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# Analytic solution of the BCS gap equation in $D$ dimensions ($D = 1, 2, 3$), at finite temperatures

ARUN BHARDWAJ and SATISH K MUTHU
Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India

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**Abstract.** The Bardeen–Cooper–Schrieffer (BCS) gap equation is solved analytically in one, two and three dimensions, for temperatures close to zero and $T_c$. We work in the weak coupling limit, but allow the interaction width $v \equiv \hbar\omega_m/E_F$ to lie in the interval $(0, \infty)$ Here, $\hbar\omega_m$ is the maximum energy of a force-mediating boson, and $E_F$ denotes the Fermi energy. We obtain expressions for $T_c$ and $\Delta C$, the jump in the electronic specific heat across $T = T_c$, in the limits $v \ll 1$ (the usual phonon pairing) and $v > 1$ (non-phononic pairing). This enables us to see how $T_c$ scales with the mediating boson cut off. Our results predict a larger jump in the specific heat for the case $v > 1$, compared to $v \ll 1$. We also briefly touch upon the role of a van Hove singularity in the density of states.

**Keywords.** BCS gap equation; phononic and non-phononic pairing; critical temperature; jump in specific heat; van Hove singularity.

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## 1. Introduction

The epochal discovery of high-temperature superconductivity by Bednorz and Müller [1], has unleashed an unprecedented level of activity. During the last eight years, an enormous amount of data has been collected [2, 3]. A correct theoretical explanation is still being sought. Among the myriad proposals dotting the theoretical landscape, the simplest seems to be to retain the usual BCS framework in which electrons form bound pairs, the so-called Cooper pairs. However, the electron-phonon interaction is replaced by other mechanisms like excitons, magnons, etc. Thus, in particular, the Debye cut-off energy $\hbar\omega_D$ is replaced by $\hbar\omega_m$, where $\hbar\omega_m$ can be greater than $E_F$. It is of considerable interest to study the usual BCS model in this limit.

The motivation for the present work stems from a paper by Walt, Quick and Llano [4], on the analytic solutions of the BCS gap equation in one, two and three dimensions, for arbitrary interaction width $v (v = \hbar\omega_m/E_F, 0 < v < \infty)$ at zero temperature. We extend these results to finite temperatures.

The article is organized as follows: In § 2, we cast the BCS gap equation at $T = 0$ in the notation of Walt, Quick and Llano. Section 3 is devoted to a solution of this equation in $D$ dimensions for $v \ll 1$. Section 4 contains solutions for the case $v > 1$. This enables us to see how $T_c$ scales with $v$. In § 5 we compare the cases $v \ll 1$ and $v > 1$ and make a rough numerical estimate of the enhancement of $T_c$. In § 6 we discuss briefly the role of the presence of a van Hove singularity in the density of states. We conclude by making some general comments on our results.

## 2. BCS theory at finite temperature

At finite temperature, the BCS gap equation in $D$ dimensions can be written as [5]

$$\Delta_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}'} V_{kk'} \tanh\left(\frac{(\Delta_{k'}^2 + \xi_{k'}^2)^{1/2}}{2kT}\right) \times \left(\frac{\Delta_{k'}}{(\Delta_{k'}^2 + \xi_{k'}^2)^{1/2}}\right). \tag{1}$$

Here $\xi_k$ represent the single-particle energies measured with respect to the chemical potential $\mu$, i.e.

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu. \tag{2a}$$

The interelectronic interaction is assumed to be of the form

$$V_{kk'} = \frac{v_0}{L^D}, \max(0, E_F - \hbar\omega_m) < \varepsilon_{k'}\varepsilon_{k'} < E_F + \hbar\omega_m$$

$$= 0, \text{ otherwise.} \tag{2b}$$

Here, $L$ is a measure of the size of the system and the coupling strength $v_0$ is taken to be positive, signifying a net attraction between the electrons; $\varepsilon_k = \hbar^2 k^2/2m$ is the single-particle kinetic energy, $E_F$ is the Fermi energy and $\hbar\omega_m$ is the maximum energy of the force-mediating boson.

In view of the model interaction given by (2b) it follows that

$$\Delta_{\mathbf{k}} = \Delta\theta(\hbar\omega_m - |\xi_k|), \tag{2c}$$

where $\theta(x)$ is the Heaviside step function and the gap parameter $\Delta$ is assumed to be isotropic (independent of $\mathbf{k}$). Hence, (1) becomes

$$1 = \frac{v_0}{2L^D}\sum_{\mathbf{k}} \frac{\theta(\hbar\omega_m - |\xi_k|)}{(\Delta^2 + \xi_k^2)^{1/2}} \tanh\left(\frac{(\Delta^2 + \xi_k^2)^{1/2}}{2kT}\right). \tag{3}$$

Changing the summation over $\mathbf{k}$ into integration, one obtains after some algebra,

$$\frac{2}{\lambda} = \int_{E_F - \hbar\omega_m, 0}^{E_F + \hbar\omega_m} d\varepsilon \frac{(\varepsilon/E_F)^{(D-2)/2}}{(\Delta^2 + (\varepsilon - E_F)^2)^{1/2}} \tanh\left(\frac{(\Delta^2 + (\varepsilon - E_F)^2)^{1/2}}{2kT}\right). \tag{4}$$

Here, $\lambda = N(0)v_0$, $N(0)$ being density of states per spin state at the Fermi level. The lower limit of integration is $E_F - \hbar\omega_m$ for the case $E_F > \hbar\omega_m$ and zero for $E_F < \hbar\omega_m$.

Introducing the dimensionless variables $\varepsilon'$, $v$ and $\delta$, defined by

$$\varepsilon' = \frac{\varepsilon}{E_F}, \tag{5a}$$

$$\nu = \frac{\hbar\omega_{\mathrm{m}}}{E_{\mathrm{F}}} \tag{5b}$$

and

$$\delta = \frac{\Delta}{E_{\mathrm{F}}}, \tag{5c}$$

equation (4) becomes

$$\frac{2}{\lambda} = \int_{1-\nu,0}^{1+\nu} \mathrm{d}\varepsilon' \frac{\varepsilon'^{(D-2)/2}}{(\delta^2 + (\varepsilon'-1)^2)^{1/2}} \tanh\left(\frac{(\delta^2 + (\varepsilon'-1)^2)^{1/2}}{2kT/E_{\mathrm{F}}}\right). \tag{6a}$$

Using the relation $\tanh x = 1 - \dfrac{2}{\mathrm{e}^{2x}+1}$, (6a) can be rewritten as

$$\frac{2}{\lambda} = \int_{1-\nu,0}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}}{(\delta^2 + (\varepsilon'-1)^2)^{1/2}} \mathrm{d}\varepsilon' - 2\int_{1-\nu,0}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}}{(\delta^2 + (\varepsilon'-1)^2)^{1/2}} \times \frac{\mathrm{d}\varepsilon'}{[\mathrm{e}^{(\delta^2+(\varepsilon'-1)^2)^{1/2}E_{\mathrm{F}}/kT} + 1]}. \tag{6b}$$

At $T = 0$, the second term in (6b) gives zero and so it reduces to

$$\frac{2}{\lambda} = \int_{1-\nu,0}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}}{(\delta_0^2 + (\varepsilon'-1)^2)^{1/2}} \mathrm{d}\varepsilon', \text{ with } \delta_0 = \delta(T=0). \tag{7}$$

The integral on the right can be easily evaluated; we have [4] for $D = 1$, 2 and 3

$$\int_{1-\nu,0}^{1+\nu} \frac{\varepsilon'^{-1/2}\mathrm{d}\varepsilon'}{(\delta_0^2 + (\varepsilon'-1)^2)^{1/2}} = 2\ln\left[\frac{8\nu}{\delta_0(1+(1-\nu)^{1/2})(1+(1+\nu)^{1/2})}\right], \quad \nu < 1$$

$$= 2\ln\left[\frac{8\nu^{1/2}}{\delta_0(1+(1+\nu)^{1/2})}\right], \quad \nu > 1$$

$$\int_{1-\nu,0}^{1+\nu} \frac{\mathrm{d}\varepsilon'}{(\delta_0^2 + (\varepsilon'-1)^2)^{1/2}} = 2\ln\left[\frac{2\nu}{\delta_0}\right], \quad \nu < 1$$

$$= 2\ln\left[\frac{2\nu^{1/2}}{\delta_0}\right], \quad \nu > 1$$

$$\int_{1-\nu,0}^{1+\nu} \frac{\varepsilon'^{1/2}\mathrm{d}\varepsilon'}{(\delta_0^2 + (\varepsilon'-1)^2)^{1/2}} = 2\ln\left[\frac{8\nu}{\delta_0(1+(1-\nu)^{1/2})(1+(1+\nu)^{1/2})}\right] + 2(1-\nu)^{1/2} + 2(1+\nu)^{1/2} - 4 \quad \nu < 1$$

$$= 2\ln\left[\frac{8\nu^{1/2}}{\delta_0(1+(1+\nu)^{1/2})}\right] + 2(1+\nu)^{1/2} - 4, \quad \nu > 1,$$

respectively.

## 3. Solution of the BCS gap equation in $D$ dimensions for $\nu \ll 1$

### 3.1 *Expression for $\delta$ near $T = 0$*

We start with eq. (6b). Since we are working near $T = 0$, the first term can be replaced using (7). This leads to

$$2\ln\left[\frac{\delta}{\delta_0}\right] = -2\int_{1-\nu}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}\,\mathrm{d}\varepsilon'}{(\delta^2 + (\varepsilon' - 1)^2)^{1/2}[\mathrm{e}^{(\delta^2 + (\varepsilon' - 1)^2)^{1/2}E_F/kT} + 1]}. \tag{8a}$$

Introducing

$$x = (\varepsilon' - 1)\frac{E_F}{kT} \tag{9a}$$

and

$$\delta' = \left(\frac{E_F}{kT}\right)\delta, \tag{9b}$$

in (8a), we have

$$\ln\left[\frac{\delta_0}{\delta}\right] = \int_{-E_F\nu/kT}^{E_F\nu/kT} \frac{(1 + kTx/E_F)^{(D-2)/2}\,\mathrm{d}x}{(\delta'^2 + x^2)^{1/2}(\mathrm{e}^{(\delta'^2 + x^2)^{1/2}} + 1)}.$$

Since close to $T = 0$, $E_F\nu/kT$ becomes very large, we can take the upper and lower limits as $\infty$, $-\infty$, respectively. Hence,

$$\begin{aligned}\ln\left[\frac{\delta_0}{\delta}\right] &= \int_{-\infty}^{\infty} \frac{(1 + kTx/E_F)^{(D-2)/2}\,\mathrm{d}x}{(\delta'^2 + x^2)^{1/2}(\mathrm{e}^{(\delta'^2 + x^2)^{1/2}} + 1)} \\ &= \int_0^{\infty} \frac{\mathrm{d}x}{(\delta'^2 + x^2)^{1/2}(\mathrm{e}^{(\delta'^2 + x^2)^{1/2}} + 1)}[(1 + kTx/E_F)^{(D-2)/2} \\ &\quad + (1 - kTx/E_F)^{(D-2)/2}].\end{aligned} \tag{8b}$$

The numerator of the integrand is now expanded binomially to obtain

$$\begin{aligned}\ln\left[\frac{\delta_0}{\delta}\right] &= \int_0^{\infty} \frac{\mathrm{d}x}{(\delta'^2 + x^2)^{1/2}(\mathrm{e}^{(\delta'^2 + x^2)^{1/2}} + 1)} \\ &\quad \times [2 + \tfrac{1}{4}(D-2)(D-4)(kTx/E_F)^2 + \ldots].\end{aligned} \tag{10}$$

Also

$$\frac{1}{\mathrm{e}^{(\delta'^2 + x^2)^{1/2}} + 1} = \mathrm{e}^{-(\delta'^2 + x^2)^{1/2}} - \mathrm{e}^{-2(\delta'^2 + x^2)^{1/2}} + \mathrm{e}^{-3(\delta'^2 + x^2)^{1/2}} - \ldots$$

Equation (10) can now be written in the form

$$\begin{aligned}\ln\left[\frac{\delta_0}{\delta}\right] &= 2\int_0^{\infty} \frac{\mathrm{d}x}{(\delta'^2 + x^2)^{1/2}}[\mathrm{e}^{-(\delta'^2 + x^2)^{1/2}} - \mathrm{e}^{-2(\delta'^2 + x^2)^{1/2}} + \mathrm{e}^{-3(\delta'^2 + x^2)^{1/2}} - \ldots] \\ &\quad + \tfrac{1}{4}(D-2)(D-4)(kT/E_F)^2 \int_0^{\infty} \frac{\mathrm{d}x\,x^2}{(\delta'^2 + x^2)^{1/2}}[\mathrm{e}^{-(\delta'^2 + x^2)^{1/2}} \\ &\quad - \mathrm{e}^{-2(\delta'^2 + x^2)^{1/2}} + \mathrm{e}^{-3(\delta'^2 + x^2)^{1/2}} - \ldots].\end{aligned} \tag{11}$$

Using [6]

$$K_\nu(xz) = \frac{(\pi)^{1/2}}{\Gamma(\nu+\frac{1}{2})}(x/2z)^\nu \int_0^\infty \frac{e^{-x(t^2+z^2)^{1/2}}}{(t^2+z^2)^{1/2}} t^{2\nu}\,dt,$$

equation (11) takes the form

$$\ln\left[\frac{\delta_0}{\delta}\right] = 2[K_0(\delta') - K_0(2\delta') + K_0(3\delta') - \ldots] + \tfrac{1}{4}(D-2)(D-4)$$

$$\times (kT/E_F)^2\left[\delta' K_1(\delta') - \frac{\delta'}{2}K_1(2\delta') + \frac{\delta'}{3}K_1(3\delta') - \ldots\right]$$

$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}K_0\left(\frac{nE_F\delta}{kT}\right) + \tfrac{1}{4}(D-2)(D-4)\left(\frac{kT}{E_F}\right)^2\left(\frac{E_F\delta}{kT}\right)$$

$$\times \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}K_1\left(\frac{nE_F\delta}{kT}\right).$$

Thus finally

$$\ln\left[\frac{\delta_0}{\delta}\right] = 2\sum_{n=1}^{\infty}(-1)^{n+1}K_0\left(\frac{nE_F\delta}{kT}\right) + \tfrac{1}{4}(D-2)(D-4)\left(\frac{\delta kT}{E_F}\right)$$

$$\times \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}K_1\left(\frac{nE_F\delta}{kT}\right). \tag{12}$$

To obtain the BCS result, we take $n = 1$ and retain only the first term. This gives

$$\ln\left[\frac{\delta_0}{\delta}\right] = 2K_0\left(\frac{E_F\delta}{kT}\right). \tag{13}$$

For large values of the argument $z$, we have [7]

$$K_0(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 - \frac{1}{8z}\right).$$

Equation (13) becomes

$$\ln\left[\frac{\delta_0}{\delta}\right] = 2\left(\frac{\pi kT}{2E_F\delta}\right)^{1/2} e^{-E_F\delta/kT}\left(1 - \frac{kT}{8E_F\delta}\right)$$

or,

$$\delta = \delta_0 - \left(\frac{2\pi\delta_0 kT}{E_F}\right)^{1/2} e^{-E_F\delta/kT}\left(1 - \frac{kT}{8\delta_0 E_F}\right). \tag{14a}$$

In more familiar notation,

$$\Delta = \Delta_0 - (2\pi\Delta_0 kT)^{1/2} e^{-\Delta_0/kT}\left(1 - \frac{kT}{8\Delta_0}\right), \tag{14b}$$

which is the usual BCS result [8].

### 3.2 Determination of the $T_c$ equation for general D

We start with (6a). Since at $T = T_c$, $\delta = 0$, (6a) becomes

$$\frac{2}{\lambda} = \int_{1-\nu}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}}{|\varepsilon' - 1|} \tanh\left(|\varepsilon' - 1| \frac{E_F}{2kT_c}\right) d\varepsilon'$$

$$= \int_{1-\nu}^{1+\nu} \frac{\varepsilon'^{(D-2)/2}}{\varepsilon' - 1} \tanh\left((\varepsilon' - 1) \frac{E_F}{2kT_c}\right) d\varepsilon'. \tag{15a}$$

Introducing the variable $y$ defined by

$$y = (\varepsilon' - 1)\frac{E_F}{2kT_c}, \tag{16}$$

into (15a) yields

$$\frac{2}{\lambda} = \int_{-E_F\nu/2kT_c}^{E_F\nu/2kT_c} \left(1 + \frac{2kT_c y}{E_F}\right)^{(D-2)/2} \frac{\tanh y}{y} dy, \tag{15b}$$

or

$$\frac{2}{\lambda} = \int_{0}^{E_F\nu/2kT_c} \frac{\tanh y}{y} \left[\left(1 + \frac{2kT_c y}{E_F}\right)^{(D-2)/2} + \left(1 - \frac{2kT_c y}{E_F}\right)^{(D-2)/2}\right] dy. \tag{15c}$$

Integrating by parts and replacing $\tanh(E_F\nu/2kT_c)$ by unity, since $E_F\nu/2kT_c$ is large, we obtain

$$\frac{2}{\lambda} = [(1+\nu)^{(D-2)/2} + (1-\nu)^{(D-2)/2}] \ln\left(\frac{E_F\nu}{2kT_c}\right) - [I_+(T_c) + I_-(T_c)], \tag{17}$$

with

$$I_+(T_c) = \int_0^{E_F\nu/2kT_c} dy \ln y \left[\left(1 + \frac{2kT_c y}{E_F}\right)^{(D-2)/2} \mathrm{sech}^2 y + \frac{kT_c}{E_F}(D-2)\left(1 + \frac{2kT_c y}{E_F}\right)^{(D-4)/2} \tanh y\right], \tag{18a}$$

and

$$I_-(T_c) = \int_0^{E_F\nu/2kT_c} dy \ln y \left[\left(1 - \frac{2kT_c y}{E_F}\right)^{(D-2)/2} \mathrm{sech}^2 y - \frac{kT_c}{E_F}(D-2)\left(1 - \frac{2kT_c y}{E_F}\right)^{(D-4)/2} \tanh y\right], \tag{18b}$$

since $\nu \ll 1$,

$$I_+(T_c) + I_-(T_c) \simeq 2\int_0^{E_F\nu/2kT_c} \ln y \, \mathrm{sech}^2 y \, dy \simeq 2\int_0^{\infty} \ln y \, \mathrm{sech}^2 y \, dy$$

$$= -2\ln\left(\frac{4e^{\gamma}}{\pi}\right),$$

$\gamma$ being the Euler–Mascheroni constant.

Hence, (17) yields

$$\frac{2}{\lambda} = [(1+\nu)^{(D-2)/2} + (1-\nu)^{(D-2)/2}]\ln\left(\frac{E_F \nu}{2kT_c}\right) + 2\ln\left(\frac{4e^{\gamma}}{\pi}\right),$$

or

$$kT_c = \frac{E_F \nu}{2}\left(\frac{4e^{\gamma}e^{-1/\lambda}}{\pi}\right)^{2/[(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}]} \tag{19}$$

This is the desired expression for $T_c$.

### 3.3 *Expression for $\delta$ near $T = T_c$*

We start from eq. (6a). Introducing $x$ and $\delta'$, as defined by eqs (9a) and (9b), we have

$$\frac{2}{\lambda} = \int_{-E_F\nu/kT}^{E_F\nu/kT} \frac{(1+kTx/E_F)^{(D-2)/2}}{(\delta'^2+x^2)^{1/2}} \tanh\left(\frac{(\delta'^2+x^2)^{1/2}}{2}\right)dx.$$

Using the well-known expansion [9]

$$\tanh\frac{x}{2} = 4x\sum_{n=0}^{\infty}\frac{1}{\pi^2(2n+1)^2+x^2},$$

we obtain

$$\frac{2}{\lambda} = 4\sum_{n=0}^{\infty}\int_{-E_F\nu/kT}^{E_F\nu/kT}\frac{(1+kTx/E_F)^{(D-2)/2}}{\pi^2(2n+1)^2+\delta'^2+x^2}dx. \tag{20a}$$

Since, near $T_c$, the size of the gap is very small, we can carry out an expansion in powers of $\delta'$. This leads us to the equation

$$\begin{aligned}\frac{2}{\lambda} &= 4\sum_{n=0}^{\infty}\int_{-E_F\nu/kT}^{E_F\nu/kT}\left(1+\frac{kTx}{E_F}\right)^{(D-2)/2}\frac{dx}{[\pi^2(2n+1)^2+x^2]}\\ &\quad -4\delta'^2\sum_{n=0}^{\infty}\int_{-E_F\nu/kT}^{E_F\nu/kT}\frac{(1+kTx/E_F)^{(D-2)/2}}{[\pi^2(2n+1)^2+x^2]^2}dx + \dots\\ &= \int_{-E_F\nu/kT}^{E_F\nu/kT}\left(1+\frac{kTx}{E_F}\right)^{(D-2)/2}\left(\frac{\tanh(x/2)}{x}\right)dx\\ &\quad -4\delta'^2\sum_{n=0}^{\infty}\int_{-E_F\nu/kT}^{E_F\nu/kT}\frac{(1+kTx/E_F)^{(D-2)/2}}{[\pi^2(2n+1)^2+x^2]^2}dx.\end{aligned} \tag{20b}$$

At $T = T_c$, we have from (15b), on putting $y = x/2$,

$$\begin{aligned}\frac{2}{\lambda} &= \int_{-E_F\nu/kT_c}^{E_F\nu/kT_c}\left(1+\frac{kT_c x}{E_F}\right)^{(D-2)/2}\left(\frac{\tanh(x/2)}{x}\right)dx\\ &= [(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}]\ln\left(\frac{E_F\nu}{2kT_c}\right)+2\ln\left(\frac{4e^{\gamma}}{\pi}\right).\end{aligned}$$

Thus (20b) becomes

$$[(1+v)^{(D-2)/2}+(1-v)^{(D-2)/2}]\ln\left(\frac{T}{T_{\mathrm{c}}}\right)$$
$$=-4\delta'^2\sum_{n=0}^{\infty}\int_0^{E_{\mathrm{F}}v/kT}\frac{\mathrm{d}x}{[\pi^2(2n+1)^2+x^2]^2}$$
$$\times\left[\left(1+\frac{kTx}{E_{\mathrm{F}}}\right)^{(D-2)/2}+\left(1-\frac{kTx}{E_{\mathrm{F}}}\right)^{(D-2)/2}\right].$$

Simplifying the numerator, using binomial expansions, we obtain

$$[(1+v)^{(D-2)/2}+(1-v)^{(D-2)/2}]\ln\left(\frac{T}{T_{\mathrm{c}}}\right)$$
$$=-8\delta'^2\left[\sum_{n=0}^{\infty}\int_0^{E_{\mathrm{F}}v/kT}\frac{\mathrm{d}x}{[\pi^2(2n+1)^2+x^2]^2}\right.$$
$$\left.+\tfrac{1}{8}(D-2)(D-4)(kT/E_{\mathrm{F}})^2\int_0^{E_{\mathrm{F}}v/kT}x^2\,\mathrm{d}x\sum_{n=0}^{\infty}\frac{1}{[\pi^2(2n+1)^2+x^2]^2}\right]. \tag{21}$$

The first integral on the R.H.S. can be easily evaluated.

Assuming the upper limit to be $\infty$, we have

$$\int_0^{\infty}\frac{\mathrm{d}x}{[\pi^2(2n+1)^2+x^2]^2}=\frac{1}{4\pi^2(2n+1)^3}. \tag{22a}$$

Using

$$\sum_{n=0}^{\infty}\frac{1}{[\pi^2(2n+1)^2+x^2]^2}=\frac{1}{8x^3}\tanh\frac{x}{2}-\frac{1}{16x^2}\mathrm{sech}^2\frac{x}{2}, \tag{22b}$$

the R.H.S. of (21) becomes

$$-8\delta'^2\left[\sum_{n=0}^{\infty}\frac{1}{4\pi^2(2n+1)^3}+\tfrac{1}{8}(D-2)(D-4)(kT/E_{\mathrm{F}})^2\right.$$
$$\left.\times\int_0^{E_{\mathrm{F}}v/kT}\left(\frac{1}{8x}\tanh\frac{x}{2}-\frac{1}{16}\mathrm{sech}^2\frac{x}{2}\right)\mathrm{d}x\right].$$

For evaluating the integral, we substitute $x/2=y$. This gives

$$I=\int_0^{E_{\mathrm{F}}v/kT}\left(\frac{1}{8x}\tanh\frac{x}{2}-\frac{1}{16}\mathrm{sech}^2\frac{x}{2}\right)\mathrm{d}x$$
$$=\frac{1}{8}\int_0^{E_{\mathrm{F}}v/2kT}\frac{\tanh y}{y}\mathrm{d}y-\frac{1}{8}\int_0^{E_{\mathrm{F}}v/2kT}\mathrm{sech}^2y\,\mathrm{d}y=\frac{1}{8}\left[\ln\left(\frac{2\mathrm{e}^{\gamma}E_{\mathrm{F}}v}{\pi kT}\right)-1\right].$$

Thus (21) takes the form

$$\tfrac{1}{2}[(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}]\ln\left(\frac{T}{T_c}\right)=-4\delta'^2\times\frac{1}{4\pi^2}\frac{7}{8}\zeta(3)$$

$$\times\left[1+\frac{\pi^2}{14\zeta(3)}(D-2)(D-4)(kT/E_F)^2\left[\ln\left(\frac{2e^{\gamma}E_F\nu}{\pi kT}\right)-1\right]\right]. \tag{23}$$

Hence,

$$\Delta^2=\pi^2k^2T_c^2\frac{8}{7\zeta(3)}\left[\frac{(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}}{2}\right]\left(1-\frac{T}{T_c}\right)$$

$$\times\left[1-\frac{\pi^2}{14\zeta(3)}(D-2)(D-4)(kT_c/E_F)^2\left[\ln\left(\frac{2e^{\gamma}E_F\nu}{\pi kT}\right)-1\right]\right]. \tag{24}$$

Neglecting the second term, which is very small compared to the first term and using $\zeta(3)=1.202$, we obtain

$$\Delta=3{\cdot}06kT_c\left[\frac{(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}}{2}\right]^{1/2}\left(1-\frac{T}{T_c}\right)^{1/2}. \tag{25}$$

### 3.4 *Jump in the electronic specific heat at $T=T_c$*

The jump in the specific heat at $T=T_c$ is given by

$$(C_s-C_n)_{T=T_c}=\Delta C=-N(0)\left.\frac{d\Delta^2}{dT}\right|_{T=T_c}, \tag{26a}$$

where the normal specific heat is given by

$$C_n=\frac{2}{3}N(0)k^2\pi^2T. \tag{26b}$$

Using (24) (neglecting the term containing $(kT_c/E_F)^2$), we obtain

$$\frac{d\Delta^2}{dT}=\pi^2k^2T_c^2\frac{8}{7\zeta(3)}\left[\frac{(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}}{2}\right]\left(\frac{1}{T_c}\right).$$

Substituting this expression into (26a), we obtain

$$(C_s-C_n)_{T=T_c}=N(0)\pi^2k^2T_c\frac{8}{7\zeta(3)}\left[\frac{(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}}{2}\right],$$

or

$$\left.\frac{C_s-C_n}{C_n}\right|_{T=T_c}=1{\cdot}43\left[\frac{(1+\nu)^{(D-2)/2}+(1-\nu)^{(D-2)/2}}{2}\right]. \tag{27}$$

## 4. Solution of the BCS gap equation in $D$ dimensions for $\nu>1$

### 4.1 *Expression for $\delta$ near $T=0$*

We again start with eq. (6b). Using (7), we have

$$2\ln\left[\frac{\delta}{\delta_0}\right]=-2\int_0^{1+\nu}\frac{\varepsilon'^{(D-2)/2}d\varepsilon'}{(\delta^2+(\varepsilon'-1)^2)^{1/2}[e^{(\delta^2+(\varepsilon'-1)^2)^{1/2}E_F/kT}+1]}. \tag{28a}$$

Introducing $x$ and $\delta'$, defined by eqs (9a) and (9b), we obtain

$$\ln\left[\frac{\delta_0}{\delta}\right]=\int_{-E_F/kT}^{E_F\nu/kT}\frac{(1+kTx/E_F)^{(D-2)/2}\mathrm{d}x}{(\delta'^2+x^2)^{1/2}(\mathrm{e}^{(\delta'^2+x^2)^{1/2}}+1)}. \tag{28b}$$

For temperatures close to $T=0$, the limits can be taken as $-\infty$ and $+\infty$, and thus we have the same result as obtained for the case $\nu \ll 1$, viz. that given by eq. (12).

### 4.2 *Determination of the $T_c$ equation for general D*

Starting from (6a) with $\delta=0$, we have

$$\frac{2}{\lambda}=\int_0^{1+\nu}\frac{\varepsilon'^{(D-2)/2}}{\varepsilon'-1}\tanh\left((\varepsilon'-1)\frac{E_F}{2kT_c}\right)\mathrm{d}\varepsilon',$$

or

$$\frac{2}{\lambda}=\int_{-E_F/2kT_c}^{E_F\nu/2kT_c}\left(1+\frac{2kT_c y}{E_F}\right)^{(D-2)/2}\frac{\tanh y}{y}\mathrm{d}y, \tag{29}$$

with $y$ defined by (16).

Following a procedure similar to the one used in obtaining (17) from (15b), we have

$$\frac{2}{\lambda}=(1+\nu)^{(D-2)/2}\ln\left(\frac{E_F\nu}{2kT_c}\right)-[I_+(T_c)+I_-(T_c)], \tag{30}$$

with

$$I_+(T_c)=\int_0^{E_F\nu/2kT_c}\mathrm{d}y\ln y\left[\left(1+\frac{2kT_c y}{E_F}\right)^{(D-2)/2}\operatorname{sech}^2 y\right.$$
$$\left.+\frac{kT_c}{E_F}(D-2)\left(1+\frac{2kT_c y}{E_F}\right)^{(D-4)/2}\tanh y\right], \tag{31a}$$

and

$$I_-(T_c)=\int_0^{E_F/2kT_c}\mathrm{d}y\ln y\left[\left(1-\frac{2kT_c y}{E_F}\right)^{(D-2)/2}\operatorname{sech}^2 y\right.$$
$$\left.-\frac{kT_c}{E_F}(D-2)\left(1-\frac{2kT_c y}{E_F}\right)^{(D-4)/2}\tanh y\right]. \tag{31b}$$

Using binomial expansions to approximate the integrands in eqs (31a) and (31b), we have

$$I_+(T_c)+I_-(T_c)=\int_0^{E_F\nu/2kT_c}\mathrm{d}y\ln y$$
$$\times\left[\left(1+(D-2)\frac{kT_c y}{E_F}\right)\operatorname{sech}^2 y+(D-2)\frac{kT_c}{E_F}\tanh y\right]$$
$$+\int_0^{E_F\nu/2kT_c}\mathrm{d}y\ln y\left[\left(1-(D-2)\frac{kT_c y}{E_F}\right)\operatorname{sech}^2 y-(D-2)\frac{kT_c}{E_F}\tanh y\right].$$

Taking the upper limits in both the integrals to be $\infty$, we obtain

$$I_+(T_c)+I_-(T_c)=2\int_0^{\infty}\ln y\operatorname{sech}^2 y\,\mathrm{d}y=-2\ln\left(\frac{4\mathrm{e}^{\gamma}}{\pi}\right).$$

Therefore, (30) takes the form

$$\frac{2}{\lambda} = (1+\nu)^{(D-2)/2} \ln\left(\frac{E_F \nu}{2kT_c}\right) + 2\ln\left(\frac{4e^{\gamma}}{\pi}\right), \tag{32}$$

or

$$kT_c = \frac{E_F \nu}{2}\left(\frac{4e^{\gamma} e^{-1/\lambda}}{\pi}\right)^{2(1+\nu)/(1+\nu)^{D/2}} \tag{33}$$

4.3 *Expression for* $\delta$ *near* $T = T_c$

The counterparts of eqs (20a) and (20b) in this case are

$$\frac{2}{\lambda} = 4\sum_{n=0}^{\infty} \int_{-E_F/kT}^{E_F \nu/kT} \frac{(1+kTx/E_F)^{(D-2)/2}}{\pi^2(2n+1)^2 + \delta'^2 + x^2} dx, \tag{34a}$$

and

$$\frac{2}{\lambda} = 4\sum_{n=0}^{\infty} \int_{-E_F/kT}^{E_F \nu/kT} \frac{(1+kTx/E_F)^{(D-2)}}{[\pi^2(2n+1)^2 + x^2]} dx - 4\delta'^2 \times \sum_{n=0}^{\infty} \int_{-E_F/kT}^{E_F \nu/kT} \frac{(1+kTx/E_F)^{(D-2)/2}}{[\pi^2(2n+1)^2 + x^2]^2} dx, \tag{34b}$$

respectively.

The first term in (34b) gives the contribution

$$(1+\nu)^{(D-2)/2} \ln\left(\frac{E_F \nu}{2kT}\right) + 2\ln\left(\frac{4e^{\gamma}}{\pi}\right), \tag{35}$$

in consonance with (32) in which $T = T_c$.

Using (34b) and (35) in conjunction with (32), we obtain

$$\begin{aligned}(1+\nu)^{(D-2)/2} \ln\left(\frac{T}{T_c}\right) &= -4\delta'^2 \sum_{n=0}^{\infty} \int_{-E_F/kT}^{E_F \nu/kT} \frac{(1+kTx/E_F)^{(D-2)/2}}{[\pi^2(2n+1)^2 + x^2]^2} dx, \\ &= -4\delta'^2 \Bigg[ 2\sum_{n=0}^{\infty} \int_0^{\infty} \frac{dx}{[\pi^2(2n+1)^2 + x^2]^2} + \frac{(D-2)}{2}\frac{kT}{E_F} \\ &\quad \times \sum_{n=0}^{\infty} \left( \int_0^{E_F \nu/kT} \frac{x\,dx}{[\pi^2(2n+1)^2 + x^2]^2} - \int_0^{E_F/kT} \frac{x\,dx}{[\pi^2(2n+1)^2 + x^2]^2} \right) \Bigg]. \end{aligned} \tag{36}$$

The first integral on the right is given by (22a). Using (22b), we have

$$\begin{aligned}\int_0^{E_F \nu/kT} x\,dx \sum_{n=0}^{\infty} \frac{1}{[\pi^2(2n+1)^2 + x^2]^2} &= \int_0^{E_F \nu/kT} \left(\frac{1}{8x^2}\tanh\frac{x}{2} - \frac{1}{16x}\operatorname{sech}^2\frac{x}{2}\right) dx \\ &\simeq \frac{1}{16}\left[1 - \frac{2kT}{E_F \nu}\right]. \end{aligned} \tag{37a}$$

Similarly

$$\int_0^{E_F/kT} x\,dx \sum_{n=0}^{\infty} \frac{1}{[\pi^2(2n+1)^2 + x^2]^2} \simeq \frac{1}{16}\left[1 - \frac{2kT}{E_F}\right]. \tag{37b}$$

Using (22a), (37a) and (37b), eq. (36) becomes

$$(1+\nu)^{(D-2)/2}\ln\left(\frac{T}{T_c}\right) = -4\delta'^2\left[\frac{1}{2\pi^2}\frac{7}{8}\zeta(3) + \frac{(D-2)}{2}(kT/E_F)^2\frac{(\nu-1)}{8\nu}\right],$$

or

$$\delta'^2 = \frac{4\pi^2}{7\zeta(3)}(1+\nu)^{(D-2)/2}\left(1-\frac{T}{T_c}\right)\left[1-\frac{\pi^2}{7\zeta(3)}(D-2)(kT_c/E_F)^2\frac{(\nu-1)}{\nu}\right].$$

Since $\delta' = (E_F/kT)\,\delta = \Delta/kT$ (see eqs (9a), (9b) and (5a), (5b), (5c)), we have finally,

$$\Delta^2 = \frac{4\pi^2}{7\zeta(3)}k^2T_c^2(1+\nu)^{(D-2)/2}\left(1-\frac{T}{T_c}\right) \times\left[1-\frac{\pi^2}{7\zeta(3)}(D-2)(kT_c/E_F)^2\frac{(\nu-1)}{\nu}\right]. \quad (38)$$

### 4.4 *Jump in the specific heat*

Using (26a) and (26b), we obtain

$$\Delta C = N(0)\frac{4\pi^2}{7\zeta(3)}k^2T_c(1+\nu)^{(D-2)/2}\left[1-\frac{\pi^2}{7\zeta(3)}(D-2)(kT_c/E_F)^2\frac{(\nu-1)}{\nu}\right].$$

Neglecting the second term in the parentheses, compared to unity, we have

$$\left.\frac{C_s-C_n}{C_n}\right|_{T=T_c} = \frac{6}{7\zeta(3)}(1+\nu)^{(D-2)/2} = (1\cdot43)\frac{(1+\nu)^{(D-2)/2}}{2}. \quad (39)$$

For $\nu > 3$ and $D = 3$, this is greater than 1·43, the usual BCS value. The specific heat measurements in high $T_c$ materials are beset with many difficulties. The lattice contribution to the specific heat dominates the electronic contribution. Moreover, it is difficult to obtain reliable values of the Sommerfeld constant, $\gamma$. Nonetheless, in some experiments on Y123 powders one gets a jump in the specific heat, which is twice the BCS prediction [10].

## 5. Comparison of $T_c$ equations for $\nu \ll 1$ and $\nu > 1$ for $D = 3$

It is instructive to compare the $T_c$ equations for the cases $\nu \ll 1$ and $\nu > 1$. For the case of $\nu \ll 1$, $D = 3$, we have from (19)

$$(kT_c)_{BCS} = \frac{E_F\nu}{2}\left(\frac{4e^{\gamma}e^{-1/\lambda}}{\pi}\right)^{2/((1+\nu)^{1/2}+(1-\nu)^{1/2})}, \quad (19a)$$

and for the case of $\nu > 1$, $D = 3$, (33) gives

$$(kT_c) = \frac{E_F\nu'}{2}\left(\frac{4e^{\gamma}e^{-1/\lambda}}{\pi}\right)^{2/(1+\nu')^{1/2}}, \quad (33a)$$

where $\nu'$ is used for $\nu$ to distinguish it from the case $\nu \ll 1$. Dividing (33a) by (19a), we obtain

$$\frac{T_c}{(T_c)_{BCS}} = \frac{\nu'}{\nu}\left(\frac{4e^{\gamma}e^{-1/\lambda}}{\pi}\right)^{2[(1/(1+\nu')^{1/2})-(1/(1+\nu)^{1/2}+(1-\nu)^{1/2})]} \quad (40a)$$

Since $\nu \ll 1$, (40a) can be simplified to give

$$\frac{T_c}{(T_c)_{BCS}} = \frac{v'}{v}(2{\cdot}27\mathrm{e}^{-1/\lambda})^{2(1/(1+v')^{1/2}-1/2)}. \tag{40b}$$

Equation (40b) tells us how $T_c$ scales with the parameter $v'$, characterizing the mediating boson energy cutoff. Clearly there is an enhancement in the value of $T_c$ compared to the BCS value. For example for $v' = 3$ we have

$$\frac{T_c}{(T_c)_{BCS}} = \frac{3}{v}. \tag{41a}$$

Since $v \ll 1$, it follows that

$$\frac{T_c}{(T_c)_{BCS}} \gg 1. \tag{41b}$$

## 6. Calculation of $T_c$ from the BCS gap equation with a van Hove singularity in the density of states

As remarked earlier, non-phononic mechanisms have been proposed to explain the high transition temperatures exhibited by the new superconducting copper oxides. Apart from this a van Hove singularity in the density of states has been proposed as a $T_c$ enhancement mechanism and as a means to explain the anomalous isotope effect in these materials [10–14].

Let us assume a logarithmic density of states

$$N(\varepsilon) = N(0)\ln\left|\frac{E_F}{\varepsilon - E_F}\right|,$$

$$N(\varepsilon) = N(0)\ln\left|\frac{1}{\varepsilon' - 1}\right|, \qquad (\varepsilon' = \varepsilon/E_F). \tag{42}$$

The equation corresponding to (6a) becomes

$$\frac{2}{\lambda} = \int_{1-v,0}^{1+v} \mathrm{d}\varepsilon' \frac{\varepsilon'^{(D-2)/2}}{(\delta^2 + (\varepsilon'-1)^2)^{1/2}} \tanh\left(\frac{(\delta^2 + (\varepsilon'-1)^2)^{1/2}}{2kT/E_F}\right) \ln\left|\frac{1}{\varepsilon'-1}\right|. \tag{43}$$

At $T = T_c$, $\delta = \Delta/E_F = 0$. Putting $1 - \varepsilon' = x(\varepsilon' < 1)$, $\varepsilon' - 1 = x(\varepsilon' > 1)$ and $xE_F/2kT_c = y$, we get the equation

$$\frac{2}{\lambda} = \int_{E_F v/2kT_c, E_F/2kT_c}^{0} \frac{\tanh y}{y}\left(1 - \frac{2kT_c y}{E_F}\right)^{(D-2)/2} \ln\left(\frac{2kT_c y}{E_F}\right)\mathrm{d}y$$
$$- \int_0^{E_F v/2kT_c} \frac{\tanh y}{y}\left(1 + \frac{2kT_c y}{E_F}\right)^{(D-2)/2} \ln\left(\frac{2kT_c y}{E_F}\right)\mathrm{d}y. \tag{44}$$

As long as $v \ll 1$, the usual BCS result follows if we take $D = 2$, since the small width of the interaction makes the problem essentially two-dimensional. In this case (44) gives

$$\frac{2}{\lambda} = -2\ln\left(\frac{2kT_c}{E_F}\right)\int_0^{E_F v/2kT_c} \frac{\tanh y}{y}\mathrm{d}y - 2\int_0^{E_F v/2kT_c} \frac{\tanh y}{y}\ln y\,\mathrm{d}y. \tag{45}$$

Assuming $\tanh y \simeq y$ for $y \leqslant 1$ and $\tanh y \simeq 1$ for $y > 1$, one obtains

$$kT_c = \frac{eE_F}{2}\exp\left[-\left(\frac{2}{\lambda} - 1 + (\ln \nu)^2\right)^{1/2}\right]. \tag{46}$$

This, of course, agrees with (3) obtained by Tsuei *et al* [11]. These authors have shown that for $\lambda = 0{\cdot}12$, $T_F = 5800\,\text{K}$ and $\nu = 0{\cdot}12$, $T_c$ enhancement is obtained with $T_c = 92\,\text{K}$. If we use the $T_c$ equation for $D = 3$, $\nu > 1$ [eq. (33a)], we get the same $T_c$ enhancement with $\nu = 6{\cdot}83$. Thus it seems that non-phononic mechanisms can lead to high values of $T_c$ without invoking van Hove singularity.

Using (46), Tsuei *et al* [11] have been able to explain the anomalous isotope effect in the superconducting Cu oxides.

## 7. Conclusions

We have obtained analytic solutions of the BCS gap equation in one, two and three dimensions, for temperatures close to zero and $T_c$. Our calculations have been carried out in the weak coupling limit ($\lambda = N(0)v_0 \ll 1$). Expressions have been obtained for $T_c$ and $\Delta C$ for the cases $\nu \ll 1$ and $\nu > 1$, respectively, where $\nu$ is a measure of the interaction width.

The scaling of $T_c$ with $\nu$, for the regime $\nu > 1$, is given by (33). Equation (41b) shows that there is an enhancement in the value of $T_c$ in this limit. However, precise numerical estimates cannot be made within the framework of the present model. We may also mention that the appearance of non-zero $T_c$ values, in dimensions lower than three, is a consequence of the mean field approximation, on which the usual BCS framework rests. In fact the non local interaction of the BCS reduced Hamiltonian violates the $f$ sum rule [15].

We also find that $\Delta C$, the jump in the electronic specific heat, across $T_c$, has a larger value than the usual BCS value. Certain experiments have revealed a similar feature in some high $T_c$ materials.

The role of van Hove singularity seems to be crucial in understanding the anomalous isotope effect in new superconducting Cu oxides. A more detailed investigation is needed to make meaningful comments on this. It involves an analysis of (44) for the case $D = 3$, $\nu > 1$.

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