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Stationary rotating string world models with a magnetic field

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Abstract. Following the techniques used by Letelier and Stachel some new physically relevant stationary solutions of string cosmology with magnetic field are presented. In these solutions, the flow vector of matter has non-zero rotation and the cosmological constant is taken to be non-zero. Previously known solutions are derived as particular cases from our class of solutions. Some string models with vanishing cosmological constant are also discussed.

Keywords. Cosmology; strings; magnetic fields.

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1. Introduction

In recent times, the study of cosmic strings has received the attention of many investigators. The present day observations of the universe do not rule out the possibility of the existence of large scale networks of strings in the early universe [1]. It is believed that cosmic strings give rise to density perturbations which lead to galaxy formations [2]. The cosmic strings possess stress energy and are coupled to the gravitational field. Detailed accounts of various features of cosmic strings are given by Vilenkin [3], Gott [4] and Garfinkle [5].

Cosmic strings in the framework of general relativity have been discussed by Letelier [6] and Stachel [7]. They have given the energy momentum tensor for string distribution in the form

$$M_{ik} = \rho v_i v_k - \lambda w_i w_k, \tag{1}$$

where

$$v^i v_i = -w^i w_i = 1, \quad v^i w_i = 0. \tag{2}$$

Here ρ denotes the energy density of a cloud of strings with particles attached to them and λ is the string tension density. The unit time-like vector v^i is the flow vector and the unit space-like vector w^i specifies the direction of the strings. The particle density associated with the configuration is given by

$$\rho_p = \rho - \lambda. \tag{3}$$

The energy conditions imply $\rho \geqslant 0, \rho_p \geqslant 0$, leaving the sign of string tension density λ unrestricted.

The energy momentum tensor E_{ik} for source-free electromagnetic fields is given by

$$E_{ik} = -g^{lm}F_{il}F_{km} + \tfrac{1}{4}g_{ik}F_{im}F^{lm}, \tag{4}$$

where

$$F_{ik} = A_{i;k} - A_{k;i} \tag{5}$$

and

$$F^{ik}_{;k} = 0. \tag{6}$$

Here F_{ik} is the electromagnetic field tensor and A_i is the electromagnetic 4-potential and a semi-colon indicates covariant differentiation. In the present paper we shall use Einstein field equations in the form

$$R_{ik} - \tfrac{1}{2}Rg_{ik} = -8\pi[M_{ik} + E_{ik}] - \Lambda g_{ik}, \tag{7}$$

where Λ is the cosmological constant.

The string cosmological models with a magnetic field are discussed by Banerjee *et al* [8], Chakraborty [9] and Tikekar and Patel [10, 11]. All these models are non-static and the flow vector v^i is irrotational. Patel and Dadhich [12] have constructed some stationary rotating string models with non-zero cosmological constant. The main purpose of the present article is to introduce an axial magnetic field in the string world models discussed earlier by Patel and Dadhich [12].

Note that rotation plays an important role in the structure and equilibrium configurations of elementary particles, and also in astrophysical bodies. For example the nature of pulsars, which are widely accepted as rotating neutron stars, can only be properly described by including the effects of rotation. During the last three decades many researchers have introduced rotation in the general theory of relativity in discussions of rotating cosmological models. Rotation is likely to significantly affect the behaviour of the gravitational field in the early universe. As we are attempting to model the early universe in terms of networks of strings, rotation should be included in our analysis to determine its effect on the field.

The occurrence of magnetic fields on galactic scales is a well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by Zeldovich *et al* [13]. Also Harrison [14] has suggested that magnetic fields could have a cosmological origin. As a natural consequence we should include magnetic fields in the energy-momentum tensor of the early universe. Therefore rotating string models with magnetic fields may have physical significance for early universe cosmology.

2. The metric and the field equations

We consider a Gödel-type stationary line element in the form

$$ds^2 = (dt + H\,dy)^2 - \alpha^2 dy^2 - dz^2 - dx^2, \tag{8}$$

where H and α are functions of x only.

We introduce the tetrad

$$\theta^1 = \mathrm{d}x, \quad \theta^2 = \alpha \mathrm{d}y, \quad \theta^3 = \mathrm{d}z, \quad \theta^4 = \mathrm{d}t + H\,\mathrm{d}y \tag{9}$$

to write the metric in the form

$$\mathrm{d}s^2 = (\theta^4)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b. \tag{10}$$

Here and in what follows the bracketed indices indicate tetrad components. Thus $g_{(ab)} = \mathrm{diag}\{-1, -1, -1, 1\}$ are tetrad components of the metric tensor g_{ik}. Using Cartan's equations of structure one can easily obtain the tetrad components $R_{(ab)}$ of the Ricci tensor for the metric (8). The nonzero $R_{(ab)}$ are given by

$$R_{(11)} = R_{(22)} = \frac{\alpha''}{\alpha} - \frac{H'^2}{2\alpha^2}, \quad R_{(44)} = -\frac{H'^2}{2\alpha^2},$$

$$R_{(24)} = -\frac{1}{2\alpha}\left(H'' - \frac{H'\alpha'}{\alpha}\right). \tag{11}$$

Here and in what follows a prime indicates differentiation with respect to x.

It is easy to see that the field equation (7) can be expressed in the tetrad form as

$$R_{(ab)} = -8\pi[\rho v_{(a)}v_{(b)} - \lambda w_{(a)}w_{(b)}] + 4\pi(p+\lambda)g_{(ab)}$$
$$- 8\pi E_{(ab)} + \Lambda g_{(ab)}. \tag{12}$$

We use comoving coordinates and take the string direction as z-direction. Therefore we have

$$v_{(a)} = (0,0,0,1) \quad w_{(a)} = (0,0,1,0). \tag{13}$$

It is an easy task to check that v_i and w_i given by (13) satisfy the condition (2).

We wish to consider an axial magnetic field along z-axis. Therefore the electromagnetic 4-potential A_i can be taken as

$$A_i = (0, \psi, 0, 0), \tag{14}$$

where ψ is a function of x only and the coordinates are named as $x^1 = x, x^2 = y, x^3 = z, x^4 = t$. Therefore the only non-vanishing component F_{12} of the electromagnetic field tensor is given by $F_{12} = -\psi'$. The Maxwell equation (6) lead to

$$\psi' = K\alpha, \tag{15}$$

where K is a constant of integration. The surviving E_{ik} are found to have the expressions

$$E_{11} = E_{44} = -E_{33} = E_{24}/H = E_{22}|_{(H^2+\alpha^2)} = \tfrac{1}{2}K^2. \tag{16}$$

The tetrad components $E_{(ab)}$ of E_{ik} can be obtained from

$$E_{(ab)} = \mathrm{e}^i_{(a)}\mathrm{e}^k_{(b)}E_{ik}, \quad \mathrm{e}^i_{(a)}\theta^a = \mathrm{d}x^i.$$

The surviving $E_{(ab)}$ are given by

$$E_{(11)} = E_{(22)} = -E_{(33)} = E_{(44)} = \tfrac{1}{2}K^2. \tag{17}$$

In view of (10), (13) and (17), the field equation (12) give the following non-trivial relations:

$$R_{(24)} = 0, \tag{18}$$

$$R_{(11)} = -4\pi(\rho+\lambda) - 8\pi E_{(11)} - \Lambda, \tag{19}$$

$$\Lambda = 4\pi(\lambda-\rho) - 8\pi E_{(33)}, \tag{20}$$

$$R_{(44)} = -8\pi E_{(44)} + 4\pi(\lambda-\rho) + \Lambda, \tag{21}$$

where $R_{(ab)}$ are given by (11).

Equation (18) is easily integrable. The solution of this equation is

$$\frac{H'}{\alpha} = \text{constant} = m, \quad \text{say}. \tag{22}$$

From (20), (21) and (22) we obtain

$$\Lambda = -\tfrac{1}{4}m^2 + 4\pi K^2. \tag{23}$$

It is interesting to note that the cosmological constant Λ depends upon the magnetic field also. From (19) and (20) we get

$$8\pi\lambda = \frac{m^2}{2} - 8\pi K^2 - \frac{\alpha''}{\alpha}, \tag{24}$$

$$8\pi\rho = m^2 - 8\pi K^2 - \frac{\alpha''}{\alpha}. \tag{25}$$

Obviously the strings satisfy the equation of state $\rho = \lambda + (m^2/16\pi)$.

From the results (24) and (25) we have

$$8\pi\rho_p = \frac{m^2}{2}. \tag{26}$$

Note that ρ_p is independent of magnetic field and is always positive.

We have checked that the velocity field v_i given by (13) has vanishing expansion, shear and acceleration. But the rotation Ω of the velocity field v_i is non-zero and it is given by

$$\Omega = \sqrt{2}m. \tag{27}$$

As the metric potential α remains undetermined, we can have many rotating string models with a magnetic field. In the next section we shall give some physically relevant explicit solutions of the equations (22)–(25).

From the results (26) and (27) it is clear that the rotation Ω and the particle density ρ_p are intimately linked. The rotating character of the string is determined by the component Ω. The vanishing of Ω implies the vanishing of ρ_p. Thus in the absence of rotation, we have $\rho = \lambda$ and consequently we regain string-dust cosmological models.

3. Explicit solutions

Case (i): $\Lambda = 0$.

If the cosmological constant Λ is zero, we have $m^2 = 16\pi K^2$. In this case we obtain

$$8\pi\lambda = -\frac{\alpha''}{\alpha}, \quad 8\pi\rho = 8\pi K^2 - \frac{\alpha''}{\alpha}, \quad \Omega = 4\sqrt{2\pi}K. \tag{28}$$

Here two particular cases are noteworthy.

Case (ia): $\alpha'' = 0$ (i.e. $\alpha = x$).

In this case string tension density λ vanishes. The line element in this case becomes

$$ds^2 = (dt + 4K\sqrt{\pi}x dy)^2 - x^2 dy^2 - dx^2 - dz^2. \tag{29}$$

This metric represents a dust-filled stationary rotating universe with magnetic field whose density is given by $\rho = K^2$. Thus the rotation and the density entirely depend upon the magnetic field. When the magnetic field is switched off, the metric (29) reduces to

$$ds^2 = dt^2 - x^2 dy^2 - dx^2 - dz^2. \tag{30}$$

The metric (30) is a flat metric. The required simple transformation is $X = x\cos y$, $Y = x\sin y$. Thus the matter-free limit of this rotating dust model is a flat metric.

Case (ib): $\alpha''/\alpha = -n^2$.

In this case we have

$$8\pi\lambda = n^2, \quad 8\pi\rho = 8\pi K^2 + n^2, \quad \Omega = 4\sqrt{2\pi}K. \tag{31}$$

When the magnetic field is removed, we get a static string-dust solution obtained earlier by Patel and Dadhich [12]. The geometry of this solution is described by the metric

$$ds^2 = \left(dt + \frac{4K\sqrt{\pi}}{n}\sin nx\, dy\right)^2 - \cos^2 nx\, dy^2 - dx^2 - dz^2. \tag{32}$$

Case (ii): $\lambda = 0$.

In this case we have

$$8\pi\rho = 8\pi\rho_p = \frac{m^2}{2}, \quad \Omega = \sqrt{2}m, \quad \frac{\alpha''}{\alpha} = \frac{m^2}{2} - 8\pi K^2. \tag{33}$$

The subcase $\alpha'' = 0$ has been considered in case (i). Here there are two subcases to be considered.

Case (iia): $m^2 > 16\pi K^2$.

Let us put $m^2 - 16\pi K^2 = 2a^2$. We have in this case

$$\Lambda = -\frac{a^2}{2}, \quad \alpha = e^{ax}, \cdot H = \frac{m}{a}e^{ax}, \quad 8\pi\rho = a^2 + 8\pi K^2. \tag{34}$$

The line element of this case is

$$ds^2 = \left(dt + \frac{m}{a}e^{ax}dy\right)^2 - e^{2ax}dy^2 - dx^2 - dz^2. \tag{35}$$

The metric (35) represents the Gödel [15] universe with a magnetic field.

Case (iib): $m^2 < 16\pi K^2$.

Let us put $m^2 - 16\pi K^2 = -2b^2$, i.e. $m^2 = 16\pi K^2 - 2b^2$. Note that here we cannot remove the magnetic field. In this case we have

$$\Lambda = \frac{b^2}{2}, \quad 8\pi\rho = 8\pi K^2 - b^2, \quad \alpha = \cos bx,$$

$$H = \frac{m}{b}\sin bx, \quad \Omega = \sqrt{2}m. \tag{36}$$

The final form of the metric of this subcase is

$$ds^2 = \left(dt + \frac{m}{b}\sin bx\, dy\right)^2 - \cos^2 bx\, dy^2 - dx^2 - dz^2. \tag{37}$$

When $\rho = 0$, i.e. $b^2 = 8\pi K^2$, the rotation vanishes and the metric (37) reduces to the metric of a simple magnetic universe discussed by Patel and Vaidya [16]. Thus the metric (37) represents a dust-filled generalization of the simple magnetic universe discussed by Patel and Vaidya [16].

Case (iii): Takabayashi strings.

Let us assume that the string distribution has the equation of state

$$\rho = (1 + W)\lambda, \tag{38}$$

where W is a positive constant. Such strings are known as Takabayashi strings.

The substitution of ρ and λ from (25) and (24) in the result (38) yields the differential equation

$$\frac{\alpha''}{\alpha} = \frac{m^2(W-1) - 16\pi W K^2}{2W}. \tag{39}$$

The solution of (39) depends upon the values of the constants m, W and K. In this case we have

$$8\pi\rho = \frac{m^2(1+W)}{2W}, \quad 8\pi\lambda = \frac{m^2}{2W}, \quad \Lambda = -\tfrac{1}{4}m^2 + 4\pi K^2. \tag{40}$$

As $W > 0$, the energy conditions are satisfied. Let us put

$$m^2(W-1) - 16WK^2 = 2W\varepsilon a^2. \tag{41}$$

Then

$$\alpha = \begin{cases} x \\ e^{ax} \\ \cos ax \end{cases} \quad \text{when} \quad \varepsilon = \begin{cases} 0 \\ 1 \\ -1 \end{cases}. \tag{42}$$

In each case we can write the explicit forms of the line elements. For the sake of brevity we shall not give these details here. When the magnetic field is switched off (i.e. $K = 0$), we get the solution for Takabayashi strings discussed by Patel and Dadhich [12].

4. Discussion

Note that the solutions presented in this paper are all homogeneous. This essentially follows because of our particular choice for the function α; other choices of α are certainly possible. Clearly there exists forms for α which would lead to inhomogeneous solutions. We provide a simple example of a solution which is inhomogeneous in nature. Let us assume that $m^2 - 8\pi K^2 = 4a^2$, i.e. $m^2 - 8\pi K^2$ is positive. Also take $\alpha = \cosh^2 ax$. Then we have

$$8\pi\rho = 2a^2 \operatorname{sech}^2 ax, \quad 8\pi\lambda = 8\pi\rho - \tfrac{1}{2}m^2,$$

$$H = \frac{m}{2}\left[x + \frac{1}{2a}\sinh 2ax\right].$$

The geometry of this inhomogeneous rotating string model is described by the line element

$$ds^2 = \left[dt + \frac{m}{2}\left(x + \frac{1}{2a}\sinh 2ax\right)dy\right]^2 - dx^2 - dz^2 - \cosh^4 ax\, dy^2,$$

where a is determined by the above relationship.

There are few exact solutions to the Einstein equations for cosmic strings available in the literature. We have made an attempt to obtain more exact solutions so that our understanding of cosmic strings may be improved. It would be interesting to find non-static generalizations of the solutions discussed above.

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