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The Painlevé property, integrability and chaotic behaviour of a two-coupled Duffing oscillators

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Abstract. Integrability and chaotic behaviour in a two-coupled Duffing oscillators are studied. The coupling is nonlinear. Painlevé test is performed to identify integrable cases of damped- and force-free system. Exact analytical solutions are given for the integrable cases. Effect of external periodic forces for (i) single well with infinite height potential, (ii) potential with a hump at the centre and (iii) single well with finite height hump potential are numerically investigated. Occurrence of multiple attractors and period doubling cascades of coexisting attractors is presented.

Keywords. Coupled Duffing oscillators; Painlevé analysis; chaos.

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1. Introduction

The evolution of many phenomena in nature is described by nonlinear ordinary or partial differential and difference equations depending on whether the concerned system is continuous or discrete. During the last decade or so, remarkable progress has been made in understanding the integrability and nonintegrability of nonlinear dynamical systems [1–4]. Integrable systems are rather limited and are in general expected to show regular behaviour whereas nonintegrable systems are capable of showing regular as well as complicated irregular motions in phase space. The knowledge of integrability of a differential equation or a system is very important in physics because we are interested to find a solution to the physical problem. Integrable limits of several physical systems were obtained by Painlevé analysis [4–10].

An ordinary differential equation is said to have the Painlevé property if all movable singularities of the solutions are poles. The term 'strong Painlevé' is used when the solution in the neighbourhood of an arbitrary singularity t^* can be expressed as $\tau = (t - t^*)^{-p}$, where p is an integer determined from the leading order, so that the movable algebraic or logarithmic branch points as well as essential singularities are excluded. Ramani, Dorizzi and Grammaticos [7] have introduced the so-called weak Painlevé property. By weak Painlevé property it is meant that the solution in the neighbourhood of the movable singularity t^* can be expressed as an expansion in powers of $\tau = (t - t^*)^{-1/q}$, where q must be a natural number that depends purely on the leading order behaviour of the singularity and the nature of the potentials. Equations having the Painlevé property might be easier to integrate or solve analytically.

Recently, a series of papers [11–15] devoted to the identification of integrability of damped anharmonic oscillator

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^3 = 0 \tag{1}$$

using the Painlevé test was published. The integrable choices of the undamped two coupled anharmonic oscillators

$$\ddot{x} = -2A_1 x - 4\alpha_1 x^3 - 2\delta x y^2, \tag{2a}$$

$$\ddot{y} = -2A_2 y - 4\alpha_2 y^3 - 2\delta x^2 y, \tag{2b}$$

has also been studied by various authors [9, 16–18]. System (2) with linear damping is written as

$$\ddot{x} = -\mathrm{d}\dot{x} - 2A_1 x - 4\alpha_1 x^3 - 2\delta x y^2, \tag{3a}$$

$$\ddot{y} = -\mathrm{d}\dot{y} - 2A_2 y - 4\alpha_2 y^3 - 2\delta x^2 y, \tag{3b}$$

where $2A_i = \Omega_{0i}$, α_i, d and δ are natural frequency, Duffing term, damping coefficient and coupling strength respectively. Equation (3) models two-coupled Duffing oscillators and has been used to model Soret driven Bénard convection [19], vibrations of a stretched string [20], motions of nonlinear circular plates [21] and so forth [2, 22, 23]. Thus the study of integrable and nonintegrable properties of (3) is not only of theoretical but also of practical interest. The choice $d = 0$ reduces to the undamped anharmonic oscillator system. When $\delta = 0$ we have decoupled Duffing oscillators.

In this paper, first we wish to apply Painlevé analysis to study the integrability of (3). It is important to investigate the dynamics of the system including chaotic behaviour in the nonintegrable limit. It is well-known that for chaotic behaviour to occur, a fixed point of a dissipative continuous dynamical system must undergo a Hopf bifurcation thereby developing a limit cycle motion. As the control parameter is varied this limit cycle may bifurcate further leading to chaotic motion. Using linear stability analysis we found that the fixed points of (3) do not undergo Hopf bifurcation for any nonzero value of the parameters. Thus system (3) cannot show chaotic behaviour. However, when the system is subjected to external periodic forces a variety of interesting behaviours such as period doubling phenomenon, coexistence of multiple attractors, chaotic motion and merging of attractors occur.

The paper is organized as follows. To be self-contained, in § 2 we briefly outline the salient features of Painlevé analysis. In § 3 we perform the Painlevé test to the coupled Duffing oscillators. The integrable limits are identified. Then we obtain the explicit analytical solution for the integrable cases in § 4. Section 5 is devoted to the study of the influence of external periodic forces. The dynamics is numerically investigated by varying the amplitude of the forces for three physically interesting potentials. We show the occurrence of multiple periodic attractors and period doubling of coexisting attractors culminating in chaos. Section 6 contains conclusions.

2. Painlevé analysis

A necessary condition for an nth order ordinary differential equation of the form

$$\dot{x}_i = F_i(x_1, \ldots, x_n; t), \quad i = 1, 2, \ldots, n, \tag{4}$$

where F_i are rational in $x_1, \ldots, x_n$ and analytic in t to have the Painlevé (P-) property is that there is a Laurent series expansion with $(n-1)$ arbitrary expansion coefficients. The P-analysis essentially consists of three steps, dealing with the dominant behaviours, the resonances and the constants of integrations, respectively [5–9].

i) *Dominant behaviours*: The first step is to determine the leading order behaviours of x in the neighbourhood of a movable singularity t^* in the form $x_i \approx a_{i0}(t-t^*)^{p_i}$, as $t \to t^*$, $a_{i0} = \text{constant}$. If all the allowed p_i's are negative integers, the solution may correspond to the strong P-property and if any of the p_i's is a rational fraction, the solution may be associated with the weak P-property. In either case, the solution takes the form of a Laurent series,

$$x_i(t) = (t-t^*)^{p_i} \sum_{k=0}^{\infty} a_{ik}(t-t^*)^k. \tag{5}$$

ii) *Resonances*: The second step is to identify the powers of (5) at which the arbitrary parameters can enter, called resonances. Apart from t^*, we have $(n-1)$ other arbitrary constants for (4). To find the resonances, we substitute

$$x_i \approx a_{i0}(t-t^*)^{p_i} + \Omega_i(t-t^*)^{p_i+r}, \quad r > 0, \quad i = 1, \ldots, n \tag{6}$$

in (4) and retain the leading order terms in Ω_i. The reduced equation will be of the form

$$Q(r)\cdot\Omega = 0, \quad \Omega = (\Omega_1, \ldots, \Omega_n), \tag{7}$$

where $Q(r)$ is an $n \times n$ matrix with r appearing only in its diagonal elements. Then the resonance values are determined from the roots of the equation $\det Q(r) = 0$.

iii) *The constants of integration*: The final step verifies that in the Laurent series (5) at the resonance values, sufficient number of arbitrary constants exist without the introduction of logarithmic branch points. To do this, we substitute the truncated expansion

$$x_i = a_{i0}(t-t^*)^{p_i} + \sum_{k=1}^{r_s} a_{ik}(t-t^*)^{p_i+k}, \tag{8}$$

where r_s is the largest resonance value in (4) and determines the integration constants. At the resonances, one usually finds some condition termed 'compatibility condition' that has to be satisfied in order to secure arbitrariness of the coefficient.

3. The Painlevé property of nonlinearly coupled Duffing oscillators

3.1 *Leading order behaviours*

Let us apply P-analysis to (3). To start with, we assume the leading orders be

$$x \approx a_0\tau^p, \quad y \approx b_0\tau^q, \quad \tau = (t-t^*) \to 0. \tag{9}$$

To determine p, q, a_0 and b_0, we use (9) in (3) and obtain pairs of leading order equations

$$a_0 p(p-1)\tau^{p-2} = -\mathrm{d}a_0 p\tau^{p-1} - 2A_1 a_0\tau^p - 4\alpha_1 a_0^3\tau^{3p} - 2\delta a_0 b_0^2\tau^{p+2q}, \tag{10a}$$

$$b_0 q(q-1)\tau^{q-2} = -\mathrm{d}b_0 q\tau^{q-1} - 2A_2 b_0\tau^q - 4\alpha_2 b_0^3\tau^{3q} - 2\delta b_0 a_0^2\tau^{2p+q}. \tag{10b}$$

These equations immediately reveal that three different types of leading orders are possible. These are

$$p=-1,\quad q=-1,\quad a_0^2=(2\alpha_2-\delta)/(\delta^2-4\alpha_1\alpha_2),$$
$$b_0^2=(2\alpha_1-\delta)/(\delta^2-4\alpha_1\alpha_2). \tag{11}$$

$$p=-1,\quad q=\tfrac{1}{2}[1+(1+(4\delta/\alpha_1))^{1/2}]\geq\tfrac{1}{2},$$
$$a_0^2=-1/(2\alpha_1),\quad b_0^2=\text{arbitrary}. \tag{12}$$

$$p=-1,\quad q=\tfrac{1}{2}[1-(1+(4\delta/\alpha_1))^{1/2}]>-1,$$
$$a_0^2=-1/(2\alpha_1),\quad b_0^2=\text{arbitrary}. \tag{13}$$

The three different solution branches, eqs (11–13) must be tested for the P-property. The next step is to carry out a resonance analysis.

3.2 *Resonances*

To find the resonances, that is, the values of the order r at which arbitrary constants will enter in the expansions of the solutions near the singularity at $t=t^*$, we write

$$x\approx a_0\tau^p+\Omega_1\tau^{p+r},\quad y\approx b_0\tau^q+\Omega_2\tau^{q+r}. \tag{14}$$

We substitute (14) in (3) to obtain resonances. Retaining leading order terms, we obtain a system of linear algebraic equation

$$M_2(r)\cdot\Omega=0,\quad \Omega=(\Omega_1,\Omega_2), \tag{15}$$

where $M_2(r)$ is a 2×2 matrix dependent on r. In order to have nontrivial set of solutions (Ω_1,Ω_2) we require the determinant of $M_2(r)$ equal to 0.

Case 1: For equation (11), the form of $M_2(r)$ is

$$M_2(r)=\begin{pmatrix}(r-1)(r-2)+8\alpha_1a_0^2-2 & 4\delta a_0b_0\\ 4\delta a_0b_0 & (r-1)(r-2)+8\alpha_2b_0^2-2\end{pmatrix} \tag{16}$$

which then leads to the equation

$$(r^2-3r-4)(r^2-3r-\mu)=0,\quad \mu=4[1+2(\alpha_1a_0^2+\alpha_2b_0^2)]. \tag{17}$$

Thus for (11) the resonances occur at

$$r=-1,4,[3\pm(9-4\mu)^{1/2}]/2. \tag{18}$$

The root -1 corresponds to the arbitrariness of t^* in (9). All the other resonances must have positive integer values as a necessary condition for (14) to be a Laurent series and (3) to possess the P-property. This happens for special values of μ. Equation (18) along with (11) then leads to the following two possibilities:

Case 1(i): $$\mu=2,\quad \alpha_1a_0^2+\alpha_2b_0^2=-1/4,\quad r=-1,1,2,4, \tag{19a}$$
$$\delta=2[(\alpha_1+\alpha_2)\pm(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2)^{1/2}]. \tag{19b}$$

Case 1(ii): $$\mu=0,\quad \alpha_1a_0^2+\alpha_2b_0^2=-1/2,\quad r=-1,0,3,4, \tag{20a}$$
$$\delta^2=4\alpha_1\alpha_2. \tag{20b}$$

The resonance analysis is already yielding valuable information. If we are interested in the P-property, this immediately restricts δ to those values given by (19b) and (20b) for the cases 1(i) and 1(ii) respectively.

Case 2: For eqs (12) and (13), the expression for $M_2(r)$ degenerates to

$$M_2(r) = \begin{pmatrix} r^2 - 3r + 8\alpha_1 a_0^2 & 0 \\ 4\delta a_0 b_0 & r^2 + 2rq - r \end{pmatrix} \tag{21}$$

so that from $\det M_2(r) = 0$, the resonance values are

$$r = -1, 0, (1 - 2q), 4. \tag{22}$$

In (22), for $(1 - 2q) \geqslant 0$ we must have $q \leqslant \frac{1}{2}$. But this is in general contradictory to the leading order singularity nature, $q \geqslant \frac{1}{2}$, eq. (12). The only consistent case $q = \frac{1}{2}$ requires both a_0 and b_0 to be arbitrary, which is not true as seen from (12). Thus, the associated P-branch can have less number of arbitrary constants.

Case 3: Using (13) in (21), we infer two possibilities:

Case 3(i): $q = 0$, and so $\delta = 0$, the uncoupled case.

Case 3(ii): $q = -\frac{1}{2}$, $3\alpha_1 = 4\delta$, $r = -1, 0, 2, 4.$ (23)

Thus, for the coupled Duffing equation (3), we identify three sets of full resonances, namely, (19), (20) and (23). The resonance analysis only tells us which coefficients should be arbitrary, and this has to be verified by checking the full recursion relations.

3.3 *Identifying the arbitrary constants of integration*

To verify the existence of a sufficient number of arbitrary constants we introduce series expansion

$$x \approx a_0 \tau^p + \sum_{k=1}^{4} a_k \tau^{p+k}, \quad y \approx b_0 \tau^q + \sum_{k=1}^{4} b_k \tau^{q+k}, \tag{24}$$

in (3) and equating the coefficients of the powers of $(\tau^{p+k-2}, \tau^{q+k-2})$ to zero, we obtain a system of linear algebraic equations for a_k and b_k. For a system containing parameters, say, $a, b, c, \ldots$ one will often find that arbitrariness is only obtained for special values of the system parameters. We will now deal with each one of the cases 1(i), 1(ii) and 3(ii) separately. For the case 1(i) we give the analysis in detail while for the remaining two cases we present main results only.

Case 1(i): The resonance values $r = 1, 2, 4$ imply that in addition to t^*, three arbitrary constants exists. Thus, for system (3) to satisfy P-property a_1(or b_1), a_2(or b_2) and a_4(or b_4) must be arbitrary which we verify here. From the coefficients of (τ^{-2}, τ^{-2}) we obtain

$$d a_0 b_0 - 2 b_0 a_1 (6\alpha_1 a_0^2 + \delta b_0^2) - 4\delta a_0 b_0^2 b_1 = 0, \tag{25a}$$

$$d a_0 b_0 - 2 a_0 b_1 (6\alpha_2 b_0^2 + \delta a_0^2) - 4\delta a_0^2 b_0 a_1 = 0. \tag{25b}$$

Then, from (25), for a_1(or b_1) to be arbitrary we require $\delta = 6\alpha_1$ (or $6\alpha_2$). For this choice of δ values from (19b) we further find that $\alpha_1 = \alpha_2$ and hence $a_0^2 = b_0^2 = -1/(8\alpha_1)$. Proceeding further, for a_2(or b_2) to be arbitrary, from the coefficients of (τ^{-1}, τ^{-1}) we obtain $A_1 = A_2$ and hence we have

$$a_2 + b_2 = (2/3)a_0(A_1 - d^2/12). \tag{26}$$

In a similar manner, equating terms of order (τ^0, τ^0) we uniquely determine the coefficients a_3 and b_3 and easily find

$$a_3 + b_3 = (1/27)da_0(9A_1 - d^2). \tag{27}$$

Finally, from the coefficients of (τ^1, τ^1) the compatibility condition for a_4(or b_4) to be arbitrary is

$$\begin{aligned} 2d(a_3 + b_3) &= -(a_2 + b_2)[A_1 + 6\alpha_1(a_1 + b_1)^2 + 12\alpha_1 a_0(a_2' + b_2)], \\ &= (1/a_0)(a_2 + b_2)[d^2 a_0/12 - (3/2)(a_2 + b_2) - A_1 a_0]. \end{aligned} \tag{28}$$

Using (26) the right hand side of (28) is found to be 0. Then, substituting $(a_3 + b_3)$ from (27) in (28) we obtain

$$a_0 d^2(9A_1 - d^2) = 0. \tag{29}$$

This means that a_4(or b_4) is arbitrary only for $d = 0$ and $d = \pm 3\sqrt{A_1}$. The choice $d = 0$ corresponds to the undamped anharmonic oscillator. Thus, the Laurent series (5) for the Duffing equation (3) with $p = -1$, $q = -1$ is seen to have three arbitrary parameters besides t^*. We know that since (3) can be rewritten as four coupled first order ordinary differential equations, its general solution is characterized by four arbitrary parameters. The above analysis shows that these are manifested in the Laurent expansion (5) by the arbitrariness of t^*, a_1(or b_1), a_2(or b_2) and a_4(or b_4). Thus, for case 1(i) the system possesses P-property for

$$\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = 0, \tag{30}$$

$$\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = \pm 3\sqrt{A_1}. \tag{31}$$

Case 1(ii): The resonance values are $r = -1, 0, 3, 4$ with the parametric condition (20b). From the leading order analysis the conditions for a_0(or b_0) to be arbitrary are

$$\alpha_1 = \alpha_2, \quad \delta = 2\alpha_1. \tag{32}$$

Comparing the coefficients of (τ^0, τ^0) in (3) we obtain the following set of equations

$$a_0 a_3 + b_0 b_3 = -d^3/(864\alpha_1), \tag{33}$$

$$\begin{aligned} 0.a_3 + 0.b_3 = d[(A_1 - A_2)\{1 + (8/3)\alpha_1^2 a_0^2(a_0^2 - b_0^2) - 2\alpha_1 a_0^2\} \\ + (2/3)\alpha_1(a_0^2 - b_0^2)(A_2 - d^2/12)]. \end{aligned} \tag{34}$$

Equation (33) implies that a_3(or b_3) is arbitrary while (34) gives the compatibility condition in terms of the parameters of the system. For (34) to be satisfied, we require

$$A_1 = A_2, \quad d^2 = 12A_2. \tag{35}$$

From the coefficients of (τ^1, τ^1) we obtain

$$0.a_4 + 0.b_4 = da_3(1 - \alpha_1 a_0^2 - \alpha_1 a_0 b_0) + db_3(1 - \alpha_1 b_0^2 - \alpha_1 a_0 b_0). \tag{36}$$

Now, using (34) and (36) we can determine both the coefficients a_3 and b_3. But from the resonance condition (20) we require either a_3 or b_3 as arbitrary. Since the coefficients a_3 and b_3 are fixed we conclude that case 1(ii) is of non-P-type for $d \neq 0$. However, for $d = 0$, the right hand side of (36) become zero and so a_4(or b_4) is arbitrary without any further restrictions on the parameters. Thus, for the case 1(ii) the system possesses P-property for

$$\alpha_1 = \alpha_2, \quad \delta = 2\alpha_1, \quad d = 0, \quad A_1 \text{ and } A_2 \text{ arbitrary.} \tag{37}$$

Case 3(ii): Proceeding as before, from the coefficients of $(\tau^{-1}, \tau^{-1/2})$ in (3) we find that b_2 is arbitrary only if

$$3\alpha_1 = 4\delta, \quad d^2 + 3A_1 - 12A_2 = 0, \quad \delta^2 - 18\delta\alpha_2 + 72\alpha_2^2 = 0. \tag{38}$$

From the last condition in (38) we obtain $\delta = 6\alpha_2$ or $12\alpha_2$. From the coefficients of τ^1 in (3) we obtain an equation containing terms b_2, and powers of b_0 and constant terms only. Since b_0 and b_2 are arbitrary we equate the coefficients of b_2 and various powers of b_0 to zero separately which leads to the condition $d = 0$. That is, case 3(ii) is non-P-type for $d \neq 0$. Thus, case 3(ii) passes P-test only if

$$3\alpha_1 = 4\delta, \quad A_1 = 4A_2, \quad \alpha_1 = 8\alpha_2, \quad d = 0, \tag{39}$$

$$3\alpha_1 = 4\delta, \quad A_1 = 4A_2, \quad \alpha_1 = 16\alpha_2, \quad d = 0. \tag{40}$$

Thus, for $d = 0$ system (3) possesses P-property for four sets of parametric restrictions given by (30), (37), (39) and (40) and when $d \neq 0$, (3) passes the P-test only for the parameters set given by (31).

4. Analytical solution for the integrable cases

For $d = 0$, Lakshmanan and Sahadevan [9, 17] explicitly constructed second integrals of motion, the first being the Hamiltonian, in order to substantiate the complete integrability. In this section we find the explicit analytical solution for the integrable cases with $d \neq 0$, namely,

$$\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = \pm 3\sqrt{A_1}. \tag{41}$$

For the above parametric choices, system (3) under the transformation $u = x + y$, $v = x - y$ decouples into two single oscillator

$$\ddot{u} \pm 3\sqrt{A_1}\dot{u} + 2A_1 u + 4\alpha_1 u^3 = 0, \tag{42}$$

$$\ddot{v} \pm 3\sqrt{A_1}\dot{v} + 2A_1 v + 4\alpha_1 v^3 = 0. \tag{43}$$

For positive damping and $\alpha_1 > 0$, under the transformation

$$W = \sqrt{2\alpha_1/A_1}\, u \exp(\sqrt{A_1}\, t), \quad Z = -\sqrt{2}\exp(-\sqrt{A_1}\, t), \tag{44}$$

eq. (42) can be reduced to

$$\mathrm{d}^2 W/\mathrm{d}Z^2 + W^3 = 0. \tag{45}$$

Equation (45) has the Jacobian elliptic function solution [24]

$$W = W_0 cn(W_0 z; k), \quad z = Z - Z_0, \quad k^2 = 1/2 \tag{46}$$

where W_0 and Z_0 are arbitrary integration constants. From (44) and (46) the solution of (42) is written as

$$u(t) = \sqrt{A_1/(2\alpha_1)}\, W_0 \exp(-\sqrt{A_1}\, t) cn(W_0 z; k),$$
$$z = -\sqrt{2} \exp(-\sqrt{A_1}\, t) - Z_0. \tag{47}$$

For $\alpha_1 < 0$, using the transformation (44) with $\alpha_1 = |\alpha_1|$, (42) becomes

$$\mathrm{d}^2 W/\mathrm{d}Z^2 - W^3 = 0 \tag{48}$$

which has the solution [24]

$$W = W_0/cn(W_0 z; k'), \tag{49a}$$

where

$$z = Z - Z_0, \quad k'^2 = 1 - k^2 = 1/2 \tag{49b}$$

and W_0, Z_0 are the arbitrary constants. The solution of (42) can now be written as

$$u(t) = \sqrt{A_1/(2|\alpha_1|)}\, W_0 \exp(-\sqrt{A_1}\, t)[cn(W_0 z; k')]^{-1},$$
$$z = -\sqrt{2} \exp(-\sqrt{A_1}\, t) - Z_0. \tag{50}$$

The solution for the negative damping $d = -3\sqrt{A_1}$ can be easily obtained from (47) and (50) by replacing t by $-t$.

5. Regular and chaotic dynamics in the coupled Duffing oscillators

In the previous two sections we were concerned with the integrability and exact solutions of the coupled Duffing oscillators in the absence of external periodic forces. In order to study the occurrence of chaotic behaviour we consider the coupled Duffing oscillators driven by external periodic forces [2], namely,

$$\ddot{x} = -\mathrm{d}\dot{x} - 2A_1 x - 4\alpha_1 x^3 - 2\delta x y^2 + f_1 \cos \Omega_1 t,$$
$$\ddot{y} = -\mathrm{d}\dot{y} - 2A_2 y - 4\alpha_2 y^3 - 2\delta y x^2 + f_2 \cos \Omega_2 t. \tag{51}$$

Equation (51) models a variety of physical systems [2, 20, 22, 23]. Elliott [20] studied the resonance behaviour and Nabergoj *et al* [23] investigated the stability of nonoscillating solution in (51) with $f_2 = 0$. Applying the method of multiple scales Nayfeh and Vakakis [21] analyzed subharmonic frequency response curves. The interaction between high and low frequency modes is analyzed by Nayfeh and Nayfeh [22]. Recently, using Melnikov analytical method we have studied the occurrence of homoclinic bifurcations [25].

Equation (51) can be written as

$$\ddot{x} = -\mathrm{d}\dot{x} - \partial V(x,y)/\partial x + f_1 \cos \Omega_1 t,$$
$$\ddot{y} = -\mathrm{d}\dot{y} - \partial V(x,y)/\partial y + f_2 \cos \Omega_2 t,$$

where the potential function V is given by

$$V(x,y) = A_1 x^2 + \alpha_1 x^4 + A_2 y^2 + \alpha_2 y^4 + \delta x^2 y^2. \tag{52}$$

Throughout our analysis we assume A_2 and α_2 to have the same signs as A_1 and α_1 respectively. Further, we assume that the coupling is weak. The shape of the potential varies with the signs of $A_1, A_2, \alpha_1, \alpha_2$. We consider the following cases.

Case 1: $A_1, A_2, \alpha_1, \alpha_2 > 0$ – single well with infinite height potential (figure 1a).
Case 2: $A_1, A_2 < 0, \alpha_1, \alpha_2 > 0$ – potential with a hump at the centre (figure 1b).
Case 3: $A_1, A_2 > 0, \alpha_1, \alpha_2 < 0$ – single well with finite height hump potential (figure 1c).
Case 4: $A_1, A_2, \alpha_1, \alpha_2 < 0$ – inverted single well potential (figure 1d).

The nature of the solutions of the system (51) depends on the shape of the potential. In cases (1) and (2) there exists globally bounded solutions since $V(x,y) \to \infty$ as $|x|$ and $|y| \to \infty$. However, for case (3) we may have unbounded solutions of exploding amplitudes for the choices of sufficiently large initial values since $V(x,y) \to -\infty$ as $|x|$ and $|y| \to \infty$. Finally, the potential with $A_1, A_2, \alpha_1, \alpha_2 < 0$ is physically uninteresting since the system has an exploding amplitude $V(x,y) \to -\infty$ as $|x|$ and $|y| \to \infty$. In the following we numerically study the occurrence of regular and chaotic dynamics in (51) for the first three potentials (cases 1–3).

5.1 *Single well with infinite height potential*

We fix the parameters at $A_1 = 0{\cdot}005$, $A_2 = 0{\cdot}01$, $\alpha_1 = 10$, $\alpha_2 = 10$, $\delta = 0{\cdot}05$, $d = 3(A_1)^{1/2}$ and $\Omega_1 = \Omega_2 = 1$. We choose $f_1 = f_2 = f$. The forcing amplitude f is varied from a small value. From our numerical studies we find the following. Figure 2a shows the occurrence of period doubling bifurcations, chaotic motion and window regions. For characterizing the regular and chaotic motion we have calculated the maximal Lyapunov exponent (λ). Variation of λ against f is plotted in figure 2b. Initially for

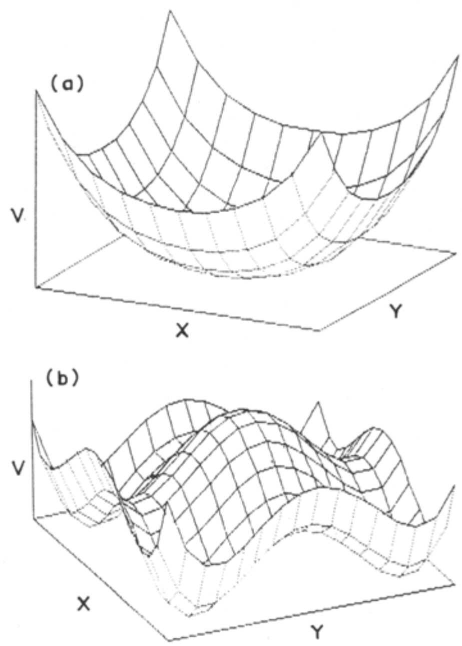

Figure 1. **(a) and (b)**

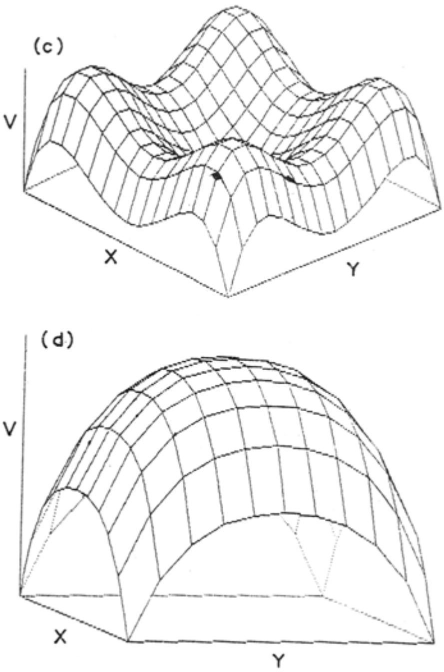

Figure 1. Potential V, equation (52), for (**a**) $A_1, A_2, \alpha_1, \alpha_2 > 0$, (**b**) $A_1, A_2 < 0$, $\alpha_1, \alpha_2 > 0$, (**c**) $A_1, A_2 > 0, \alpha_1, \alpha_2 < 0$ and (**d**) $A_1, A_2, \alpha_1, \alpha_2 < 0$.

small values of f the system exhibits symmetrical orbit. A typical orbit is shown in figure 3a for $f = 0{\cdot}2$. As the parameter f is varied the symmetrical orbit loses its stability and experiences a symmetry breaking bifurcation. The resulting asymmetric orbit is shown in figure 3b for $f = 0{\cdot}8$. As f is further increased the asymmetrical attractor undergoes period doubling cascade to chaos. Period-1, 2, 4 and 8 orbits are found in the interval (0–0·896), (0·896–1·016), (1·016–1·04) and (1·04–1·052) respectively. Onset of chaos is found at $f = 1{\cdot}064$. A feature of chaotic regime is the presence of windows of periodic solutions interspersed throughout the range of their existence. Period-3 window occurs for $f \in (1{\cdot}364–1{\cdot}652)$ in which there is no chaotic behaviour. The developed chaos disappears at $f \approx 2{\cdot}072$ by a period-1 limit cycle. In the chaotic regime as the parameter f is increased the size of the attractor increases gradually as shown in the bifurcation diagram (2a). In figure 4 the Poincaré map of the chaotic attractor in $x - \dot{x}$ plane is plotted for $f = 1{\cdot}7$.

5.2 *Potential with a hump at the centre*

We now fix $A_1 = -0{\cdot}5$, $A_2 = -0{\cdot}055$, $\alpha_1 = 0{\cdot}25$, $\alpha_2 = 0{\cdot}025$, $d = 0{\cdot}4$, $\delta = 0{\cdot}025$, $\Omega_1 = \Omega_2 = 1$. For small values of f coexistence of two limit cycle orbits occur. As the parameter f is increased both the orbits exhibit a cascade of period doubling leading to chaotic motion. Each coexisting attractor possesses its own basin of attraction, defined as the set of initial conditions from which the system evolves to a particular orbit.

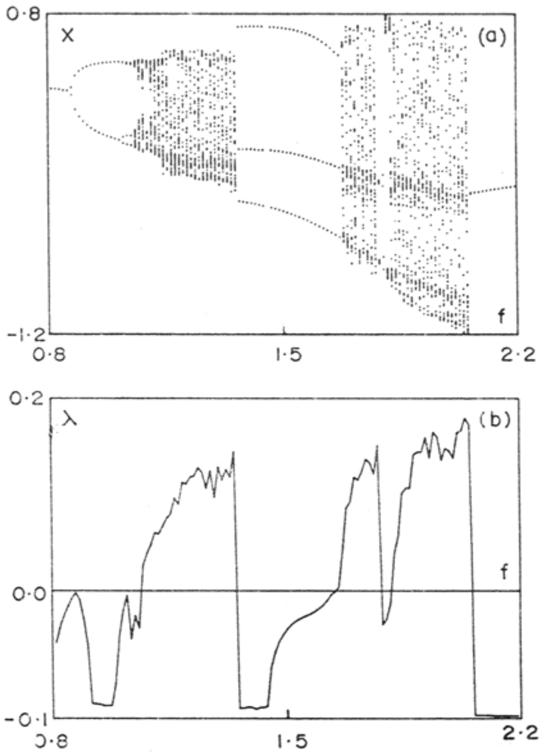

Figure 2. (**a**) Bifurcation diagram illustrating period doubling route to chaos. (**b**) Maximal Lyapunov exponent against the parameter f corresponding to the figure (2a).

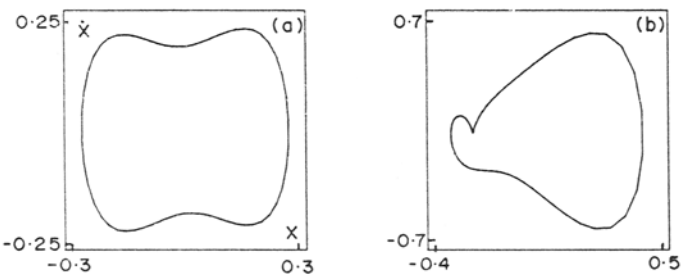

Figure 3. Phase portrait of (**a**) symmetrical orbit for $f = 0{\cdot}2$ and (**b**) asymmetrical orbit for $f = 0{\cdot}8$.

Figures 5a and 5b shows the successive bifurcations of two coexisting attractors. Interestingly, both the attractors underwent bifurcations at the same f values. For example, both the period-1 orbits bifurcate to period-2 orbits at $f \approx 0{\cdot}255$ and period-4 at $f \approx 0{\cdot}26475$. Chaotic motion is first observed at $f = 0{\cdot}267$. The chaotic attractors disappeared at $f = 0{\cdot}281$. Due to crisis two period-1 limit cycles are found. Moreover, these orbits underwent period doubling which is clearly seen in figure 5. Further, we note a sudden expansion in the size of the chaotic attractors at $f \approx 0{\cdot}2715$ and $0{\cdot}3088$.

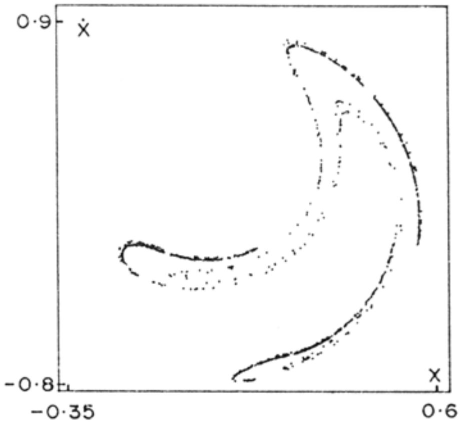

Figure 4. Poincaré map of the chaotic attractor for $f = 1{\cdot}7$.

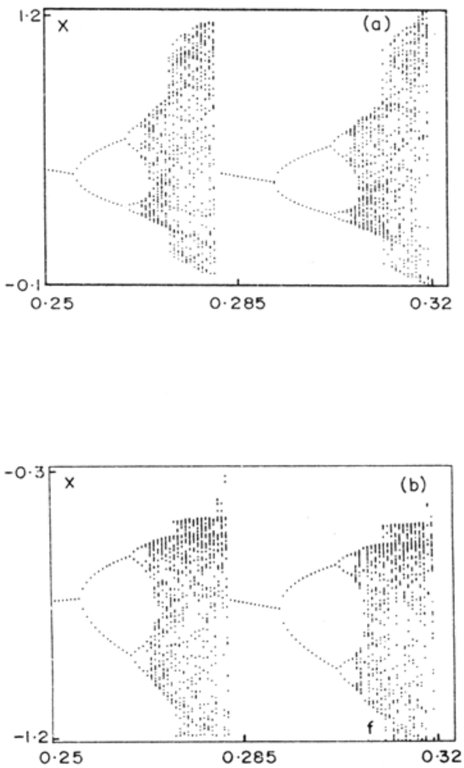

Figure 5. Figures showing successive bifurcations of two coexisting period-1 attractors.

For $f > 0{\cdot}3165$ cross-well chaos is observed. Here, the two chaotic attractors merge into a single attractor. This is shown in figure 6 for $f = 0{\cdot}6$.

5.3 *Single well with finite height hump potential*

Next we consider the case $A_1, A_2 > 0$ and $\alpha_1, \alpha_2 < 0$. The dynamics of the system has been investigated for the following fixed parameters $A_1 = 0{\cdot}5, A_2 = 0{\cdot}55, \alpha_1 = -1, \alpha_2 = -0{\cdot}975$, $\delta = 0{\cdot}025, d = 0{\cdot}4$ and $\Omega_1 = \Omega_2 = 0{\cdot}526$ thereby varying f as done in the other two potential well cases. In this potential well also we have found coexistence of more than one stable periodic orbit and period doubling bifurcations. In contrast to the case 2 potential where both the period-1 orbits underwent period doubling bifurcations at same f value, here the attractors are found to undergo period doubling bifurcations at different f values. Figure 7 shows the phase portrait of three coexisting period-1 orbits for $f = 0{\cdot}1143$. When f is increased the limit cycle labelled as c in figure 7 alone persists while the other two become unstable at certain f values and undergo cascades of period doubling bifurcations. Bifurcations of the period-1 orbit a (b) to period-2 occur at $f = 0{\cdot}1143375(0{\cdot}11448)$; to period-4 at $f = 0{\cdot}114481(0{\cdot}1146225)$; to period-8 at $f = 0{\cdot}1145184(0{\cdot}114645)$.

Figure 8 shows successive period doubling process leading to chaotic motion of coexisting period-1 attractors. The Poincaré map of the two coexisting chaotic attractors at $f = 0{\cdot}11466$ is given in figure 9. For comparison, same scales in x and $\dot{x}$ coordinates are used. For clarity, in figure 9c we show the magnification of the chaotic attractor shown in figure 9b. As the parameter f is increased beyond a certain critical

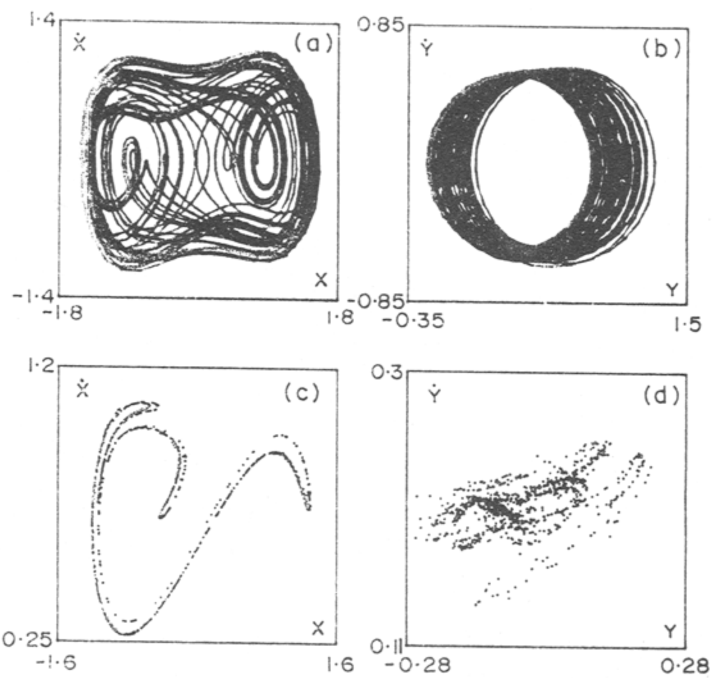

Figure 6. Phase portrait (**a, b**) and Poincaré map (**c, d**) of the chaotic attractor for $f = 0{\cdot}6$.

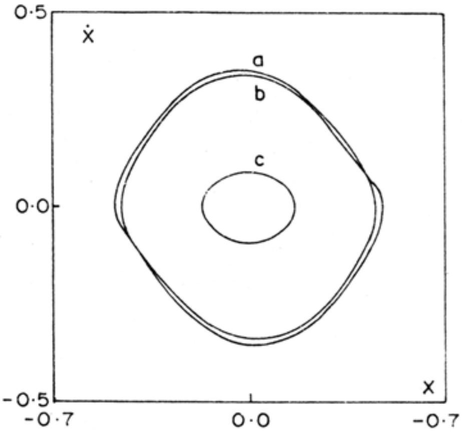

Figure 7. Phase portrait of three different coexisting period-1 attractors for $f = 0{\cdot}1143$. The initial conditions used are $(x, \dot{x}, y, \dot{y}) = (0, 0{\cdot}35, 0, 0)$ (a), $(-0{\cdot}06, 0{\cdot}35, 0, 0)$ (b) and $(0{\cdot}1, 0, 0, 0)$ (c).

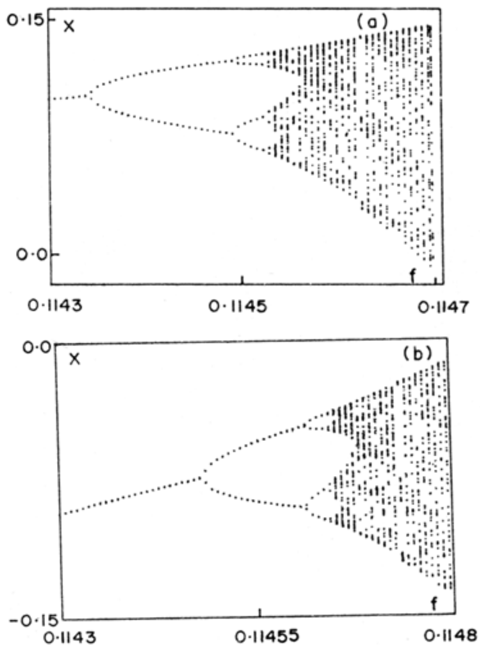

Figure 8. Period doubling bifurcations culminating in chaos of two coexisting attractors.

value of f, both the chaotic attractors merge together and form a single large chaotic attractor. This is caused by a crisis in which both the attractors fuse together and form a large attractor. Figure 9d shows the Poincaré map of such a chaotic attractor. By comparing figures (9a), (9b) and (9d) we note that the large attractor is indeed a mixture

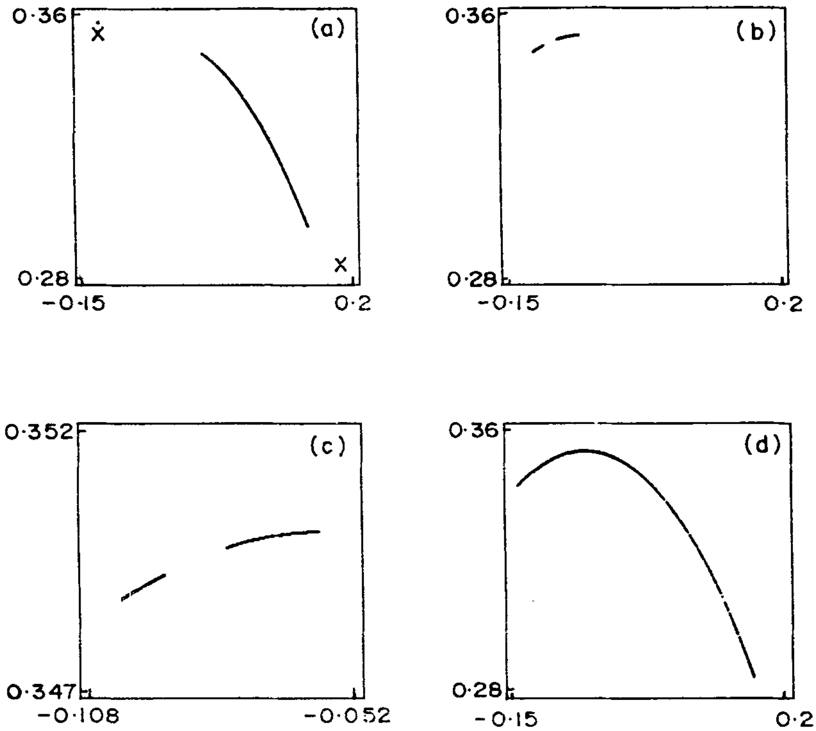

Figure 9. (**a, b**) Poincaré map of the two coexisting chaotic attractors for $f = 0{\cdot}11466$. (**c**) Enlargement of the attractor shown in (b). (**d**) Poincaré map of the chaotic attractor for $f = 0{\cdot}11482$.

of the two coexisting attractors found at lower f values. The coexistence of limit cycle c along with the chaotic attractors provides a mode of physical regulation as it allows to switch to a periodic regime upon suitable perturbation.

6. Conclusions

In this paper we have applied the singularity structure analysis (Painlevé analysis) to the nonlinearly coupled Duffing oscillators. Specific sets of parameters for which the system becomes integrable are obtained. For the integrable cases explicit analytical solution is constructed. For the system (3) with $d = 0$ four integrable choices were identified (cf. eqs (30), (37), (39) and (40)), namely,

i) $\alpha_1 = \alpha_2$, $\delta = 6\alpha_1$, $A_1 = A_2$,
ii) $\alpha_1 = \alpha_2$, $\delta = 2\alpha_1$, A_1 and A_2 arbitrary,
iii) $\alpha_1 = 16\alpha_2$, $\delta = 6\alpha_2$, $A_1 = 4A_2$,
iv) $\alpha_1 = 8\alpha_2$, $\delta = 6\alpha_2$, $A_1 = 4A_2$.

However, when the damping term is added the system is found to be integrable only for $\alpha_1 = \alpha_2, \delta = 6\alpha_1, A_1 = A_2$ and $d = \pm 3\sqrt{A_1}$. Further, we have studied the occurrence of chaotic motion for the three potential wells for specific parameter values. Approximate theories of nonlinear oscillation can be used to get much insight into the occurrence of chaotic motion and locate the chaotic regions in the various parameter space of (51). Chaotic attractor of (51) at critical bifurcations such as onset of chaos, band merging, crisis and intermittency can be characterized by the dynamical structure functions [26, 27]. It is also of interest to study the effect of the coupling parameter δ on both the regular and chaotic dynamics of uncoupled systems. These will be investigated in future.

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