') \\ & \times \int_{t_0}^{\tau'} \mathrm{d}t''' \exp(\bar{\gamma}^* t''') \langle F(t'') F^+(t''') \rangle,\end{aligned} \tag{50}$$

which, upon using (36) and carrying out the integrals, reduces to

$$\begin{aligned}X(t,t') = {} & \frac{\langle |\dot{Z}(t_0)|^2 \rangle}{\gamma^2 + \omega_c^2} \{1 + \exp[-\gamma(t + t' - 2t_0) - i\omega_c(t - t')] \\ & - \exp[-\bar{\gamma}(t - t_0)] - \exp[-\bar{\gamma}^*(t' - t_0)]\} \\ & + \frac{\gamma}{m\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \hbar\omega \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right] \frac{1}{\gamma^2 + (\omega + \omega_c)^2} \\ & \times \left\{ \frac{1}{\omega^2} [1 + \exp(i\omega(t - t')) - \exp(i\omega(t - t_0)) - \exp(i\omega(t' - t_0))] \right. \\ & + \frac{1}{\gamma^2 + \omega_c^2} [1 + \exp(-\bar{\gamma} t - \bar{\gamma}^* t' + 2\gamma t_0) \\ & - \exp(-\bar{\gamma}(t - t_0)) - \exp(-\bar{\gamma}^*(t' - t_0))] \\ & - \frac{i}{\omega\bar{\gamma}} [1 - \exp(-i\omega(t' - t_0))] [\exp(-\bar{\gamma}(t - t_0)) - 1] \\ & \left. + \frac{i}{\omega\bar{\gamma}^*} [1 - \exp(i\omega(t - t_0))] [\exp(-\bar{\gamma}^*(t' - t_0)) - 1] \right\}.\end{aligned} \tag{51}$$

Taking $t' = t$, and setting $t_0 = 0$, we have from (51)

$$
\begin{aligned}
&\langle x^2(t) + y^2(t) \rangle \\
&= \frac{\langle \dot{x}(0)^2 + \dot{y}(0)^2 \rangle}{\gamma^2 + \omega_c^2} \{1 + \exp(-2\gamma t) - 2\exp(-\gamma t)\cos(\omega_c t)\} \\
&+ \frac{1}{m\pi} \int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2 + (\omega + \omega_c)^2} \hbar\omega \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1\right] \\
&\times \left\{ \frac{4}{\omega^2} \sin^2(\tfrac{1}{2}\omega t) + \frac{1}{\gamma^2 + \omega_c^2} [1 + \exp(-2\gamma t) - 2\exp(-\gamma t)\cos(\omega_c t)] \right. \\
&+ \frac{2}{\omega(\gamma^2 + \omega_c^2)} [2\omega_c \sin^2(\tfrac{1}{2}\omega t) - \gamma \sin(\omega t) \\
&+ \exp(-\gamma t)(\omega_c \cos(\omega + \omega_c)t - \omega_c \cos(\omega_c t) \\
&\left. + \gamma \sin(\omega + \omega_c)t - \gamma \sin(\omega_c t)] \right\}. \qquad (52)
\end{aligned}
$$

In order to gain insight into the result in (52) it is instructive to consider certain special cases. The simplest case is of course, that of a free particle ($\omega_c = 0$) in the classical ($\hbar = 0$) Brownian motion. We now have

$$
\begin{aligned}
\langle x^2(t) + y^2(t) \rangle &= \langle \dot{x}(0)^2 + \dot{y}(0)^2 \rangle \left[\frac{1 - \exp(-\gamma t)}{\gamma}\right]^2 \\
&+ \frac{2k_B T}{m} \left\{ \frac{2}{\gamma^2} [\gamma t - (1 - \exp(-\gamma t))] - \left[\frac{1 - \exp(-\gamma t)}{\gamma}\right]^2 \right\}. \qquad (53)
\end{aligned}
$$

Using the equipartition theorem

$$
\langle \dot{x}(0)^2 + \dot{y}(0)^2 \rangle = \frac{2k_B T}{m},
$$

we finally arrive at the familiar result [14]

$$
\langle x^2(t) + y^2(t) \rangle = \frac{4k_B T}{m\gamma^2} [\gamma t - 1 + \exp(-\gamma t)]. \qquad (54)
$$

Evidently, the right hand side of (54) is quadratic in t for short times (i.e. $\gamma t \ll 1$) and is diffusive for long times (i.e. $\gamma t \gg 1$) with the diffusion coefficient (in two dimensions) being $4k_B T/m\gamma$.

The next case to consider is that of a *free* particle ($\omega_c = 0$) in quantum Brownian motion [11]. We have from (52),

$$\langle x^2(t)+y^2(t)\rangle = \langle \dot{x}(0)^2+\dot{y}(0)^2\rangle\left[\frac{1-\exp(-\gamma t)}{\gamma}\right]^2 + \frac{1}{m\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma}{\gamma^2+\omega^2}\hbar\omega\coth\left(\frac{\hbar\omega}{2k_B T}\right) \times \left\{\frac{4}{\omega^2}\sin^2(\tfrac{1}{2}\omega t)+\left[\frac{1-\exp(-\gamma t)}{\gamma}\right]^2 - \frac{2}{\omega}\left[\frac{1-\exp(-\gamma t)}{\gamma}\right]\sin(\omega t)\right\}. \quad (55)$$

Following our discussion in § 3, it is not surprising that (55) is identical to the one derived by Hakim and Ambegaokar (cf. eq. (69a) of [11]), for "factorized" initial condition. Hakim and Ambegaokar have argued that at zero temperature and at long times ($\gamma t \gg 1$), the mean-squared displacement is proportional to $\ln(\gamma t)$ – reminiscent of Sinai-type anomalous diffusion [15]. The logarithmic dependence crosses over to a linear, diffusive one at any finite temperature, for times longer than ν^{-1} where ν has been defined in § 1.

Our final limiting case is that of a charged particle, in classical ($\hbar = 0$) Brownian motion in the presence of a magnetic field. Employing the equipartition theorem (cf. eq. (53)) once again and evaluating the relevant integrals by contour methods, (52) now yields

$$\langle x^2(t)+y^2(t)\rangle = \frac{4k_B T}{m(\gamma^2+\omega_c^2)^2}\{\gamma t(\gamma^2+\omega_c^2)-(\gamma^2-\omega_c^2) + \exp(-\gamma t)[(\gamma^2-\omega_c^2)\cos(\omega_c t)-2\gamma\omega_c\sin(\omega_c t)]\}, \quad (56)$$

the expected result (cf. eq. (40) of I). Asymptotically, for $t \gg \gamma^{-1}$,

$$\langle x^2(t)+y^2(t)\rangle = \frac{4k_B T}{m}\frac{\gamma t}{(\gamma^2+\omega_c^2)}. \quad (57)$$

Since under the same limiting condition, the mean-squared displacement in the direction of the magnetic field is given by the $\omega_c = 0$ limit of eq. (57), we have

$$\langle \mathbf{r}^2(t)\rangle = \frac{2k_B T}{m\gamma}\left(1+\frac{2\gamma^2}{\gamma^2+\omega_c^2}\right)t, \quad (58)$$

which agrees with (47) and (48).

The analysis presented above clearly indicates that the diffusive behaviour in the general case, given by (52) is rather complex, and is not amenable to analytical investigation. The dynamics now is dominated at low temperatures by quantum effects, going over to classical diffusive behaviour at high temperatures.

5. Conclusion

This paper is complementary to I in that we have extended the treatment of diffusive charged particle dynamics in a magnetic field to the quantum regime. We have used an earlier derivation of a quantum Langevin equation by Ford and coworkers. We have solved this equation exactly and the resultant solution has been employed to calculate velocity auto-correlation function and the mean-squared displacement. The present approach has been based on a direct analysis of the equation of motion for observables, as opposed to a master equation for the density matrix or its path integral formulation as made popular in recent years by Leggett and coworkers. However the equivalence between the two approaches has been established by rederiving for the special case of a free quantum particle, the results obtained earlier by Hakim and Ambegaokar.

Our analysis in §4 on the mean-squared displacement shows that the diffusive behaviour in the quantum case is rather complicated. This, of course, is not unexpected, as even a free particle is known to have anomalous diffusion, especially at low temperatures.

One immediate application of our present treatment is to the study of Landau diamagnetism in the presence of dissipation. It is well known that diamagnetism is purely a quantum phenomenon [16]. On the other hand, in real materials, electrons are expected to undergo scattering due to phonons, impurities, ion imperfections, etc. Scattering would naturally lead to dissipative loss and hence the present results are eminently suitable to tackle the issue of what happens to Landau diamagnetism when (quantum) friction is large. The required analysis inter-alia would be automatically a generalization of the corresponding classical treatment of Jayannavar and Kumar [17]. Details of this analysis will be reported in a forthcoming paper.

Acknowledgements

A preliminary version of the work reported in this and the preceding paper (I) was presented in lectures given by SD at an SERC School organized in December 1995 on "Magnetic Reconnection in Plasma Physics" at the Institute for Plasma Research, Gandhinagar. SD is extremely grateful to Dr. P K Kaw and Dr. A Sen for their generous hospitality, and to the participants in the School for lively discussions, which helped clarify many of the subtleties.

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