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Cosmic strings in Bianchi II, VIII and IX spacetimes: Integrable cases

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Abstract. We investigate the integrability of cosmic strings in Bianchi II, VIII and IX spacetimes using a Lie symmetry analysis. The behaviour of the gravitational field is governed by solutions of a single second order nonlinear differential equation. We demonstrate that this equation is integrable and admits an infinite family of physically reasonable solutions. Particular solutions obtained by other authors are shown to be special cases of our class of solutions.

Keywords. Cosmology; strings; integrable.

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1. Introduction

Topologically stable defects such as vacuum domain walls, strings and monopoles are produced during phase transitions in the early universe [1]. Domain walls and monopoles are not important in the study of cosmological models at later times. On the other hand strings can lead to many interesting astrophysical consequences. Strings may be one of the sources of density perturbations that are required for the formation of large scale structures in the universe ([2], [3]). They possess stress energy and hence couple to the gravitational field. Various features of cosmic strings have been discussed by Vilenkin [1], Gott [4] and Garfinkle [5].

The general relativistic treatment of strings was initiated by Letelier [6] and Stachel [7]. Subsequently many relativistic exact solutions were found which describe homogeneous string cosmological models with different Bianchi symmetries. Krori *et al* [8] and Chakraborty and Nandy [9] have considered models with Bianchi types II, VIII and IX spacetimes. Bianchi type I string based models are studied by Banerjee *et al* [10]. Tikekar and Patel [11] have discussed some Bianchi type VI_0 string models with and without magnetic fields. More recently a number of exact solutions, in the presence of a magnetic field and also with vanishing magnetic field, in Bianchi type III spacetimes were obtained by Tikekar and Patel [12]. Maharaj *et al* [13] investigated the integrability of cosmic strings in Bianchi type III spacetimes using a symmetry analysis and extended the class of solutions studied in [12].

Tikekar *et al* [14] have obtained a new class of physically relevant inhomogeneous solutions for string cosmology endowed with cylindrical symmetry on the background of singularity-free cosmological spacetimes. Patel and Beesham [15] have also obtained a new class of plane symmetric inhomogeneous string cosmological models.

The purpose of the present paper is to study the integrability of cosmic strings in the context of Bianchi types II, VIII and IX spacetimes, Essentially the solution of the field equations reduces to integrating a single second order nonlinear ordinary differential equation. We show that this equation has a rich structure and admits many solutions, some of which may lead to new physically significant string models.

2. The field equations

We consider the general Bianchi type II, VIII and IX spacetimes given by the line element

$$ds^2 = dt^2 - A^2(dr + 4m^2 d\phi)^2 - B^2K^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where A and B are functions of time, t, and m and K are functions of θ satisfying the differential equations

$$\frac{4m}{K^2\sin\theta}\frac{dm}{d\theta} = \lambda_1 \tag{2.2}$$

and

$$\frac{d^2K}{d\theta^2} - \frac{1}{K}\left(\frac{dK}{d\theta}\right)^2 + \cot\theta\frac{dK}{d\theta} - K = \mu K^3. \tag{2.3}$$

Here λ_1 and μ are constants, μ being proportional to the curvature of the two-dimensional surface with the metric

$$d\Sigma^2 = K^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4}$$

The metric (2.1) with $\lambda_1 \neq 0$ represents

(i) a Bianchi type II spacetime if $\mu = 0$ and $K = \mathrm{cosec}\theta$,
(ii) a Bianchi type VIII spacetime if $\mu = 1$ and $K = \tan\theta$ and
(iii) a Bianchi type IX spacetime if $\mu = -1$ and $K = 1$.

The energy-momentum tensor is given by

$$T_{ik} = \rho v_i v_k - \lambda w_i w_k, \quad v_i v^i = -w_i w^i = 1, \quad v^i w_i = 0 \tag{2.5}$$

for a cloud of strings, In (2.5) ρ, the proper energy density, and λ, the string tension density, are related by

$$\rho = \rho_p + \lambda, \tag{2.6}$$

where ρ_p is the particle density of the configuration. We use comoving coordinates and take the string fibres along the r-direction. One can easily check that the Einstein field equations

$$R_{ik} - \frac{1}{2}Rg_{ik} = -8\pi T_{ik} \tag{2.7}$$

corresponding to the string distribution for the metric (2.1) reduce to the system

$$8\pi\rho = \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{\lambda_1^2 A^2}{B^4} - \frac{\mu}{B^2} \tag{2.8}$$

$$8\pi\lambda = 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{3\lambda_1^2 A^2}{B^4} - \frac{\mu}{B^2} \tag{2.9}$$

$$0 = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\lambda_1^2 A^2}{B^4}. \tag{2.10}$$

Here and in the sequel an overdot indicates differentiation with respect to t. The particle density is given by

$$8\pi\rho_p = -2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{2\lambda_1^2 A^2}{B^4}. \tag{2.11}$$

We generate many solutions to (2·8–2·10) in subsequent sections.

Here it should be noted that the expansion scalar Θ and the shear scalar σ for the velocity field v_i have the general expressions

$$\Theta = \frac{2\dot{B}}{B} + \frac{\dot{A}}{A}, \quad \sigma^2 = \frac{2}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2. \tag{2.12}$$

3. The model equation

We have three equations (2·8–2·10) for four unknown functions ρ, λ, A and B. In order to obtain explicit solutions of the system we must impose one additional constraint. We assume that

$$A = B^n, \tag{3.1}$$

where n is a real constant, so that (2.10) becomes

$$(n+1)\frac{\ddot{B}}{B} + n^2\frac{\dot{B}^2}{B^2} + \lambda_1^2 B^{2(n-2)} = 0. \tag{3.2}$$

Chakraborty and Nandy [9] have provided solutions of (3.2) for $n = 0$ and $n = 2$. Our aim here is to find all possible solutions of the differential equation (3.2).

Equation (3.2) can be written as the simpler form

$$y'' + y^{\nu} = 0 \tag{3.3}$$

by means of the transformation

$$y = B^{\alpha} \quad x = \beta t, \tag{3.4}$$

where

$$\alpha = \frac{n^2+n+1}{n+1}, \quad \beta^2 = \frac{(n+1)^2}{\lambda_1^2(n^2+n+1)}, \quad \nu^2 = \frac{3n^2-n-3}{n^2+n+1} \tag{3.5}$$

and we take $\beta \geq 0$. In (3.5) we require that $n \neq -1$ and $\lambda_1 \neq 0$. From (3.2) we see that $n = -1$ leads to the degenerate case

$$\frac{\dot{B}^2}{B^2} + \lambda_1^2 B^{-6} = 0 \tag{3.6}$$

with solution

$$B(t) = (K_1 - 3(-\lambda_1^2)^{1/2} t)^{-3}, \tag{3.7}$$

where K_1 is the single arbitrary constant of integration. If $\lambda_1 = 0$, the solution of (3.2) would be

$$B = (K_1 + K_2 t)^{(n+1)/(n^2+n+1)}. \tag{3.8}$$

Henceforth we consider the general case $n \neq -1$ and $\lambda_1 \neq 0$.

Equation (3.3) is an Emden–Fowler equation [16–19] of index ν. Analysis of the Lie point symmetries of (3.3) using Program LIE [20] shows that for general ν there are two Lie point symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{3.9}$$

$$G_2 = x\frac{\partial}{\partial x} - \frac{2}{\nu - 1}\frac{\partial}{\partial y}. \tag{3.10}$$

There are three particular values of ν for which the number of point Lie symmetries is greater than two.

When $\nu = 1$, (3.3) is linear, has eight Lie point symmetries and is trivially integrable. For $\nu = 1, n = 2, -1$ the second of which values has been already treated. For $n = 2$ the solution of (3.2) is

$$B(t) = (K_1 \sin\gamma t + K_2 \cos\gamma t)^{3/7}, \tag{3.11}$$

where

$$\gamma^2 = \frac{4}{3}\lambda_1^2. \tag{3.12}$$

When $\nu = 0$, (3.3) is also linear and has eight point symmetries. For $\nu = 0$, n takes the values $(1 \pm \sqrt{37})/6$ and the corresponding solutions of (3.2) are

$$B(t) = \left\{K_1 + K_2\left[\left(\frac{296 + 27\sqrt{37}}{252\lambda_1^2}\right)^{1/2} t\right] - \frac{1}{2}\left[\frac{296 + 27\sqrt{37}}{252\lambda_1^2}t^2\right]\right\}^{3/(11-\sqrt{37})}, \tag{3.13}$$

i.e. only one solution is obtained. When $\nu = -3$, (3.3) is a special form of the Ermakov–Pinney equation [21, 22] and has the three point symmetries

$$G_1 = \frac{\partial}{\partial x},$$

$$G_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$G_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{3.14}$$

the Lie algebra of which is well-known to be $sl(2, R)$. For $\nu = -3, n = 0, -1/3$ and the solution of (3.3) is [22]

$$y = K_1 + 2K_2 x + K_3 x^2 \quad K_1 K_3 - K_2^2 = 1. \tag{3.15}$$

It follows that

$$B(t)=K_1+2K_2\left(\frac{t}{\lambda_1}\right)+K_3\left(\frac{t}{\lambda_1}\right)^2 \quad \text{for } n=0 \tag{3.16}$$

and

$$B(t)=\left(K_1+2K_2\left(\frac{2t}{\lambda_1\sqrt{7}}\right)+K_3\left(\frac{2t}{\lambda_1\sqrt{7}}\right)^2\right)^{6/7} \quad \text{for } n=-1/3. \tag{3.17}$$

That covers the algebraically special values of ν (and so n).

For general ν the algebra of G_1(3.9) and G_2(3.10) is A_2 in the Mubarekzyanov classification scheme [23–27]. Since A_2 is a solvable algebra and $G_1 \propto G_2$, the algebra is that of Lie's type IV [28, p. 424]. However, we do not use the standard representation of a second order equation invariant under a type IV algebra since the form (3.3) is more suitable for the purposes of the present discussion.

Chakraborty and Nandy [9] have presented some particular solutions to (3.2) in Bianchi II and VIII spacetimes. Their $n=0$ solution corresponds with our (3.16) and their $n=2$ to our (3.11).

4. General treatment of (3.3)

For general ν (3.3) possesses the two symmetries (3.9) and (3.10) so that its solution can be reduced to an algebraic equation and a quadrature. The normal subgroup, G_1, is used to reduce (3.3) to a first order equation. The zeroth order and first order differential invariants are obtained from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}y'}{0} \tag{4.1}$$

and are

$$u=y, \quad v=y'. \tag{4.2}$$

The reduced equation is

$$vv'+u^{\nu}=0, \tag{4.3}$$

where now $'$ denotes differentiation with respect to u. In the new coordinates G_2 is

$$X_2=2u\frac{\partial}{\partial u}+(\nu+1)v\frac{\partial}{\partial v} \tag{4.4}$$

(up to a constant multiplier). Its invariants are found from the solution of

$$\frac{\mathrm{d}u}{2u}=\frac{\mathrm{d}v}{(\nu+1)v}=\frac{\mathrm{d}v'}{(\nu-1)v'} \tag{4.5}$$

and are

$$p=vu^{-(\nu+1)/2}, \quad q=v'u^{-(\nu-1)/2} \tag{4.6}$$

so that (4.3) is reduced to the algebraic equation

$$qp=-1. \tag{4.7}$$

The solution of the original equation, (3.3), can thereby be expressed as the inversion of the quadrature

$$x - x_0 = \begin{cases} \displaystyle\int \frac{dy}{[I - (2/(v+1))y^{v+1}]^{1/2}} \\ \displaystyle\int \frac{dy}{[I - \log y]^{1/2}}, \end{cases} \tag{4.8}$$

in which x_0 and I are constants of integration and (4.8b) corresponds to the special case $v = -1 (n = \pm 2^{-1/2})$. The integral in (4.8b) is related to the exponential integral, $Ei(ax)$, which cannot be expressed in terms of a finite number of terms [29, p. 93]. Hence it does not lead to a closed form solution of (3.3). In the cases $v = 2, 3$ ($n = (3 \pm \sqrt{29})/2, -3/2$) the integral in (4.8a) can be evaluated as an incomplete elliptic integral and the solution of (3.3) (and so (3.2)) is given in terms of elliptic functions.

5. General results

Apart from the special case of the degenerate solution (3.6) the field variables can be written directly in terms of $y(x)$ with x being replaced by $\beta^{-1}t$. Whatever the outcome of the quadratures in (4.8) we can write down some general results. (We omit the special case $v = -1$ to avoid what is virtually repetition.) The energy density (2.8) is

$$8\pi\rho = -\frac{2n+1}{\alpha^2 y^2}\left(I - \frac{2}{v+1}y^{v+1}\right) - \lambda_1^2 y^{2(n-2)/\alpha} - \mu y^{-2/\alpha}, \tag{5.1}$$

the string tension (2.9) is

$$8\pi\lambda = \frac{2}{\alpha}y^{v-1} + \frac{3-2\alpha}{\alpha^2 y^2}\left(I - \frac{2}{v+1}y^{v+1}\right) - \lambda_1^2 y^{2(n-2)/\alpha} - \mu y^{-2/\alpha} \tag{5.2}$$

and the particle density (2.11) is

$$8\pi\rho_p = \frac{2}{\alpha}y^{v-1} + \frac{2}{\alpha^2 y^2}(n - 1 + \alpha)\left(I - \frac{2}{v+1}y^{v+1}\right). \tag{5.3}$$

The expansion scalar is

$$\Theta = \frac{n+2}{\alpha y}\left(I - \frac{2}{v+1}y^{v+1}\right)^{1/2} \tag{5.4}$$

and the shear scalar is

$$\sigma^2 = \frac{2}{3}\frac{(n-1)^2}{\alpha^2 y^2}\left(I - \frac{2}{v+1}y^{v+1}\right). \tag{5.5}$$

The expressions (5.1–5.5) give the physical parameters of interest once y is known. We note the appearance of the constant of integration, I. In (4.8) we can take $I = 0$ as a special case when $v \leqslant -1$ which occurs for $-1/\sqrt{2} \leqslant n \leqslant 1/\sqrt{2}$. Then (4.8a) is easily integrated to

$$y = \frac{(x - x_0)^{2/(1-\nu)}}{\left[\left(\frac{1-\nu}{2}\right)\left(\frac{1+\nu}{-2}\right)^{1/2}\right]}. \tag{5.6}$$

(That (4.8b) cannot be evaluated in closed form has already been noted.) This gives a range of solutions for the field variables for n in the interval specified albeit with only one parameter present.

6. Solutions of the governing equation

The solutions of the field equations (2.8–2.11) and the evaluation of the field variables, etc (4.2–4.6) have been reduced under the assumption (3.1) to the evaluation of the integral (4.8),

$$x - x_0 = \begin{cases} \displaystyle\int \frac{\mathrm{d}y}{[I - (2/(\nu+1))y^{\nu+1}]^{1/2}} \\ \displaystyle\int \frac{\mathrm{d}y}{[I - \log y]^{1/2}}, \end{cases} \tag{6.1}$$

It is well-known that (6.1b) cannot be evaluated in closed form under any circumstances since it is a variant of the exponential-integral function [29, p. 93]. However, (6.1a) is known to be evaluable as a standard integral for $\nu = -3, -2, 0, 1, 2, 3$.

There is a sequence for which (6.1a) can be evaluated in closed form [13]. Let

$$\nu = -\frac{m}{m+2}, \quad m \in \mathscr{Z}. \tag{6.2}$$

Them (6.1a) is

$$\int \frac{\mathrm{d}y}{[I - (m+2)y^{2/(m+2)}]^{1/2}}. \tag{6.3}$$

For $m + 2 \geqslant 0$ the first nontrivial value of $m = 1$. The appropriate substitution is

$$y = \left[\left(\frac{I}{m+2}\right)^{1/2} \sin u\right]^{m+2} \tag{6.4}$$

so that (6.3) becomes

$$x - x_0 = (m+2)I^{(m+4)/2} \int^{u(y)} \sin^{m+1} u \, \mathrm{d}u \tag{6.5}$$

which can be evaluated in closed form for all integral $m \geqslant 1$. Inversion is not generally possible apart from locally. The one exception is $m = 2$. However, (6.4) with (6.5) does define a parametric solution. It is a simple matter to express (4.2–4.6) in terms of the parameter u through (6.4).

For $m+2 \leqslant 0$ we replace m by $-p-2$, $p \in \mathscr{Z}^+$. Then (6.3) becomes

$$x - x_0 = \int \frac{\mathrm{d}y}{[I + py^{-2/p}]^{1/2}}. \tag{6.6}$$

We rewrite (6.6) as

$$x - x_0 = \int \frac{y^{1/p}\mathrm{d}y}{[Iy^{2/p} + p]^{1/2}} \tag{6.7}$$

and the integral is evaluated in closed form by the substitution

$$Y = \left[\left(\frac{p}{I}\right)^{1/2} \sinh u\right]^p. \tag{6.8}$$

Finally we note that there is another set of values of v for which (3.3) is integrable. If

$$v = \frac{p+2}{p}, \quad p \in \mathscr{Z}^+, \tag{6.9}$$

(3.3) possesses the Painlevé property [30] and is integrable in the sense of Painlevé [31]. Unfortunately the evaluation of the quadrature is by no means obvious.

7. Conclusion

In this paper we have extended our previous analysis [13] of Bianchi type III cosmic strings to cosmic strings in Bianchi types II, VIII and IX spacetimes. The procedure followed here is similar to our analysis in [13]. The evolution of our models is governed by a single nonlinear ordinary differential equation. On utilizing the Lie symmetry analysis we reduce the behaviour of the gravitational field to the quadrature (4.8). A detailed investigation of (4.8) shows that it may be evaluated as a standard integral only for certain values of v which contain, as a proper subset, the cases considered by Chakraborty and Nandy [9]. In addition we present a particular sequence for which the integral may be evaluated in closed form; in general the solution can only be put into parametric form and inversion is only possible locally. Our analysis is an attempt to obtain more exact solutions of cosmic strings so that our understanding of these objects may be improved. It is hoped that some of the solutions presented here will prove helpful in building physically reasonable models of cosmic strings in the early universe.

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