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Geometric phase *a la* Pancharatnam

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Abstract. In mid-1950s, Pancharatnam [1] encountered the geometric phase associated with the evolution along a geodesic triangle on the Poincaré sphere. We generalize his 3-vertex phase and employ it as the fundamental building block, to geometrically construct a general ray-space expression for geometric phase. In terms of a reference ray used to specify geometric phase, we delineate clear geometric meanings for gauge transformations and gauge freedom, which are generally regarded as mere mathematical abstractions.

Keywords. Geometric phase; Pancharatnam triangle; parallel transportation.

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While studying evolutions of optical polarization states, Pancharatnam [1] made three seminal contributions, about 30 years ahead of their time, to the understanding of phases between distinct states. His deceptively simple, yet incisive physical observation that two states are in phase when the intensity of their superposition is maximum, led to a completely general phase definition [1–5], now known as the Pancharatnam connection. Secondly, Pancharatnam made the first explicit recognition of geometric phase by deriving the solid-angle expression for the invariant phase associated with a geodesic triangle on the Poincaré sphere. Pancharatnam's third contribution, which is not so well known as the first two, consisted in showing that if two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are subjected to an analysis along a ray $|\psi_0\rangle$ prior to superposition, the resultant interference pattern will display an additional phase equal to the geometric phase for the geodesic triangle formed by the three rays on the Poincaré sphere.

Just over a decade ago, Berry provided a general quantal framework [6] for geometric phase, independently of Pancharatnam's earlier work [1], for an eigen state of a Hamiltonian whose parameters are cycled adiabatically. This paper triggered an intense activity [7–9] in the field. Geometric phase is now recognized to be the Hamiltonian-independent, nonintegrable component of the total phase, depending exclusively on the geometry in the ray space. Geometric phase identifies with the phase anholonomy of a parallel transported [4, 9–11] quantal system and is manifested in a wide spectrum of physical phenomena [12–16]. Recently, geometric phase has been subjected to a group theoretic [17] and a kinematic [18] treatment. Here, we opt for the natural, i.e. geometric, treatment of geometric phase. We reword Pancharatnam's results in the modern quantum physics language [3–4, 9] and show that his triangle phase is the only fundamental input needed to obtain the most general expression for geometric phase purely geometrically.

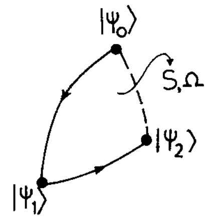

Figure 1. Pancharatnam geodesic triangle in the ray space. Two successive filtering measurements on the ray $|\psi_0\rangle$ along rays $|\psi_1\rangle$ and $|\psi_2\rangle$ yield an invariant phase dependent solely on the geometry of the geodesic triangle. The surface S spanned by the triangle is the solid angle Ω for the special case of a two-sphere.

On a polarization state represented by a ray $|\psi_0\rangle$, Pancharatnam considered a pair of successive filtering measurements carried out along rays $|\psi_1\rangle$ and $|\psi_2\rangle$ (figure 1), the three rays being mutually nonorthogonal. These operations are represented by shorter geodesics joining $|\psi_0\rangle$ to $|\psi_1\rangle$ and $|\psi_1\rangle$ to $|\psi_2\rangle$ respectively on the Poincaré sphere. Pancharatnam realized that a filtering measurement is a phase-preserving projection. He however recognized that the two successive projections yield a state which has a pure geometric phase $\Phi_G^\Delta = -\Omega/2$ with reference to the initial state, Ω denoting the solid angle spanned by the geodesic triangle completed by joining $|\psi_2\rangle$ to $|\psi_0\rangle$ with the shorter geodesic. The phase Φ_G^Δ resulting from the two successive phase-preserving operations brings out the non-integrability of geometric phase, recognized by Pancharatnam [1] as 'an unexpected geometrical result'. It is remarkable that Pancharatnam derived his result for a nonadiabatic, nonunitary and noncyclic evolution, never invoking any specific equation or Hamiltonian to effect the evolution. The result therefore has a completely general applicability.

Pancharatnam's results for a two-state system can be extended to a general quantal system. The filtering measurement on the initial state $|\Psi_0\rangle$ made along the ray $|\psi_1\rangle$ yields the state $|\psi_1\rangle\langle\psi_1|\Psi_0\rangle = \rho_1|\Psi_0\rangle$, viz. the component of the initial wavefunction along $|\psi_1\rangle$. The pure state density operator $\rho = |\Psi\rangle\langle\Psi|/\langle\Psi|\Psi\rangle$ used here, with the familiar properties: $\rho^\dagger = \rho, \mathrm{Tr}\,\rho = 1, \rho^2 = \rho$, is a ray space quantity, i.e. it is uniquely determined by the ray $|\psi\rangle$ regardless of the phase or norm of the wavefunction $|\Psi\rangle$. The second projection then yields the state $\rho_2\rho_1|\Psi_0\rangle$ with a phase

$$\Phi_G^\Delta = \arg\langle\Psi_0|\rho_2\rho_1|\Psi_0\rangle = \arg\mathrm{Tr}\,\rho_0\rho_2\rho_1, \tag{1}$$

with respect to the initial state $|\Psi_0\rangle$, according to the Pancharatnam connection [1, 3–5]. The two-state geometric phase $-\Omega/2$ thus generalizes to the argument of the Bargmann invariant [19] $\mathrm{Tr}\,\rho_0\rho_2\rho_1$, a pure geometric quantity associated with the geodesic triangle in the ray space of a general quantal wavefunction $|\Psi\rangle$.

Since $\mathrm{Tr}\,\rho_0\rho_2\rho_1 = \mathrm{Tr}\,\rho_1\rho_0\rho_2 = \mathrm{Tr}\,\rho_2\rho_1\rho_0$, the same geometric phase is obtained no matter which vertex of the triangle one begins with, as far as the sequence of filtering measurements is maintained. In an actual evolution of a system, represented in the ray space by the geodesics $|\psi_0\rangle \to |\psi_1\rangle$ and $|\psi_1\rangle \to |\psi_2\rangle$, produced by an appropriate Hamiltonian, the total phase in general has a dynamical component which depends on the actual Hamiltonian. The remaining, i.e. geometric phase however is invariant irrespective of the Hamiltonian [7], regardless of adiabaticity or unitarity of the evolution and whether or not the evolution includes [18] the geodesic $|\psi_2\rangle \to |\psi_0\rangle$. If any two of the projection operators coincide, the three-vertex geometric phase equals zero as the triangle collapses to a single geodesic and its area vanishes.

We now make a simple physical observation which will yield a crucial ingredient of geometric phase. If the sequence of the filtering measurements is reversed, the phase acquired just changes sign, i.e.

$$\Phi_G^{-\Delta} = \arg \mathrm{Tr}\, \rho_0 \rho_1 \rho_2 = -\Phi_G^\Delta. \tag{1a}$$

This observation readily leads to the expression

$$\tan \Phi_G^\Delta = i \frac{\mathrm{Tr}\, \rho_0 [\rho_1, \rho_2]}{\mathrm{Tr}\, \rho_0 \{\rho_1, \rho_2\}}, \tag{2}$$

in terms of the commutator (square) and anticommutator (curly) brackets between ρ_1 and ρ_2. The three-vertex geometric phase Φ_G^Δ (2), originating directly from the commutator between the projection operators, depends exclusively on the ray space geometry and is gauge independent.

If the rays $|\psi_1\rangle$ and $|\psi_2\rangle$ are separated infinitesimally, so that $\rho_1 = \rho$ and $\rho_2 = \rho + \mathrm{d}\rho$, say, the infinitesimal 3-vertex geometric phase becomes [20]

$$\mathrm{d}\Phi_G^\Delta = i \frac{\mathrm{Tr}\, \rho_0 [\rho, \rho + \mathrm{d}\rho]}{\mathrm{Tr}\, \rho_0 \{\rho, \rho + \mathrm{d}\rho\}} = \frac{i}{2} \frac{\mathrm{Tr}\, \rho_0 [\rho, \mathrm{d}\rho]}{\mathrm{Tr}\, \rho_0 \rho} = i \frac{\mathrm{Tr}\, \rho_0 (\rho - \frac{1}{2}\mathbf{1}) \mathrm{d}\rho}{\mathrm{Tr}\, \rho_0 \rho}, \tag{2a}$$

since $\rho \mathrm{d}\rho + (\mathrm{d}\rho)\rho = \mathrm{d}(\rho^2) = \mathrm{d}\rho$. Here $\mathbf{1}$ signifies the unity operator. The infinitesimal 3-vertex phase (2a) forms the smallest possible building block with which we will presently build the general geometric phase.

Before harnessing (2a), we highlight the special physical significance of the commutator between ρ and its differential appearing in (2a). As noted earlier, geometric phase is the phase acquired by a parallel transported [4, 9–11] state. The specific Hermitian Hamiltonian which parallel transports [21–22] a state $|\Psi\rangle$, is $i\hbar[\mathrm{d}\rho/\mathrm{d}t, \rho]$. Over each infinitesimal step $|\psi\rangle \to (\exp[\mathrm{d}\rho, \rho])|\psi\rangle = (\cos(\mathrm{d}\rho) + \sin(\mathrm{d}\rho))|\psi\rangle$ (since $\rho \mathrm{d}\rho |\Psi\rangle = 0$, cf. [22]), in the state evolution effected by this Hamiltonian, the phase is preserved, as $\cos(\mathrm{d}\rho)|\psi\rangle$ is along and in phase with $|\psi\rangle$ whereas $\sin(\mathrm{d}\rho)|\psi\rangle$ is orthogonal to $|\psi\rangle$. To the first order, this step in the evolution is equivalent to a projection $(\rho + \mathrm{d}\rho)|\Psi\rangle$. Parallel transportation along a geodesic produces an identically null phase [21–22] until a ray $|\bar{\psi}_0\rangle$, orthogonal to the initial ray, is reached. In the parallel transportation along the two geodesics $|\psi_0\rangle \to |\psi\rangle$ and $|\psi\rangle \to (\exp[\mathrm{d}\rho, \rho])|\psi\rangle$ therefore, the phase of the final state $(\exp[\mathrm{d}\rho, \rho])|\Psi\rangle$, as prescribed by the Pancharatnam connection [1–5], is $\arg\langle\Psi_0|(1 + [\mathrm{d}\rho, \rho])|\Psi\rangle$. This equals $\mathrm{d}\Phi_G^\Delta$ (2a) obtained above geometrically, since $|\Psi\rangle$ is in phase with $|\Psi_0\rangle$ ($\langle\psi_0|\psi\rangle$ real) due to the geodesical parallel transportation. The operator $i\hbar[\mathrm{d}\rho/\mathrm{d}t, \rho]$ was introduced 45 years ago as a generator of adiabatic [23] evolutions.

We now derive the geometric phase in terms of the infinitesimal 3-vertex phase (2a). Consider a completely arbitrary evolution of a general quantal system, represented in the ray space by an open curve C (figure 2) from $|\psi_1\rangle$ to $|\psi_2\rangle$. The acquired phase Φ can be measured from the interference pattern obtained by superposing the initial and final

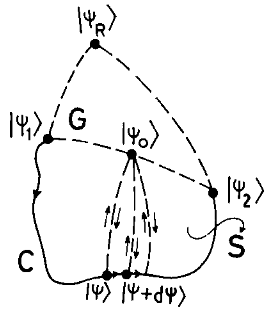

Figure 2. The curve C in the ray space representing an arbitrary evolution from $|\psi_1\rangle$ to $|\psi_2\rangle$ is divided into infinitesimal geodesic segments. The ray $|\psi_0\rangle$ on the shorter geodesic G, joining the ends of C, is chosen as the reference. The sum of geometric phases associated with the geodesic triangles having the infinitesimal segments of C as bases and $|\psi_0\rangle$ as the vertex, equals the geometric phase acquired along C. If the reference is shifted to $|\psi_R\rangle$, an additional geometric phase arising from the geodesic triangle $|\psi_R\rangle \to |\psi_1\rangle \to |\psi_2\rangle$, will accrue. S is the surface spanned by the closed curve C + G.

states. Pancharatnam showed that if the states are passed through an analyzer oriented along the ray $|\psi_R\rangle$, say, prior to the superposition, the resulting interference pattern (cf. eq. (10) in [1]) displays a phase $(\Phi + \Phi_G^A)$ where Φ_G^A is the 3-vertex geometric phase associated with the geodesic triangle $|\psi_R\rangle \to |\psi_1\rangle \to |\psi_2\rangle$. This can be understood from figure 2. The projection of the final state on the initial state represented by the shorter geodesic G joining $|\psi_2\rangle$ to $|\psi_1\rangle$ does not alter the phase [1]. Hence the $|\psi_R\rangle$ analysis of the two states, represented by the shorter geodesics joining them to $|\psi_R\rangle$, effectively corresponds to an additional evolution along the geodesic triangle $|\psi_2\rangle \to |\psi_R\rangle \to |\psi_1\rangle$, generating the extra phase.To obtain the correct phase Φ, therefore, the analyzer state should be selected such that Φ_G^A vanishes, e.g. by making the geodesic triangle collapse to a single geodesic. This is achieved with any arbitrary $|\psi_R\rangle$ if the evolution is cyclic $(|\psi_1\rangle = |\psi_2\rangle)$. For a non-cyclic evolution however, one can select $|\psi_R\rangle$ to lie anywhere on the shorter geodesic G joining $|\psi_2\rangle$ and $|\psi_1\rangle$. We therefore choose such a reference ray $|\psi_0\rangle$ on G (figure 2).

We divide the curve C into infinitesimal geodesic segments and join their ends to $|\psi_0\rangle$ with shorter geodesics to form infinitesimal geodesic triangles. Along each side shared by two such contiguous triangles (figure 2), the virtual evolutions to and fro $|\psi_0\rangle$ contribute equal and opposite phases $\arg\langle\psi|\psi_0\rangle$ and $\arg\langle\psi_0|\psi\rangle$. The successive triangle evolutions thus yield Φ_G for the evolution along C + G, which is identical to that along C (cf. also [18]). The infinitesimal extent of each segment implies a sequence of infinitely dense filtering measurements [9, 24] along C, yielding an effective unitary transformation of purely geometric nature. Hence the geometric component Φ_G of the phase Φ acquired over the evolution along C just equals the sum of the individual $d\Phi_G^A$ (2a), i.e.

$$\Phi_G = i \int_{\substack{\rho_1 \\ C}}^{\rho_2} \frac{\mathrm{Tr}\,\rho_0(\rho - \frac{1}{2}1)\mathrm{d}\rho}{\mathrm{Tr}\,\rho_0\rho}. \tag{3}$$

If C is a single (shorter) geodesic between $|\psi_1\rangle$ and $|\psi_2\rangle$, the numerator of the integrand in (3) vanishes [20] identically and a null geometric phase results. This is geometrically clear as G then retraces C.

The phase Φ_G is determinate if the rays $|\psi_1\rangle$ and $|\psi_2\rangle$ are not mutually orthogonal. If the curve C passes through a ray $|\bar{\psi}_0\rangle$ orthogonal to $|\psi_0\rangle$, the denominator of the integrand in (3) vanishes, but so does the numerator. The contribution from the corresponding infinitesimal triangle however never diverges (eqs. (1), (2)). If the infinitesimal segment of the triangle is centred at the orthogonal ray $|\bar{\psi}_0\rangle$, the triangle approaches a finite slice between two geodesics of length π each between $|\psi_0\rangle$ and $|\bar{\psi}_0\rangle$, as the segment length tends to zero. We can then write without any loss of generality,

$$|\psi_k\rangle = |\bar{\psi}_0\rangle\cos\delta\exp(i\beta_k) + |\psi_0\rangle\sin\delta, \quad k = 1,2, \tag{4}$$

so that the 3-vertex geometric phase (1) for the slice tends to

$$\Delta\Phi_G(|\psi_0\rangle,|\bar{\psi}_0\rangle) = \beta_1 - \beta_2, \tag{4a}$$

as $\delta\to 0$, i.e. as the segment $\psi_1\to\psi_2$ approaches zero. Thus unless the curve retraces itself at $|\bar{\psi}_0\rangle$ $(\beta_2=\beta_1)$, the contribution from this slice to $\Phi_G(\rho_0)$ is finite, viz. a jump (4a) as C crosses $|\bar{\psi}_0\rangle$.

If the geodesic continues through the orthogonal ray $|\bar{\psi}_0\rangle$ without a kink, a switch-over to a distinct geodesic with the label $\beta_2=\beta_1\pm\pi$ occurs and a $\mp\pi$ jump (4a) in $\Phi_G(\rho_0)$ results. This can be understood from the Pancharatnam triangle phase (1). When $|\psi_1\rangle$ and $|\psi_2\rangle$ lie on the same side of $|\bar{\psi}_0\rangle$ on the geodesic $|\psi_0\rangle\to|\bar{\psi}_0\rangle$, the shorter geodesic G (figure 1) between $|\psi_2\rangle$ and $|\psi_0\rangle$ closing the triangle just retraces the geodesic $|\psi_0\rangle\to|\psi_1\rangle\to|\psi_2\rangle$, enclosing a null area and yielding $\Phi_G^{\Delta}=0$. However if $|\psi_2\rangle$ lies on the other side of $|\bar{\psi}_0\rangle$, the shorter geodesic G continues in the same direction as C and the closed curve C + G encloses a finite slice generating a geometric phase jump of $\mp\pi$. Thus while an evolution along a geodesic [1, 5, 18] yields a null geometric phase until the ray orthogonal to the initial ray is reached, a geometric phase jump of $\mp\pi$ occurs just when the geodesic crosses the orthogonal ray without a kink. Such π phase jumps for a 2-state system have been discussed [5, 25] previously.

For a 2-state system such as a spin-1/2 particle, the ray space is a 2-sphere and the labels β_k in (4) are the azimuthal angles ϕ_k of the geodesics, measured in the plane transverse to the direction of the ray $|\psi_0\rangle$. Therefore $\Phi_G(\rho_0)$ for a 2-state jumps by $\phi_1-\phi_2=-\frac{1}{2}\Omega_{\text{slice}}$ as C crosses $|\bar{\psi}_0\rangle$. A case in point is the first experiment [26, 27] clearly demarcating dynamical and geometric phases. Here an interferometer is illuminated with a beam of neutrons in the $|\uparrow\rangle$ state. Two identical spin flippers F_1 and F_2 placed in the two arms of the interferometer take the neutron state to $|\downarrow\rangle$. A relative translation between F_1 and F_2 generates a pure dynamical phase, while their relative rotation $\delta\beta$ about the $|\uparrow\rangle$-direction produces a pure geometric phase $\Phi_G=\delta\beta$ (cf. figure 2b in [26]). If $|\psi_0\rangle$ is selected to be the initial, i.e. the $|\uparrow\rangle$ ray, the net geometric phase Φ_G (3) is just the phase jump $\delta\beta$ caused by the kink $\delta\beta$ of the curve at $|\downarrow\rangle$. This experiment, inclusive of a direct verification [27] of Pauli anticommutation, has been performed [21, 28].

We now redefine the ray $|\psi\rangle$ with reference to a fixed ray $|\psi_R\rangle$ as

$$|\tilde{\psi}\rangle = \frac{\rho|\psi_R\rangle}{|\langle\psi|\psi_R\rangle|} = |\psi\rangle e^{i\arg\langle\psi|\psi_R\rangle}, \tag{5}$$

which is in phase [1] with $|\psi_R\rangle$. The representation $|\tilde{\psi}\rangle$ of the ray differs from $|\psi\rangle$ only by the shown phase factor. Equation (5) represents a gauge transformation, which we associate here geometrically with a change of the reference ray $|\psi_R\rangle$. Since the inner products of both the rays $|\tilde{\psi}\rangle$ and $|\tilde{\psi}+\mathrm{d}\tilde{\psi}\rangle$ with $|\psi_R\rangle$ are real, for the Pancharatnam triangle $|\psi_R\rangle \to |\psi\rangle \to |\psi + \mathrm{d}\psi\rangle$,

$$\mathrm{d}\Phi_G^\Delta = \arg\langle\tilde{\psi}+\mathrm{d}\tilde{\psi}|\tilde{\psi}\rangle = \arg(1+\langle \mathrm{d}\tilde{\psi}|\tilde{\psi}\rangle) = -i\langle \mathrm{d}\tilde{\psi}|\tilde{\psi}\rangle. \tag{6}$$

This can also be directly verified by differentiating (5) and taking the inner product with $|\tilde{\psi}\rangle$ to get the purely imaginary quantity

$$\langle\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle = -\langle \mathrm{d}\tilde{\psi}|\tilde{\psi}\rangle = \frac{\mathrm{Tr}\,\rho_R(\rho-\frac{1}{2}\mathbf{1})\mathrm{d}\rho}{\mathrm{Tr}\,\rho_R\rho} = -i\mathrm{d}\Phi_G^\Delta, \tag{6a}$$

(cf. eq. (2a)). Hence the integral

$$\Phi_{G,\rho_R} = i\int_{\substack{\tilde{\psi}_1\\ \mathrm{C}}}^{\tilde{\psi}_2} \langle\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle, \tag{7}$$

yields the geometric phase that would be observed if the states $|\Psi\rangle$ were analyzed [1] along the reference ray $|\psi_R\rangle$ (figure 2) prior to superposition. Since all $\tilde{\psi}$ are in phase with ψ_R due to the gauge transformation (5), the total phase (7) is just the integral along C of the relative phase between neighbouring $\tilde{\psi}$ rays. As discussed earlier, this would yield, apart from the phase Φ_G (3) acquired along C, an additional phase Φ_G^Δ associated with the geodesic triangle $|\psi_R\rangle \to |\psi_1\rangle \to |\psi_2\rangle$. If the evolution is cyclic, i.e. the curve C is closed ($\rho_1 = \rho_2$), Φ_{G,ρ_R} equals the correct phase Φ_G (3) regardless of the choice of $|\psi_R\rangle$, as discussed before, even though the integrand in (7) is a gauge-dependent, i.e. $|\psi_R\rangle$-dependent, connection 1-form. Equation (7) then reduces to the nonadiabatic Φ_G derived for the special case [7] of a cyclic and unitary evolution. Thus for a cyclic evolution, the reference ray $|\psi_R\rangle$ may be selected anywhere in the ray space, i.e. the gauge freedom is complete. For an open curve C however, the relations (7) and (3) become identical, if $|\psi_R\rangle$ is confined to the shorter geodesic G (figure 2) joining the ends of C. Thus even for noncyclic evolutions, the gauge freedom survives, albeit in a restricted form and the integral (7) over the open curve C yields the integral over the closed curve C + G, i.e. the correct Φ_G.

Whether the evolution is cyclic or not, the integral (3, 7) of the connection 1-form for Φ_G transforms to the integral

$$\Phi_G = i\int_S \langle \mathrm{d}\tilde{\psi}|\wedge|\mathrm{d}\tilde{\psi}\rangle = i\int_S \langle \mathrm{d}\Psi|\wedge|\mathrm{d}\Psi\rangle = i\int_S \mathrm{Tr}\,\rho\mathrm{d}\rho\wedge\mathrm{d}\rho, \tag{8}$$

of the gauge invariant curvature 2-form (for a normalized state $|\Psi\rangle$) over the surface S spanned by C (+ G, if necessary), vide a Stokes-like theorem. The relation (8) is known [29, 24] for cyclic evolutions.

Nonintegrability along a general curve in the ray space is a salient feature of geometric phase, yet results (3) and (7) show that geometric phase with respect to a fixed reference ray is triangle-integrable, in the spirit of the surface integral (8). This triangle-integrability of Φ_G was recognized earlier (eq. (10) in [17] and eq. (4.37) in

[18]). Relation (18) in [17] was derived group theoretically for $d\Phi_G^A$, which coincides with eq. (6) read with (5) above. In eqs (3.17) and (4.34) of [18], only a formal irreducible expression

$$\Phi_G = \arg\mathrm{Tr}\left(\rho_1 P\exp\left(\int_{\rho_1}^{\rho_2} \mathrm{d}\rho\right)\right),$$

was obtained in terms of the ordered sequence of noncommuting operators. As discussed, this expression originates from the unitary operation $P\exp(\int_{\rho_1}^{\rho_2}[\mathrm{d}\rho,\rho])$ effected by the parallel transport Hamiltonian, if each infinitesimal step of the evolution is evaluated only to the first order. The noncommutation between the successive infinitesimal operators makes the geometric phase nonintegrable. We have exploited the triangle-integrability of Φ_G by considering virtual evolutions to and fro the reference ray $|\psi_0\rangle$, after each infinitesimal step of the actual evolution to arrive at a simple closed-form integral (3) for Φ_G, in contrast to [17, 18]. Further, we have explicitly identified the integrands in (3) and (7) with the invariant 3-vertex phase of the associated infinitesimal Pancharatnam triangle. A special version of (3), with the reference ray $|\psi_0\rangle$ fixed at the initial ray $|\psi_1\rangle$, applicable to any general evolution,has been obtained previously [20] by invoking the Pancharatnam connection continuously.

The key requirement for generating geometric phase has also emerged here. During the formative years of geometric phase immediately following the publication of [6], geometric phase was believed to arise only in very special evolutions obeying several constraints. It is now realized that for producing geometric phase, the state evolution need not be adiabatic [7], cyclic [4], unitary [8] or be governed by a specific equation [18]. Does geometric phase then have any prerequisite? The relation (2) for the Pancharatnam triangle phase implies that a ray space characterized by noncommuting density operators is necessary for producing geometric phase.

In conclusion, we have presented a geometric formalism for the general geometric phase using Pancharatnam's ideas [1]. The 3-vertex invariant phase for the infinitesimal Pancharatnam triangle is the only basic input here. This has obviated the need to invoke any Hamiltonian or governing equation for effecting a state evolution and hence to make reference to dynamical phase, enabling us to confine the discussion of geometric phase strictly to the ray space. We have thence arrived at a general, yet simple, closed-form expression for geometric phase in terms of just the density operator.

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