PRAMANA © Printed in India Vol. 46, No. 3,
— journal of March 1996
physics pp. 219–221

An identity for 4-spacetimes embedded into E_5

JOSÉ L LÓPEZ-BONILLA and H N NÚÑEZ-YÉPEZ*
Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana-Azcapotzalco, Apartado Postal 21-726, Coyoacán 04000 D.F., México
*Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apartado Postal 21-726, Coyoacán 04000 D.F., México

MS received 10 January 1995

Abstract. We show that if a 4-spacetime V_4 can be embedded into E_5 then, if b_{ij} is the second fundamental form tensor associated with V_4, the quantity (trace b)$\cdot b_{ij}^{-1}$ depends only on intrinsic geometric properties of the spacetime. Such fact is used to obtain a necessary condition for the embedding of a V_4 into E_5.

Keywords. Embedding of Riemannian 4-spaces; local and isometric embedding.

PACS Nos **04·20; 04·90**

1. Introduction

Let us consider a 4-spacetime local and isometrically embedded into E_5. This means that there exists the second fundamental form tensor $b_{ij} = b_{ji}$ which fulfils the Gauss and the Codazzi equations [1–3]

$$R_{ijkc} = \varepsilon(b_{ik}b_{jc} - b_{ic}b_{jk}) \tag{1}$$

and

$$b_{ij;k} = b_{ik;j}, \tag{2}$$

where $\varepsilon = \pm 1$, R_{ijkc} is the V_4 Riemann tensor and ;r stands for covariant derivative. It is well-known that whenever $\det b^i_j \neq 0$ then (1) implies (2); furthermore, it is not difficult to obtain the following relationship [3–7]

$$-24\det(b^r_c) = K_2 \equiv {}^*R^{*ijkr}R_{ijkr}, \tag{3}$$

K_2 being one of the Lanczos invariants [8, 9] defined in terms of the double dual [10] of Riemann tensor

$${}^*R^{*ij}{}_{rc} \equiv \tfrac{1}{4}\eta^{ijaq}R_{aq}{}^{me}\eta_{merc}, \tag{4}$$

where η_{abcd} is a Levi–Civita symbol. This work deals with the case in which $K_2 \neq 0$; according to (3), this implies that the inverse matrix to the second fundamental form, b_{ij}^{-1}, exists. We offer a proof that it is possible to find a relation between ${}^*R^*_{ijkc}$ and b_{ar}^{-1} which is analogous to (1). Among other things this implies that (trace $\mathbf{b}$)$\cdot\mathbf{b}^{-1}$ depends only on the intrinsic geometry of V_4.

2. Gauss equation for $*R^*_{ijkl}$

For any class-one spacetime we can use (1) to obtain the identity [2, 11]

$$pb_{ij} = \frac{K_2}{48} g_{ij} - \frac{1}{2} R_{imnj} G^{mn}, \tag{5}$$

where $p \equiv \varepsilon b^{ar} G_{ar}/3$, $G_{ac} = R_{ac} - Rg_{ac}/2 = *R^{*i}{}_{aci}$ is the Einstein tensor, $R_{ij} = R^r{}_{ijr}$ is the Ricci tensor, and $R = R^c_c$ is the scalar curvature. In [2] González *et al* have shown that p is determined by the intrinsic geometry of V_4. Furthermore, when $p = 0$, using (5) is trivial to establish that **b** is determined by the intrinsic geometry of the spacetime and, therefore, that (trace **b**)$\mathbf{b}^{-1}$ is also an intrinsic quantity. The problem appears when the case $p \neq 0$ is considered [6, 12, 13] for then the previous statements become non-trivial. It is the purpose of this work to show that (trace **b**)$\mathbf{b}^{-1}$ is in fact intrinsic irrespective of the value of p. To this end an equation of the type (1) but relating $*R^*_{ijkl}$ to b^{-1}_{ar} is needed.

Yakupov [14, 15] has shown that for every class-one V_4 the following equation holds good

$$*R^{*imnj} R_{acnj} = \frac{K_2}{12} (\delta^i_a \delta^m_c - \delta^i_c \delta^m_a). \tag{6}$$

Substituting (1) into (6) we get

$$\varepsilon \frac{K_2}{24} (\delta^i_a \delta^m_c - \delta^i_c \delta^m_a) = *R^{*ijmnj} b_{an} b_{jc}; \tag{7}$$

assuming $K_2 \neq 0$ then, on multiplying (7) by $b_r^{-1a} b_q^{-1c}$, we easily get

$$\frac{24}{K_2} *R^*_{imrq} = \varepsilon (b^{-1}_{ir} b^{-1}_{mq} - b^{-1}_{iq} b^{-1}_{mr}). \tag{8}$$

This expression has the same structure as Gauss equation (1) hence illustrating the analogous role, the curvature tensor and its double dual play. The problem of embedding for a class-one 4-spacetime is thus reduced to analyzing (1) or (8).

González *et al* [2] have studied how (1) implies (5), in an analogous way from (8) we may get

$$b^r_r b_a^{-1j} = -\frac{1}{2} \delta^j_a + \frac{12}{K_2} *R^*_{amn}{}^j R^{mn}. \tag{9}$$

This exhibits that (trace **b**)$\cdot\mathbf{b}^{-1}$ is a quantity which only depends on the intrinsic geometry of V_4 irrespective of the value of p.

Equation (9) is interesting since it allows establishing a new necessary condition for the embedding of a V_4 (but with $K_2 \neq 0$) into E_5. This condition can be obtained by first contracting i with j in (5), in this way we get

$$p \,\mathrm{trace}\, b = \frac{1}{2} \left(\frac{K_2}{6} - R^{cq} G_{cq} \right); \tag{10}$$

then, on multiplying (5) and (9), we obtain

$$24{}^*R^*_{imnj}R^{i\ \ c}_{\ rt}R^{mn}G^{rt} + K_2\left[3\left(\frac{K_2}{8} + R^2/2 - R^{mn}R_{mn}\right)\delta^c_j - RR^c_j + 2R^c_nR^n_j\right] = 0, \tag{11}$$

where we used (10). We have established that every class-one spacetime with $K_2 \neq 0$ should comply with (11), which is thus a necessary condition for embedding that we have not found previously published.

Acknowledgements

This work has been partially supported by CONACyT grant 4846–E9406.

References

[1] D Kramer, H Stephani, M Mac Callum and E Herlt, *Exact solutions of Einstein's field equations* (Cambridge University Press, 1980)
[2] G González, J López-Bonilla and M A Rosales, *Pramana – J. Phys.* **42**, 85 (1994)
[3] J López-Bonilla, J Morales and M A Rosales, *Braz. J. Phys.* **24**, 522 (1994)
[4] T Y Thomas, *Acta Math.* **67**, 169 (1936)
[5] D Ladino, J López-Bonilla, J Morales and G Ovando, *Rev. Mex. Fis.* **36**, 354 (1990)
[6] R Fuentes, J López-Bonilla, G Ovando and T Matos, *Gen. Relativ. Gravit.* **21**, 777 (1989)
[7] D Ladino, J López-Bonilla, J Morales and G Ovando, *Rev. Mex. Fis.* **36**, 354 (1990)
[8] C Lanczos, *Ann. Math.* **39**, 842 (1938)
[9] V Gaftoi, J López-Bonilla, D Navarrete and G Ovando, *Rev. Mex. Fis.* **36**, 503 (1990)
[10] C Lanczos, *Rev. Mod. Phys.* **34**, 379 (1962)
[11] H F Goenner, *Tensor New Series* **30**, 15 (1976)
[12] O Chavoya, D Ladino, J López-Bonilla and J Fernández, *Rev. Colomb. Fis.* **23**, 15 (1991)
[13] J López-Bonilla, J M Rivera and H Yee-Madeira, *Braz. J. Phys.* **25**, 80 (1995)
[14] M Sh Yakupov, *Sov. Phys. Dokl. (Engl. Transl.)* **13**, 585 (1968)
[15] R Fuentes and J López-Bonilla, *Acta Mex. Cienc. Technol. IPN* **3**, 9 (1985)