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Current algebra results for the $\bar{B} - D$ systems

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Abstract. Using the equal time commutation relations for the components of the vector and axialvector currents and keeping single particle states we obtain relations for the weak form factors for the $\bar{B} - D$ systems. In the heavy quark effective theory (HQET) limit these relations determine the Isgur–Wise function.

Keywords. Heavy quark effective theory; current algebra; Isgur–Wise function; weak form factors of heavy quarks.

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1. Introduction

In the last few years a very interesting approach to physics of hadrons containing a heavy quark has been developed [1–8]. Theoretically this has led to the formulation of a heavy quark effective theory (HQET). In this approach new symmetries appear which have led to interesting predictions. In particular, the 6 form factors, which in general would determine the hadronic matrix elements in the semileptonic decays

$$\bar{B}(p) \to D(p')e\nu, \ \bar{B}(p) \to D^*(p')e\nu, \tag{1}$$

get related. All of them can be expressed in terms of only one unknown scalar function $\xi(v \cdot v')$, called the Isgur–Wise function. The argument of ξ is the Lorentz scalar $\omega = v \cdot v'$ where v_μ and v'_μ are the four velocities of the $\bar{B}$ and D (or D^*) mesons.

In this paper, we use equal time commutators (ETC) for the time components of vector (V_0) and axialvector (A_0) currents, keeping only single particle states, to derive two relations (see (14) and (20)) between the form factors entering in the $\bar{B}$ decays in (1). The eqs. (14) and (20) so obtained are strictly speaking inequalities because of the contribution of the multiparticle states. There results of current algebra in the HQET limit reduce to the result

$$\xi(\omega) \leqslant \sqrt{\frac{2}{1+\omega}}, \tag{2}$$

a result noted earlier [5, 6, 7]. Though this result is quoted in the literature [7], the derivation seems to use dispersion relations. In this, we note that the result follows from all the equal time commutators ($x_0 = y_0$) $[X_0(x), Y_0(y)]$, where X, Y can be either time or space component of vector or the axialvector operator. For the spatial components we assume that the Schwinger terms a complex c number.

2. Determination of sum rules

We assume that the b and c heavy quark currents satisfy equal time commutation relations a' la Gell-Mann [9].

(a) Consider the equal time commutator (ETC) $(x_0 = y_0)$:

$$[V_0^+(x), V_0^-(y)] = 2V_0^3(x)\delta^3(\mathbf{x}-\mathbf{y}), \tag{3}$$

for the time components of the vector currents V_μ for b and c quarks. In (3), $V_0^+ = \bar{c}\gamma_0 b$, $V_0^- = \bar{b}\gamma_0 c$ and $V_0^3 = \frac{1}{2}(\bar{c}\gamma_0 c - \bar{b}\gamma_0 b)$. The $V_0 - V_0$ commutator is not expected to have Schwinger terms.

Starting with (3), for an arbitrary $\mathbf{q}$, we obtain [10]

$$\left[\int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} V_0^+(x), \int d^3y e^{i\mathbf{q}\mathbf{y}} V_0^-(y)\right] = 2\int d^3x d^3y e^{-i\mathbf{q}(\mathbf{x}-\mathbf{y})} V_0^3(x)\delta^3(\mathbf{x}-\mathbf{y}). \tag{4}$$

Sandwich (4) between the states $|\bar{B}^0(p)\rangle$ and $\langle\bar{B}^0(p')|$ with 4-momenta p and p' respectively where $\bar{B}^0 \sim b\bar{d}$.

The r.h.s. of (4) R then becomes (since $\int d^3x V_0^3(x)$ is the generator of the 3rd component of the heavy quark flavour symmetry)

$$R \equiv 2\left\langle \bar{B}^0(p')\left|\int d^3x V_0^3(x)\right|\bar{B}^0(p)\right\rangle = 2(-\tfrac{1}{2})\langle\bar{B}^0(p')|\bar{B}^0(p)\rangle = -\delta^3(\mathbf{p}-\mathbf{p}'). \tag{5}$$

Last equality specifies our normalization for the meson states.

In the l.h.s. insert a complete set of intermediate states and use translation invariance $V(x) = e^{-iP\cdot x}V(O)e^{iP\cdot x}$ and perform the x and y integrations to obtain, the l.h.s. of (4) to be

$$\begin{aligned} L \equiv (2\pi)^6 \sum_n [\langle\bar{B}^0(p')|V_0^+(O)|n, p_n\rangle\langle n, p_n|V_0^-(O)|\bar{B}^0(p)\rangle\delta^3(\mathbf{p}'+\mathbf{q} \\ -\mathbf{p}_n)\delta^3(\mathbf{p}+\mathbf{q}-\mathbf{p}_n) - \langle\bar{B}^0(p')|V_0^-(O)|n, p_n\rangle\langle n, p_n|V_0^+(O)|\bar{B}^0(p)\rangle\delta^3 \\ (\mathbf{p}'-\mathbf{q}-\mathbf{p}_n)x\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{p}_n)]. \end{aligned} \tag{6}$$

Of course, $L = R$, because of (4).

We now approximate (6) by keeping single-particle states in the sum over n. Since $V_0^+ \sim \bar{c}\gamma_0 b$, only the 2nd term will contribute and the possible states which can contribute are $D^+(c\bar{d})$ and $D^{*+}(c\bar{d})$ with $J^P = 0^-$ and 1^- respectively. In this approximation

$$L = -[L_D + L_{D^*}], \tag{7}$$

where

$$\begin{aligned} L_D = (2\pi)6 \int d^3p_D \langle B^0(p')|V_0^-(O)|D^+, p_D\rangle\langle D^+, p_D|V_0^+(O)|\bar{B}^0(p)\rangle \\ \times \delta^3(\mathbf{p}'-\mathbf{q}-\mathbf{p}_D)\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{p}_D) \\ = (2\pi)^6\delta^3(\mathbf{p}'-\mathbf{p})\langle\bar{B}^0(p)|V_0^-(O)|D^+, p-q\rangle\langle D^+, p-q|V_0^+(O)|\bar{B}^0(p)\rangle, \end{aligned} \tag{7a}$$

where $p_D = p - q$ has been made explicit.

Similarly

$$L_{D^*} = (2\pi)^6 \delta^3(\mathbf{p}-\mathbf{p}') \sum_{\lambda} \langle \bar{B}^0(p)| V_0^-(O)|D^{*+}, \varepsilon(\lambda), p-q\rangle \langle D^{*+}, \varepsilon(\lambda), p-q| V_0^+(O)|\bar{B}^0(p)\rangle. \tag{7b}$$

The transition amplitudes in (7a) and (7b) can be written in terms of form factors (in usual notation) with our normalization as follows

$$(2\pi)^3\sqrt{4p^0k^0}\langle D^+(k^0)| V_\mu^+(O)|\bar{B}^0(p)\rangle = f_+(Q^2)(p+k)_\mu + f_-(Q^2)(p-k)_\mu \tag{8}$$

and

$$(2\pi)^3\sqrt{4p^0k^0}\langle D^{*+}(k^0)| V_\mu^+(O)|\bar{B}(p)\rangle = ig(Q^2)\varepsilon_{\mu\nu\lambda\sigma}\varepsilon^{*\nu}(p+k)^\lambda(p-k)^\sigma. \tag{9}$$

Here, $Q = p - k$ and the form factors $f_\pm$ and g are real. Since, in (7a, b), $k = p - q, Q^2 = q^2$, one obtains

$$L_D = \frac{\delta^3(\mathbf{p}'-\mathbf{p})}{4p^0(p^0-q^0)}[f_+(q^2)(2p_0-q_0)+f_-(q^2)q_0]^2 \tag{10}$$

and

$$L_{D^*} = \frac{\delta^3(\mathbf{p}'-\mathbf{p})}{4p^0(p^0-q^0)}[2g(q^2)]^2\sum_{\lambda}\varepsilon_{0\alpha\beta\gamma}\varepsilon_{0\mu\nu\sigma}\varepsilon^{*'}(\lambda)\varepsilon^{\mu}(\lambda)p^\beta q^\gamma p^\nu q^\sigma. \tag{11}$$

Performing the sum over λ and D^* polarization is straightforward and yields

$$L_{D^*} = \frac{\delta^3(\mathbf{p}'-\mathbf{p})}{4p^0(p^0-q^0)}[2g(q^2)]^2[\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\cdot\mathbf{q})^2]. \tag{12}$$

To obtain a covariant result [10], we use the standard technique of going to the infinite momentum frame in (10) and (12), namely

$$p_0, |\mathbf{p}| \to \infty, \nu = p\cdot q \text{ fixed.} \tag{13}$$

To ensure fixed ν we choose $\mathbf{q}$ such that $\mathbf{p}\cdot\mathbf{q}$. This choice gives $\nu = p_0 q_0$ and implies $q_0 \to 0$ and $-\mathbf{q}^2 = q^2$. Putting together (4–12) and implementing (13) yields

$$1 = f_+^2(q^2) - g^2(q^2)q^2. \tag{14}$$

This is the general basic result, for any finite q^2. If we keep the multiparticle states this becomes $1 \geqslant f^2+(q^2) - g^2(q^2)q^2$. Any estimation of the contribution from these multiparticle states (such as D) would involve a detailed analysis of experimental data, not all of which is available. We choose not to go into such details in this brief report.

In the HQET, the form factors $f_\pm$ and g in (8) and (9) are expressible in terms of the single function $\xi(\omega)$ and because of the spin symmetry the $J^P = 0^-$ and 1^- states are degenerate in mass, so we take $m_{D^*} = m_D$ below. In the leading order, HQET gives

$$f_\pm(q^2) = \xi(\omega)\frac{m_B \pm m_D}{2\sqrt{m_B m_D}}, \tag{15}$$

$$g(q^2) = \frac{\xi(\omega)}{2\sqrt{m_B m_D}}, \tag{16}$$

where $\omega = v{\cdot}v'$ and $p_\mu = m_B v_\mu$ and $p'_\mu = m_D v'_\mu$ are the four momenta of $\bar{B}$ and D (or D^*). Also $q^2 = (p-p')^2 = m_B^2 + m_D^2 - 2m_B m_{D^{(*)}}$. Substituting (15) and (16) in the sum rules (14) immediately yields (2).

(b) Next we consider the ETC for the time components of the axial vector currents, viz.

$$[A_0^+(x), A_0^-(y)] = 2\delta^3(\mathbf{x}-\mathbf{y})\, V_0^3(x). \tag{17}$$

Any possible operator Schwinger terms, present additionally in the right hand side of (17), would be neglected. These might be expected to involve long-range effects and could go away in the heavy quark limit.

Following the same procedure as in (3), and keeping the single particle contribution (viz. D^* in this case) one obtains

$$1 = (2\pi)^6 \sum_\lambda \langle \bar{B}^0(p)|A_0^-(O)|D^{*+}, \varepsilon(\lambda), p-q\rangle \langle D^{*+}, \varepsilon(\lambda), p-q|A_0^+(O)|\bar{B}(p)\rangle. \tag{18}$$

The 3 real form factors f, $a_\pm$ for the A_μ transition amplitude are defined through

$$\begin{aligned}(2\pi)^3 \sqrt{4p_0k_0}\,\langle D^{*+}, \varepsilon(\lambda), k|A_\mu^+(O)|\bar{B}^0(p)\rangle \\ = f(Q^2)\varepsilon_\mu^* + [a_+(Q^2)(p+k)_\mu + a_-(Q^2)(p-k)_\mu](\varepsilon^*{\cdot}p),\end{aligned} \tag{19}$$

where $Q = p-k$. Substituting this in (18) and going to the infinite momentum frame (see eq. (13) and following) gives the sum rule

$$\begin{aligned}4m_{D^*}^2 = [f(q^2)]^2 + 2f(q^2)a_+(q^2)[m_B^2 - m_{D^*}^2 - q^2] \\ + [a_+(q^2)]^2[-4m_B^2 m_{D^*}^2 + (m_B^2 + m_{D^*}^2 - q^2)^2].\end{aligned} \tag{20}$$

As remarked above, (20) should read $4m_{D^*}^2 \geqslant \cdots$ because of the multiparticle states. This relation is expected to hold for a general a_+ and f for any finite q^2.

In HQET, to leading order [4]

$$f(q^2) = \frac{\xi(\omega)}{\sqrt{4m_B m_{D^*}}}[(m_B + m_{D^*})^2 - q^2], \tag{21}$$

$$a_+(q^2) = -\frac{\xi(\omega)}{\sqrt{4m_B m_{D^*}}}, \tag{22}$$

where, as before $\omega = v{\cdot}v'$ and v and v' are the 4 velocities of the $\bar{B}$ and D^*. Using (21–22) in the relation (20) again yields (2) for $\xi(\omega)$. It is gratifying that both the $V_0 - V_0$ and $A_0 - A_0$ ETC give the same result for $\xi(\omega)$.

(c) If one uses the ETC between V_0 and A_0 and evaluates the matrix element between $\bar{B}^{0*}$ and D^* states and saturate it with single particle states, one again obtains an answer which is consistent with (2) in the HQET limit.

Finally we note that the same results follow on the use of commutation relation $[A_1^a(x), A_1^b(y)] = i\varepsilon^{abc} V_0^C \delta^3(x-y) + \text{S.T.}$ or $[V_1^a(x), V_1^b(y)] = i\varepsilon^{abc} V^C \delta^3(x-y) + \text{S.T.}$ and consider its matrix element between states $|B(p)\rangle$ with momentum $p = (p^0, p^1, 0, 0)$ and take the infinite momentum limit. Here we assume the Schwinger term is of the form $c\delta^{\alpha\beta}(\partial/\partial x_1)\delta^3(\mathbf{x}-\mathbf{y})$, where c is a complex number.

The above results are a foretaste of the interesting relations or bounds one can obtain from current algebra for the weak form factors of the $\bar{B} - D$ system. The current algebra provides a straightforward, simple and systematic approach to obtain constraints among form factors which are more general than those obtained from HQET.

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