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A spherically symmetric gravitational collapse-field with radiation

P C VAIDYA and L K PATEL
Department of Mathematics, Gujarat University, Ahmedabad 380 009, India

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Abstract. An interior spherically symmetric solution of Einstein's field equations corresponding to perfect fluid plus a flowing radiation-field is presented. The physical 3-space t = constant of our solution is spheroidal. Vaidya's pure radiation field is taken as the exterior solution. The inward motion of the collapsing boundary surface follows from the equations of fit. An approximation procedure is used to get a generalization of the standard Oppenheimer–Snyder model of collapse with outflow of radiation. One such explicit solution has been given correct to second power of eccentricity of the spheroidal 3-space.

Keywords. General relativity; collapse with radiation.

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1. Introduction

Gravitational collapse is one of the important problems in which general relativity can play a significant role. The problem has many interesting astrophysical applications. It is well known that the formation of compact stars is usually preceded by an epoch of radiative collapse. In the collapse problems, the surface of the star divides the entire space-time into two different regions: the region inside the surface of the star, called the interior region, filled with matter and flowing radiation, and the region outside that surface called the exterior region which will usually be filled with pure radiation. These two regions must be matched smoothly across the surface of the star.

Historically Oppenheimer and Snyder [1] were the first to discuss the gravitational collapse of dust ball with static Schwarzschild exterior. Since then the study of relativistic models describing collapsing bodies has received considerable attention. Vaidya [2, 3] and Lindquist *et al* [4] studied outgoing radiation from collapsing bodies. Many attempts have been made to formulate and solve the relativistic equations for collapse [5]. Misner [6] obtained the basic equations of spherical collapse allowing for a simplified heat transfer process in which internal energy is converted into an outward flux of neutrinos. Santos and his collaborators [7–10] have carried out a detailed analysis of non-adiabatic collapse of spherical radiating bodies and have used this analysis to propose models for radiating collapsing spherical bodies with heat flow [11]. Vaidya and Patel [12] have presented a radiating collapse solution based on Schwarzchild interior solution.

In the present paper we discuss a new spherically symmetric collapse solution with radiation whose physical 3-space $t = \text{constant}$ is spheroidal. The space-times with spheroidal physical 3-space have been discussed in detail by Vaidya and Tikekar [13] (see also Tikekar [14]).

2. The interior space-time

Vaidya and Tikekar [13] have shown that the metric

$$ds^2 = e^{\nu(r)}dt^2 - \left(\frac{1-kr^2/R^2}{1-r^2/R^2}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

can represent the interior of a superdense star with total mass of about $3{\cdot}5\, M_{\odot}$. If the mass exceeds that limit equilibrium is not possible and gravitational collapse must follow. It is our aim to study this collapse.

Put $r = R\sin\lambda$ and rewrite the metric as

$$ds^2 = e^{\nu}dt^2 - R^2[\{\cos^2\lambda + (1-k)\sin^2\lambda\}d\lambda^2 + \sin^2\lambda(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

with $k = 1 - b^2/R^2$ where the 3-space $dt = 0$ is

$$\frac{x^2+y^2+z^2}{R^2} + \frac{w^2}{b^2} = 1.$$

For a contracting situation we assume R and b to be functions of t such that $b^2/R^2 = 1-k$ is a constant. So if e is the eccentricity of the spheroidal 3-space, our assumption implies that during gravitational contraction this eccentricity of the spheroidal 3-space remains constant.

We introduce a new co-ordinate r by setting $\sin\lambda = r$ and choose $e^{\nu} = 1$. We therefore consider the spherically symmetric space-time given by the line-element

$$ds^2 = dt^2 - R^2(t)\left[\frac{(1-kr^2)}{(1-r^2)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{1}$$

where R is an arbitrary function of time t and k is a constant. The metric (1) is an obvious generalization of the Oppenheimer–Snyder metric. We name the coordinates as $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$. It is a routine matter to compute the Einstein tensor G^i_k for metric (1). The surviving components of G^i_k are listed below for ready reference

$$-G^1_1 = \frac{1-k}{R^2(1-kr^2)} + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2},$$

$$-G^2_2 = -G^3_3 = \frac{1-k}{R^2(1-kr^2)^2} + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2},$$

$$-G^4_4 = \frac{2(1-k)}{R^2(1-kr^2)^2} + \frac{1-k}{R^2(1-kr^2)} + 3\frac{\dot{R}^2}{R^2}. \tag{2}$$

Here and in what follows, an overhead dot indicates differentiation with respect to time t.

Einstein's field equations are

$$R^i_k - (1/2)\delta^i_k R = G^i_k = -8\pi T^i_k, \tag{3}$$

where T^i_k are the components of energy momentum tensor.

We assume that the material contents of the space-time is a mixture of perfect fluid and outflowing radiation. The expression for T^i_k for such a distribution is given by

$$T^i_k = (p+\rho)v^i v_k - p\delta^i_k + \sigma w^i w_k \tag{4}$$

with

$$v^i v_i = 1, \quad w_i w^i = 0, \quad v^i w_i = 1, \tag{5}$$

where p, ρ, σ are respectively the fluid pressure, the matter density and the density of flowing radiation. We take v^i and w^i in the form $v^i = (v^1, 0, 0, v^4)$ and $w^i = (w^1, 0, 0, w^4)$. Then the condition (5) imply

$$\begin{aligned} &-e^{\alpha}(v^1)^2 + (v^4)^2 = 1, \quad -e^{\alpha}(w^1)^2 + (w^4)^2 = 0, \\ &-e^{\alpha}v^1 w^1 + v^4 w^4 = 1, \quad e^{\alpha} = R^2(1-kr^2)/(1-r^2). \end{aligned} \tag{6}$$

Equation (6) can be used to find $e^{\alpha/2}v^1, v^4, e^{\alpha/2}w^1$ and w^4 in terms of a single parameter n. Thus we get

$$\begin{aligned} &e^{\alpha/2}v^1 = -\sinh n, \quad v^4 = \cosh n, \\ &w^1 e^{\alpha/2} = w^4 = \cosh n - \sinh n, \end{aligned} \tag{7}$$

where n is a function of co-ordinates to be determined from the field equations. Using (2) and (7) we have seen that the field equation (3) give a system of four non-trivial equations. These four equations are sufficient to determine four physical parameters ρ, p, σ and n. They are given by

$$8\pi\rho = \frac{(1-k)(3-2kr^2)}{R^2(1-kr^2)^2} + 3\frac{\dot{R}^2}{R^2} \tag{8}$$

$$8\pi p = \frac{-(1-k)}{R^2(1-kr^2)} - 2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}, \tag{9}$$

$$8\pi\sigma = \frac{\dfrac{k(1-k)}{R^2(1-kr^2)^2}\left(\dfrac{(1-k)(2-kr^2)}{R^2(1-kr^2)^2} - 2\dfrac{\ddot{R}}{R} + 2\dfrac{\dot{R}^2}{R^2}\right)}{\dfrac{-2(1-k)}{R^2(1-kr^2)} - 2\dfrac{\ddot{R}}{R} + 2\dfrac{\dot{R}^2}{R^2}} \tag{10}$$

$$\tanh n = \frac{\dfrac{k(1-k)r^2}{R^2(1-kr^2)^2}}{\dfrac{(1-k)(2-kr^2)}{R^2(1-kr^2)^2} - 2\dfrac{\ddot{R}}{R} + 2\dfrac{\dot{R}^2}{R^2}}. \tag{11}$$

It can be seen that

$$T^1_1 - T^2_2 = -((p+\rho)\sinh^2 n + \sigma(\cosh n - \sinh n)^2)$$

which is negative. But using the expressions (2) one can see that

$$8\pi(T^1_1 - T^2_2) = \frac{-k(1-k)r^2}{R^2(1-kr^2)^2}.$$

$T_1^1 - T_2^2$ being negative implies that $k(1-k)$ is positive. Therefore we must have

$$0 < k < 1. \tag{12}$$

When $k=0$, we get $\sigma = 0$, $n=0$ and the above solution reduces to Oppenheimer–Snyder solution. When $k=1$, then $\sigma=0$, $n=0$. In this case we get Einstein–de Sitter universe.

3. Equations of fit

We take the contracting boundary of sphere to be $r = a(t)$. For $r \geqslant a(t)$ we have Vaidya's radiating star metric [3].

$$\mathrm{d}s^2 = [1 - 2m/S + 2\dot{S}/\dot{u}]\dot{u}^2\,\mathrm{d}t^2 - [1 - 2m/S + 2\dot{S}/\dot{u}]u'^2\,\mathrm{d}r^2 - S^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{13}$$

where m is an undetermined function of u and S is an undetermined function of t and r, and

$$u'[1 - 2m/S + \dot{S}/\dot{u}] = -S'. \tag{14}$$

An overhead dash denotes differentiation with respect to r.

We shall use the standard system of equations of fit viz. at $r = a(t)$

$$\text{(i) } p = 0, \quad \text{(ii) } v^1/v^4 = \dot{a}, \quad \text{(iii) } g_{ik} \text{ continuous.} \tag{15}$$

For $r \leqslant a(t)$ our interior metric is (1). Let us put $S = rR(t)$ so that the external metric is

$$\mathrm{d}s^2 = \left\{1 - \frac{2m}{rR} + \frac{2\dot{R}r}{\dot{u}}\right\}(\dot{u}^2\mathrm{d}t^2 - u'^2\mathrm{d}r^2) - R^2r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{16}$$

From (1) and (16) it is clear that g_{22} and all its derivatives are continuous over $r = a(t)$. We use the notation $[X]$ to denote the value of X on the boundary $r = a(t)$. We now consider the continuity of g_{11} and g_{44}. That leads to

$$\left[u'^2\left\{1 - \frac{2m}{rR} + \frac{2\dot{R}r}{\dot{u}}\right\}\right] = [e^\alpha] = \frac{R^2(1-ka^2)}{(1-a^2)} \tag{17}$$

and

$$\left[\dot{u}^2\left\{1 - \frac{2m}{rR} + \frac{2R\dot{r}}{\dot{u}}\right\}\right] = 1. \tag{18}$$

We rewrite (18) as

$$[\dot{u}^2]\left[1 - \frac{2m}{Rr}\right] + 2\dot{R}a[\dot{u}] = 1 \tag{19}$$

For the external metric (13) we have $g^{11}u'^2 + g^{44}\dot{u}^2 = 0$. Using the continuity of g^{11} and g^{44} we have $[u'^2] = -[e^{\alpha/2}\dot{u}]$. Taking the boundary values of the relation (14) and using $[u'] = -[e^{\alpha/2}\dot{u}]$ we get

$$[\dot{u}^2]\left[1 - \frac{2m}{Rr}\right] + 2\dot{R}a[\dot{u}] = R[e^{-\alpha/2}\dot{u}]. \tag{20}$$

From (19) and (20) we get

$$[\dot{u}] = \frac{1}{a\dot{R} + R[e^{-\alpha/2}]} \tag{21}$$

and

$$[u'] = \frac{-[e^{\alpha/2}]}{a\dot{R} + R - [e^{-\alpha/2}]}. \tag{22}$$

The result (15(ii)) gives

$$\dot{a} = -[e^{\alpha/2}\tanh n]. \tag{23}$$

Also vanishing of pressure p at $r = a(t)$ gives

$$\frac{1-k}{R^2(1-ka^2)^2} + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0. \tag{24}$$

The function u satisfies (14). Taking the boundary values on both sides and substituting $[\dot{u}] = [u']e^{-\alpha/2}$ we get

$$[u']\left[1 - \frac{2m}{S}\right] - [\dot{S}]e^{\alpha/2} = -R. \tag{25}$$

Using (22) in this equation we get

$$[\dot{S}]^2 = R^2 e^{-\alpha} - \left[1 - \frac{2m}{S}\right].$$

Finally we have

$$[\dot{S}]^2 = \frac{-(1-k)a^2}{1-ka^2} + \left[\frac{2m}{S}\right]. \tag{26}$$

S is the radius of the sphere as seen by an external observer and $\dot{S}^2 = 2m/S$ – (function of time t). Thus equations (14), (21), (22) and (26) among themselves give us the boundary values of $S', \dot{S}, u'$ and $\dot{u}$. The functions $S = rR(t)$ and u are continuous across the boundary. Therefore we have, on the boundary, the values of u and S and their first derivatives. This will enable us to write the march of functions u and S in the external solution. The function m is arbitrary.

Equation (8) shows that ρ is always positive. Differentiating (9) we find that $8\pi p'$ is always negative. Clearly at the centre $8\pi p$ is positive. As $8\pi p'$ is negative p continuously decreases from the origin to the boundary $r = a$.

It can be verified that

$$8\pi\sigma = -\frac{(G_2^2 - G_1^1)(G_2^2 - G_4^4)}{2G_2^2 - G_1^1 - G_4^4}.$$

Since p is positive throughout, G_2^2 is positive. We have verified that $G_2^2 - G_1^1$ is negative and $G_2^2 - G_4^4$ is positive. The denominator is $8\pi(p + \rho)$ and hence is positive. This shows that the radiation density σ remains positive throughout the distribution.

4. An approximate solution for small k

We have seen that if e is the eccentricity of the spheroidal 3-space, then our parameter k is e^2. In what follows we shall try to integrate eqs (23) and (24) and get solutions correct up to e^2.

We have already obtained in (8), (9), (10) and (11) expressions for ρ, p, σ and n. We now regard k to be a small parameter and so rewrite these expressions correct to the first power of k. They are

$$8\pi p = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{1}{R^2} + \frac{k(1-2r^2)}{R^2} \tag{27}$$

$$8\pi\rho = 3\frac{\dot{R}^2}{R^2} + \frac{3}{R^2} - \frac{k(3-4r^2)}{R^2} \tag{28}$$

$$8\pi\sigma = kr^2/R^2 \tag{29}$$

$$\tanh n = \frac{kr^2/R^2}{-\frac{2\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{2}{R^2}}. \tag{30}$$

Vanishing of the pressure p at boundary $r = a(t)$ will now give

$$\frac{2\ddot{R}}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} - \frac{k(1-2a^2)}{R^2} = 0 \tag{31}$$

which is (24) when k^2 and higher powers are neglected. Here $a = a(t)$ is given by (23). Neglecting k^2 and higher powers of k (23) becomes

$$\dot{a} = \frac{(-ka^2/R^3)(1-a^2)^{1/2}}{-\frac{2\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{2}{R^2}}. \tag{32}$$

When $k = 0$, we have Oppenheimer–Snyder solution, σ vanishes and a becomes a constant. To the first power of k we take

$$a(t) = c + kF(t) \tag{33}$$

where c is the constant value of a when $k = 0$. Then (31) becomes

$$\frac{2\ddot{R}}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} - \frac{k(1-2c^2)}{R^2} = 0 \tag{34}$$

which admits a first integral

$$\dot{R}^2 = -1 + B/R + k(1-2c^2), \tag{35}$$

where B is a constant of integration. Therefore we can rewrite p, ρ, σ and n, using (33) and (35). They are given by

$$8\pi p = 2k(c^2 - r^2)/R^2, \quad 8\pi\rho = 3B/R^3 - 2k(3c^2 - 2r^2)/R^2,$$
$$8\pi\sigma = kr^2/R^2, \quad \tanh n = kRr^2/3B. \tag{36}$$

From (32) and (33) we find that

$$a(t) = c + kF(t)$$

$$= c + k\left\{\frac{-tc^2\sqrt{1-c^2}}{3B} + c_1\right\} \tag{37}$$

where c_1 is another constant of integration. We must now find R as a function of t correct to first power of k. We have already obtained the first integral (35). Let $R = R_0(t)$ be the solution of (31) when $k = 0$. Then (31) can be integrated up to first power of k. The solution is

$$R = R_0(t) + k\left\{(1-2c^2)R_0 - \frac{3}{2}(1-2c^2)t\dot{R}_0 + D\dot{R}_0\right\} \tag{38}$$

where D is a constant of integration. From (20) and (21) it is easy to see that

$$a^2\dot{R}^2 = R^2[e^{-\alpha}] - 1 + 2[m]/Ra \tag{39}$$

Using (33) and (35) one can verify that

$$2[m] = Bc^3 - kR_0c^5 + 3kBc^2F(t) \tag{40}$$

with $F(t)$ given in (37).

One can now show that σ is continuous across the boundary. On the interior side

$$8\pi[\sigma]_i = kc^2/R_0^2. \tag{41}$$

On the exterior side $8\pi[\sigma]_e$ is given by (Vaidya [3]).

$$8\pi[\sigma]_e = \frac{-[m_u][\dot{u}^2]}{4\pi a^2R^2}, \quad m_u = \mathrm{d}m/\mathrm{d}u. \tag{42}$$

We can find $[m_u]$ from (40) and we have already found $[\dot{u}^2]$. So we can find $[\sigma]_e$. Up to the first power of k, $8\pi[\sigma]_e$ is given by

$$8\pi[\sigma]_e = kc^2/R_0^2. \tag{43}$$

The result (41) and (43) establish the continuity of the radiation density σ across the boundary $r = a(t)$. Though we have proved this continuity using approximation – neglecting k^2 and higher power of k, we have verified that this continuity does hold good in the general solution discussed earlier.

Lastly since we are working in co-ordinates which are co-moving in the limit $k = 0$, we can find the finite co-ordinate time in which the radius Ra of the distribution would tend to zero. We have

$$\dot{R}^2 = -1 + B/R + k(1-2c^2).$$

Therefore we have

$$\dot{R} = -(B - R\mu)^{1/2}/R^{1/2},$$

$$\mu = 1 - k(1-2c^2),$$

$$1/\mu = 1 + k(1-2c^2). \tag{44}$$

$\dot{R}$ vanishes when $R = B/\mu$. The time required for R to diminish from the value which makes $\dot{R}$ zero to the value $R = 0$ is given by

$$t = \int_{B/\mu}^{0} \frac{-R^{1/2}\mathrm{d}R}{(B - R\mu)^{1/2}}$$
$$= B\pi/2\{1 + 3k(1 - 2c^2)/2\}. \tag{45}$$

5. Conclusion

In the above analysis a model describing a radiating collapsing sphere is studied. Vaidya's radiating star solution is taken as the exterior solution. The equations of fit are explicitly derived. An approximate solution corresponding to small values of the parameter k is presented. This approximate solution represents a radiating generalization of the well-known Oppenheimer–Snyder solution. This solution has an interesting property that the radiation density is continuous across the moving boundary of the sphere.

References

[1] J R Oppenheimer and H Snyder, *Phys. Rev.* **56**, 455 (1939)
[2] P C Vaidya, *Proc. Indian Acad. Sci.* **A33**, 264 (1951)
[3] P C Vaidya, *Astrophys. J.* **144**, 943 (1966)
[4] R W Lindquist, R A Schwartz and C W Misner, *Phys. Rev.* **B137**, 1364 (1965)
[5] C W Misner and D H Sharp, *Phys. Rev.* **B136**, 571 (1964)
[6] C W Misner, *Phys. Rev.* **B137**, 1360 (1965)
[7] N O Santos, *Mon. Not. R. Astron. Soc.* **216**, 403 (1985)
[8] A K G de Oliveira, N O Santos and C Kolassis, *Mon. Not. R. Astron. Soc.* **216**, 1001 (1985)
[9] A K G de Oliveira, J A de Pacheco and N O Santos, *Mon. Not. R. Astron. Soc.* **220**, 405 (1986)
[10] A K G de Oliveira and N O Santos, *Astrophys. J.* **312**, 640 (1987)
[11] D Kramer, *J. Math. Phys.* **33**, 1458 (1992)
[12] P C Vaidya and L K Patel, *J. Indian Math. Soc.* **61**, 87 (1995)
[13] P C Vaidya and R Tikekar, *J. Astrophys. Astron.* **3**, 325 (1982)
[14] R Tikekar, *J. Math. Phys.* **31**, 2454 (1990)