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Magnetized cylindrically symmetric universe in general relativity

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Abstract. A magnetized cylindrically symmetric universe with two degrees of freedom in which the free gravitational field is Petrov type I degenerate, is obtained. The magnetic field is due to an electric current produced along the x-axis. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity. The behaviour of the model when magnetic field tends to zero and other physical aspects of the model are also discussed.

Keyword. General relativity.

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1. Introduction

Stewart [1] investigated the models as more stable if the matter is supposed to be anisotropic. Ellis and MacCallum [2] obtained a class of homogeneous models for perfect fluid distribution. The anisotropic magnetic fluid models have significant contribution in the evolution of galaxies and stellar bodies. Primordial magnetic fields of cosmological origin have been speculated by Asseo and Sol [3]. FRW models are approximately valid as present day magnetic field strength is very small. In early universe the strength must have been appreciable. The breakdown of isotropy is due to magnetic field. Thorne [4] investigated an LRS Bianchi type I cosmological models containing a magnetic field. Jacobs [5, 6] investigated Bianchi type I cosmological models with magnetic field satisfying barotropic equation of state. Collins [7] gave a qualitative analysis of Bianchi type I models with magnetic field. Roy *et al* [8] obtained Bianchi type I cosmological model with perfect fluid and magnetic field introducing space-like unit vector field orthogonal to the coordinate time. Recently, Bali [9] investigated a magnetized cosmological model in which expansion θ in the model is proportional to σ_1^1, the eigenvalue of shear tensor σ_i^j for perfect fluid distribution. Bali and Tyagi [10] investigated stiff magnetofluid cosmological model for perfect fluid distributions. Bali and Ali [11], to appear in IJTP obtained magnetized cosmological model with two degrees of freedom for perfect fluid distribution in general relativity. Roy and Prakash [12] obtained a plane symmetric cosmological model with an incident magnetic field for perfect fluid distribution in which the gravitational field is of Petrov type I degenerate. Roy *et al* [13] investigated a cylindrically symmetric universe with two degrees of freedom in general relativity in which free gravitational field is Petrov type I degenerate.

In this paper we have obtained a magnetized cylindrically symmetric universe with two degrees of freedom in general relativity in which free gravitational field is of Petrov type I degenerate. The magnetic field is due to an electric current along x-axis. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity.

Let us consider an anisotropic homogeneous universe with two degrees of freedom in the form

$$dS^2 = A^2(dx^2 - dt^2) + C^2dz^2 + (B^2 + D^2)dy^2 + 2CDdydz, \tag{1.1}$$

where the metric potentials A, B, C, D are function of time alone. The energy momentum tensor is got in the form

$$T_i^j = (\varepsilon + p)v_i v^j + p g_i^j + E_i^j, \tag{1.2}$$

where E_i^j is the electromagnetic field given by Lichnerowicz (1967) as

$$E_i^j = \bar{\mu}[|h|^2(v_i v^j + \tfrac{1}{2}g_i^j - h_i h^j)], \tag{1.3}$$

where ε is the density, p the pressure and v^i the flow vector satisfying

$$g_{ij}v^i v^j = -1. \tag{1.4}$$

$\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\varepsilon_{ijkl}F^{kl}v^j, \tag{1.5}$$

where F_{kl} is the electromagnetic field tensor and ε_{ijkl} the Levi-Civita tensor density. A semi colon stands for covariant differentiation. We assume that the coordinates are comoving so that $v^1 = 0 = v^2 = v^3$ and $v^4 = 1/A$. We take the incident magnetic field to be in the direction of x-axis, so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = F_{13} = 0$, by virtue of (1.5). We also find that $F_{14} = F_{24} = F_{34} = 0$ due to the assumption of the inifinite conductivity of the fluid. Thus, the only non-vanishing component of F_{ij} is F_{23}. The first set of Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0$$

leads to $F_{23} = \text{constant} = H(\text{say})$.

Hence

$$h_1 = \frac{AH}{\bar{\mu}BC} \tag{1.6}$$

The Einstein's field equations

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + \Lambda g_i^j = -8\pi T_i^j$$

for the line element (1.1) are

$$\frac{1}{A^2}\left\{\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} - \frac{B_4C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C}\right\} - \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda = 8\pi\left(p - \frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{1.7}$$

$$-\frac{1}{A^2}\left\{\frac{A_{44}}{A}+\frac{B_{44}}{B}-\frac{A_4^2}{A^2}\right\}-\frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44}+\frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B}-\frac{3C_4}{C}\right)$$
$$-\frac{3}{4}\left[\frac{CD_4-DC_4}{ABC}\right]^2-\Lambda=8\pi\left(p+\frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{1.8}$$

$$-\frac{1}{A^2}\left\{\frac{A_{44}}{A}+\frac{C_{44}}{C}-\frac{A_4^2}{A^2}\right\}+\frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44}-\frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B}-\frac{3C_4}{C}\right)$$
$$+\frac{1}{4}\left[\frac{CD_4-DC_4}{ABC}\right]^2-\Lambda=8\pi\left(p+\frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{1.9}$$

$$\frac{1}{A^2}\left\{\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right\}-\frac{1}{4}\left[\frac{CD_4-DC_4}{ABC}\right]^2+\Lambda=8\pi\left(\varepsilon+\frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{1.10}$$

and

$$\left(\frac{D}{C}\right)_{44}-\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B}-\frac{3C_4}{C}\right)=0. \tag{1.11}$$

2. Solution of the field equations

Equations (1.7)–(1.11) are five equations in six unknowns A, B, C, D, ε and p. For complete determination of this set, we need an extra condition. We assume that the free gravitational field is Petrov type I degenerate which leads to

$$C_{12}^{12}=C_{13}^{13}.$$

This implies that

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}+\frac{2A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right)=\left[\frac{CD_4-DC_4}{BC}\right]^2. \tag{2.1}$$

Equation (1.11) leads to

$$\frac{CD_4-DC_4}{BC}=\frac{K}{C^2}, \tag{2.2}$$

where K is a constant of integration. From eqs (1.7), (1.8), (1.9) and (1.11), we have

$$\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}-\frac{B_4C_4}{BC}-\frac{C_{44}}{C}+\left(\frac{A_4}{A}\right)_4=-\frac{1}{2}\left[\frac{CD_4-DC_4}{BC}\right]^2-\frac{8\pi H^2A^2}{\bar{\mu}B^2C^2}, \tag{2.3}$$

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\left[\frac{CD_4-DC_4}{BC}\right]^2. \tag{2.4}$$

From eqs (2.1), (2.2) and (2.4), we have

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\frac{K^2}{C^4} \tag{2.5}$$

and

$$\frac{A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right)=0. \tag{2.6}$$

Since $B \neq C$, (2.6) on integration leads to

$$A = \text{constant} = \alpha(\text{say}). \tag{2.7}$$

From (2.3), (2.4) and (2.7), we have

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{2B_4C_4}{BC}=\frac{L}{B^2C^2}, \tag{2.8}$$

where $L=\alpha^2 b$ and $b=16\pi H^2/\bar{\mu}$.

Putting $B=e^{u+v/2}$ and $C=e^{u-v/2}$ in (2.5) and (2.8), we have

$$e^u v_4=\sqrt{M-K^2e^{2v}} \tag{2.9}$$

and

$$\mu f f' = L, \tag{2.10}$$

where M is a constant of integration. Equation (2.10) leads to

$$f^2=\log(N\mu)^{2L}, \tag{2.11}$$

where

$$(BC)=\mu \text{ and } \mu_4=f(\mu). \tag{2.12}$$

Also equation (2.9) leads to

$$e^{-v}=\frac{K}{\sqrt{M}}\cosh\phi, \tag{2.13}$$

where $\phi=P-\sqrt{2M}/L\{\log(N\mu)^L\}^{1/2}$.

From (2.2), (2.12) and (2.13), we have

$$\frac{D}{C}=R-\frac{\sqrt{M}}{K}\tanh\phi, \tag{2.14}$$

where P and R are constants of integration.

From (2.12), (2.13) and (2.14), we have

$$B^2=\frac{\sqrt{M}}{K}\mu\,\text{sech}\,\phi, \tag{2.15}$$

$$C^2=\frac{K}{\sqrt{M}}\mu\cosh\phi, \tag{2.16}$$

$$D^2=\frac{K\mu}{\sqrt{M}}\left(R-\frac{\sqrt{M}}{K}\tanh\phi\right)^2\cosh\phi \tag{2.17}$$

and

$$CD=\frac{K}{\sqrt{M}}\mu\left(R-\frac{\sqrt{M}}{K}\tanh\phi\right)\cosh\phi. \tag{2.18}$$

Hence, the metric (1.1) reduces to the form

$$dS^2 = \alpha^2(dx^2 - dt^2) + \frac{\mu}{\sqrt{M}} K \cosh\phi \, dz^2 + K\left\{\left(\frac{\mu R^2}{\sqrt{M}} + \frac{\mu\sqrt{M}}{K^2}\right)\cosh\phi - \frac{2\mu R}{K}\sinh\phi\right\}dy^2 + \frac{2\mu}{\sqrt{M}} K\left(R\cosh\phi - \frac{\sqrt{M}}{K}\sinh\phi\right)dy\,dz. \quad (2.19)$$

After suitable transformations, the metric reduces to the form

$$dS^2 = \alpha^2\left(dX^2 - \frac{dT^2}{\log T}\right) + 2T\cosh\phi \, dZ^2 + \left\{\left(2TR^2 + \frac{T}{2}\right)\cosh\phi - 2TR\sinh\phi\right\}dY^2 + 4T\left(R\cosh\phi - \frac{1}{2}\sinh\phi\right)dY\,dZ. \quad (2.20a)$$

In the absence of magnetic field, the metric (2.20a) reduces to the form

$$dS^2 = \alpha^2\left(dX^2 - \frac{dT^2}{\log T}\right) + 2T\cosh\phi \, dZ^2 + \left[\left(2TR^2 + \frac{T}{2}\right)\cosh\phi - 2TR\sinh\phi\right]dY^2 + 4T\left(R\cosh\phi - \frac{1}{2}\sinh\phi\right)dY\,dZ, \quad (2.20b)$$

where $\phi = P - (\log T)^{1/2}$.

3. Some physical and geometrical features

The pressure and density for the model (2.20a) are given by

$$8\pi p = \frac{1}{\alpha^2 T^2}\left(\frac{1}{4}\log T - \frac{9}{16} + \frac{\alpha}{8}\right) - \Lambda, \quad (3.1)$$

$$8\pi\varepsilon = \frac{1}{\alpha^2 T^2}\left(\frac{1}{4}\log T - \frac{1}{16} - \frac{\alpha}{8}\right) + \Lambda. \quad (3.2)$$

The model has to satisfy the reality condition (Ellis (1971)) (i) $\varepsilon + p > 0$ and (ii) $\varepsilon + 3p > 0$.

The condition (i) leads to

$$T^{1/2} > \exp\left(\frac{5}{8}\right). \quad (3.3)$$

The condition (ii) leads to

$$T > \exp\left(\frac{7}{4} - \frac{\alpha}{4} + 16\pi\Lambda\alpha^2 T^2\right) \quad (3.4)$$

which gives condition on Λ. From (3.3) and (3.4), we find that reality conditions are satisfied.

The scalar of expansion θ calculated for the flow vector v^i is given by

$$\theta = \frac{1}{\alpha T}(\log T)^{1/2}. \quad (3.5)$$

The rotation ω is identically zero and the components of shear tensor is given by

$$\sigma_1^1 = -\frac{1}{3\alpha T}(\log T)^{1/2}, \tag{3.6}$$

$$\sigma_2^2 = \frac{1}{\alpha T}\left[\frac{1}{6}(\log T)^{1/2} - LR\right], \tag{3.7}$$

$$\sigma_3^3 = \frac{1}{\alpha T}\left[\frac{1}{6}(\log T)^{1/2} + LR\right], \tag{3.8}$$

$$\sigma_3^2 = \frac{1}{2\alpha T}. \tag{3.9}$$

The non-vanishing components of the conformal curvature tensor are

$$C_{12}^{12} = C_{13}^{13} = -\tfrac{1}{2}C_{23}^{23}$$
$$= \frac{1}{6\alpha^2 T^2}\left(\frac{1}{2}\log T - \frac{3}{8}\right). \tag{3.10}$$

The model (2.20a) does not exist between $T=0$ and $T=1$. At $T=1$, $\theta=0$. Hence it starts from rest and goes on expanding till $T=\infty$ when θ is zero again. The model in general represents expanding, shearing and non-rotating universe since $\lim_{T\to\infty}$, $\sigma/\theta \neq 0$. Hence, the model does not approach isotropy for large values of T. The ratio of magnetic energy to material energy is given by

$$\frac{E_4^4}{\varepsilon} = \frac{\sqrt{2}L^{3/2}\alpha T}{4\left[\frac{1}{2}\log T - \frac{L}{8} - \frac{L\alpha}{4} + 8\pi\Lambda a^2 T^2\right]}. \tag{3.11}$$

Since $\lim_{T\to 0} E_4^4/\varepsilon \to 0$, the material energy is more dominant than magnetic energy.

In the absence of magnetic field, the pressure and density for the model (2.20b) are given by

$$8\pi p = \frac{1}{\alpha^2 T^2}\left(\frac{1}{4}\log T + \frac{\alpha}{8} - \frac{9}{16}\right) - \Lambda, \tag{3.12}$$

$$8\pi \varepsilon = \frac{1}{\alpha^2 T^2}\left(\frac{1}{4}\log T - \frac{\alpha}{8} - \frac{1}{16}\right) + \Lambda. \tag{3.13}$$

The expansion θ, the components of shear tensor and non-vanishing components of conformal curvature tensor in the absence of magnetic field are given by

$$\theta = \frac{1}{\alpha T}(\log T)^{1/2}, \tag{3.14}$$

$$\sigma_1^1 = -\frac{1}{3\alpha T}(\log T)^{1/2}, \tag{3.15}$$

$$\sigma_2^2 = \frac{1}{6\alpha T}(\log T)^{1/2}, \tag{3.16}$$

$$\sigma_3^3 = \frac{1}{6\alpha T}(\log T)^{1/2}, \tag{3.17}$$

$$\sigma_3^2 = \frac{1}{2\alpha T} \tag{3.18}$$

and

$$C_{12}^{12} = C_{13}^{13} = -\frac{1}{2}C_{23}^{23}$$
$$= \frac{1}{6\alpha^2 T^2}\left(\frac{1}{2}\log T - \frac{3}{8}\right). \tag{3.19}$$

The reality condition $(\varepsilon + p) > 0$ leads to

$$T^{1/2} > \exp\left(\frac{5}{8}\right).$$

The condition $(\varepsilon + 3p) > 0$ leads to

$$T > \exp\left(\frac{7}{4} - \frac{\alpha}{4} + 16\pi\Lambda\alpha^2 T^2\right)$$

which gives condition on Λ. Thus reality conditions are satisfied for T. The model in the absence of magnetic field starts expanding with a big bang at $T = 0$ and it stops at $T = 1$. In the absence of magnetic field, the model in general represents expanding, shearing, non-rotating and Petrov type I degenerate universe in general. Since $\lim_{T \to \infty} \sigma/\theta$ does not tend to zero, the model does not approach isotropy for large values of T.

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