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A q deformation of Gell-Mann–Okubo mass formula

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Abstract. We explore the possibility of deforming Gell-Mann–Okubo (GMO) mass formula within the framework of a quantized enveloping algebra. A small value of the deformation parameter is found to provide a good fit to the observed mass spectra of the π, K and η mesons.

Keywords. Deformation; mass formula; chiral symmetry breaking.

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The study of quantum groups has aroused [1] much interest of late. A glance through the literature reveals [2] that several deformed algebraic structures have been developed to modify various physical systems. Since the idea of a quantum group is more easily accessible via an enveloping algebra – the latter almost always corresponding to a deformation [3, 4] of a Lie structure of some sort, it is worthwhile looking for deformations of those schemes in which Lie algebras are of potential relevance. In this note we consider a deformation of the underlying SU(3) algebra of the quark model and look for the consequences.

One can deform [4] the full SU(3) group as defined by say, Jimbo and Drinfeld or Fairlie and Nuyts. Alternatively since SU(3) of the quark model is effectively described by the constituent subgroups namely, the isospin, U spin and V spin, deformation of all or any one of these subsectors may be carried out. In the following we enquire how deforming a particular SU(2) influences the whole of SU(3) (minimal deformation).

SU(3) consists of two diagonal matrices λ_3 and λ_8. Assuming $\lambda_3 = \mathrm{diag}(1, -1, 0)$ along with $m_u = m_d$ and noting that the members of a U spin multiplet enjoys the same electromagnetic properties we perform, as a first step, deformation of the V spin sector of SU(3) only. The idea is to retain the properties of the I and U spin invariances even after deformation. The amusing point is that because of an interplay between the SU(2) subalgebras, a q deformation of the V spin brings about modifications in the I spin and U spin sectors too. As a consequence we are led to a deformed GMO mass rule for the π, K and η mesons involving a single deformation parameter.

In our scheme the deformation variable q plays essentially the role of a phenomenological parameter offering an extra freedom to fit the GMO formula with the observed masses. In the literature, correction factors have been introduced from time to time to match the experiment: for example, soft meson corrections, extended PCAC, form factors with a suitable phenomenological parameter, etc. Deformation of the underlying algebra of the quark model provides another possibility. It should be stated that in principle we could have deformed all the SU(2) subsectors of SU(3). But this would have only increased the number of deformation parameters corresponding to each subsector (maximal deformation). In our simple-minded approach we have avoided dealing with such extra parameters.

The framework of our analysis rests on the GMOR scheme [5] of chiral symmetry breaking which, for concreteness, we deform according to the oft-studied [6a] quantum algebra of Witten and Woronowicz [3,4]. It is needless to mention that we could have adopted any other form of enveloping algebra yielding results [7] similar to the present one.

Witten–Woronowicz SU(2) quantum algebra is given by

$$[W_0, W_+]_q = W_+, [W_-, W^0]_q = W_-, [W_+, W_-]_{1/q^2} = W_0 \tag{1a}$$

for a triplet of operators $W_\pm$, W_0 and the deformed brackets are to be read $[A, B]_q = qAB - q^{-1}BA$ for a pair of operators A and B. The q commutations (1) reduce to those of the conventional Lie algebra SU(2) or SU(1, 1) in the limits $q = 1$ or -1 respectively and so may be looked upon as a deformed map of either SU(2) or SU(1, 1).

The advantage in working with the algebra (1a) is that the generators can be given a matrix representation. Moreover, it has recently been shown by Lorek and Wess [6b] that the deformed scheme (1a) admits of a co-multiplication rule

$$\begin{aligned}
\Delta(W_\pm) &= W_\pm \otimes 1 + \tau_3^{1/2} \otimes W_\pm \\
\Delta(W_0) &= W_0 \otimes 1 + \tau_3 \otimes W_0 - q\lambda\tau_3^{1/2}\, W_+ \otimes W_- - \frac{\lambda}{q}\tau_3^{1/2}\, W_- \otimes W_+ \\
\tau_3 &= 1 - \frac{\lambda}{q}(q + 1/q)\, W_0 - \frac{\lambda^2}{q^2}(q + 1/q)\, W_+ W_- \\
\lambda &= q - 1/q.
\end{aligned} \tag{1b}$$

Further within the framework of (1a) these authors have demonstrated that interacting systems may be interpreted in terms of a free system based on q-deformed kinematics.

Curtright and Zachos [8] have worked out explicitly invertible functionals of the SU(2) generators which deform SU(2) continuously into some quantum algebra. For the deformed brackets (1) they have found the following matrix representations

$$\begin{aligned}
W_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Big/ (q + q^{-1})^{1/2}, \quad W_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Big/ (q + q^{-1})^{1/2} \\
W_0 &= \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \Big/ (q + q^{-1}).
\end{aligned} \tag{2}$$

Since for $q = 1$, $W_0 \to j_0$ and $W_\pm \to j_\pm$ with the SU(2) algebra $[j_0, j_\pm] = \pm j_\pm, [j_+, j_-] = j_0$ holding good, it is clear that the deformed matrices (2) convert the SU(2) generators into operators which obey the q-brackets of the quantized algebra (1) for all values of q.

Given this background we ask the question as to whether the underlying algebra of the quark model can be mapped onto some quantum algebra say, the one provided by (1).

We shall presently see that even for an infinitesimal deformation $\mathrm{Tr}(\lambda_8^d \lambda_3) \neq 0$ where λ_8^d is the deformed λ_8 matrix. The interesting point is that an infinitesimal deformation (that is in terms of $O(\log q)$) does not affect the π and K but alters only the η mass.

In analogy with (2) let us propose that the deformed V spin obeys the following quantized algebra

$$[V_0^d, V_+^d] = V_+^d, [V_-^d, V_0^d] = V_-^d$$

$$[V_+^d, V_-^d]_{1/q^2} = V_0^d \tag{3}$$

where a superscript d indicates a deformed quantity.

It is straightforward to work out the representations of V_+^d and V_0^d which satisfy (3). These turn out to be

$$V_+^d = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big/ (q+q^{-1})^{1/2}, \quad V_-^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Big/ (q+q^{-1})^{1/2}$$

$$V_0^d = \mathrm{diag}\left(\frac{1}{q^2}, 0, -q^2\right) \Big/ (q+q^{-1}). \tag{4a, b, c}$$

Since λ_3 has not been deformed it follows from (4c) that the deformed λ_8 matrix is

$$\lambda_0^d = \mathrm{diag}\left[-1 + \frac{4}{q(q^2+1)}, 1, \frac{-4q^3}{q^2+1}\right] \Big/ \sqrt{3}. \tag{5}$$

If the deformation is taken to be infinitesimal $q = 1 + \varepsilon$, $\varepsilon = \log q$ then

$$\lambda_8^d = \mathrm{diag}(1 - 4\varepsilon, 1, -2 - 4\varepsilon)/\sqrt{3} \tag{6}$$

along with

$$\lambda_8^d = \sqrt{2}/3\, \mathrm{diag}(1, 1, 1 - 4\varepsilon). \tag{7}$$

Note that although $\mathrm{Tr}(\lambda_0^d \lambda_8^d)$ and $\mathrm{Tr}(\lambda_0^d \lambda_3)$ vanish, $\mathrm{Tr}(\lambda_8^d \lambda_3)$ does not vanish. Henceforth we shall work with infinitesimal deformation only.

We now turn to the pseudoscalar mass spectra and chiral symmetry breaking. Because of the deformed λ_8 and λ_0 matrices the symmetry breaking Hamiltonian density in the GMOR scheme reads

$$H_d = c_0^d(\bar{q}\lambda_0^d q) + c_3^d(\bar{q}\lambda_3^d q) + c_8^d(\bar{q}\lambda_8^d q) \tag{8}$$

where $q = (u, d, s)^T$ and the symmetry breaking parameters $c_i (i = 0, 3, 8)$ in terms of the conventional quark masses are

$$c_0^d = \frac{1}{3}\sqrt{3/2}\left[\left(1 + \frac{8\varepsilon}{3}\right)(m_u + m_d) + \left(1 - \frac{4\varepsilon}{3}\right)m_s\right] + O(\varepsilon^2)$$

$$c_3^d = \frac{2\varepsilon + 3}{6}m_u + \frac{2\varepsilon - 3}{6}m_d - \frac{2\varepsilon}{3}m_s + O(\varepsilon^2)$$

$$c_8^d = \frac{\sqrt{3}}{6}\left[\left(1 - \frac{10\varepsilon}{3}\right)(m_u + m_d) - 2\left(1 + \frac{2\varepsilon}{3}\right)m_s\right] + O(\varepsilon^2). \tag{9}$$

Notice that the parameter $c_3^{\rm d} \neq 0$ for $m_u = m_d$ [naive SU(2)] and $c_8^{\rm d} \neq 0$ for $m_u = m_d = m_s$ [naive SU(3)] as a consequence of deformation. Thus even an infinitesimal deformation induces SU(2) and SU(3) breakings: in the I spin sector it is roughly of the order of $\frac{2}{3}(m - m_s)$. This is reminiscent of Oakes observation [9] made years ago that an isospin conserving Hamiltonian density when rotated about the 7th axis in the SU(3) space picks an isospin violating piece in a natural way.

To obtain the pseudoscalar mass spectra we use the Heisenberg's equation namely, $\delta_\mu A_\mu^j = i[H'_{\rm d}, O_5^j]$ and the PCAC relation $\delta_\mu A_\mu^j = f_j m_j^2 \phi_j$, j stands for π, K and η mesons respectively and f_j's are the corresponding decay constants.

From (8) we find

$$f_\pi m_\pi^2 = m Z_\pi^{1/2} + O(\varepsilon^2) \tag{10a}$$

$$f_K m_{K^+}^2 = f_K m_{K^0}^2 = \tfrac{1}{2}(m + m_s) Z_K^{1/2} + O(\varepsilon^2) \tag{10b}$$

$$f_\eta m_\eta^2 = \tfrac{1}{3}[(1 - 4\varepsilon)m + 2(1 + 4\varepsilon)m_s] Z_\eta^{1/2} + O(\varepsilon^2). \tag{10c}$$

In deriving (10a), (10b) and (10c) we have, in the context of our approximation, identified the meson particle states with those of the undeformed exact SU(3). The quantities $Z_p^{1/2} (p = \pi, K$ and $\eta)$ are the respective wave function renormalization constants.

Since the particle states are assumed to be eigenstates of exact symmetry we may equate $Z_\pi^{1/2} = Z_K^{1/2} = Z_\eta^{1/2}$ and obtain from (10)

$$4f_K m_K^2 - 3f_\eta m_\eta^2 - f_\pi m_\pi^2 = 4\varepsilon(m - 2m_s) Z^{1/2} + O(\varepsilon^2). \tag{11}$$

We thus get a deformed GMO mass formula in the limit $f_\pi = f_K = f$

$$4m_K^2 - 3m_\eta^2 - m_\pi^2 = 4\varepsilon\left(1 - 2\frac{m_s}{m}\right) m_\pi^2 + O(\varepsilon^2). \tag{12}$$

The rhs of (12) gives the deformation corrections. Inserting the masses for π, K and η we get $\varepsilon \approx -0{\cdot}02$ for $m_s/m \approx 20$–25 which shows that the deformation is truly infinitesimal. It may be checked that the infinitesimal nature of deformation persists even for other choices of deformed algebras enlisted in [8]. Indeed following the expressions in (10)–(12) deformation may be looked upon [10] as a kind of perturbation.

Let us end by remarking that recently some attempts have been made to seek phenomenological applications [11–13] of q deformations. These include the study [11] of the transition from $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times U(1)$ to $\mathrm{SU}(2) \times U(1)$ of the standard model by q deformation and an attempt [12] to understand the problem of the identity between the superdeformed bands in neighbouring odd and even nuclei. The spirit of the present work comes close to these: by treating the deformation variable as a phenomenological parameter but confining ourselves to the constraints of an enveloping algebra we have shown that a small deformation can account for the 20% discrepancy in the fit to the GMO formula. Our scheme of deformation can be extended to other low-energy theorems by quantizing not only SU(3) but also XU(3) × SU(3) algebra of currents.

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