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Objectification problem, CHSH inequalities for a system of two spin-1/2 particles

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Abstract. The weak objectification and Bell/CHSH inequalities are studied for a particular type of set of states of two spin-1/2 particles. The restriction on interference term which allows Bell/CHSH inequalities to be satisfied are found out.

Keywords. Bell-CHSH inequalities; interference term; weak objectification; classicality of probabilities.

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1. Introduction

A quantum mechanical observable is generally non-objective unless the system is prepared in an eigenstate or gemenge of eigenstates. The hypothetical assignment of eigenvalues of some non-objective observable to a system which is actually in a superposition of eigenstates known as weak objectification, is incompatible with quantum mechanics [1, 2].

Hypothetical value assignment is also the subject of discussion about hidden variable theories underlying quantum mechanics. It has been shown that application of EPR reality criterion and locality on a collection of observables is equivalent to the existence of their joint distribution. But in certain important special cases the existence of joint probabilities is equivalent to the validity of a set of probability relations, namely Bell/CHSH inequalities [3, 4], the experimental violation of which can be taken as evidence against objectification. Recently it has been shown that the same set of inequalities is necessary and sufficient conditions that these probabilities can be obtained in the range of classical probability measure [5, 6]. One could use such a criterion of the classicality of probabilities in order to postulate the objectivity of properties of a physical system.

It has been shown that weak objectification and the classicality of probabilities lead to different consequences [2]. For example, in the case of two spin-1/2 particles in a singlet state, it has been shown that joint probability may be satisfied even in cases where weak objectification fails. The validity domain of Bell's inequalities has been shown in figure 1 of [2].

The main cause of non-objectification, either in the sense of weak objectification or classicality condition for probabilities, is the existence of interference term in a pure state. Here, considering two parameter family of states of two spin-1/2 particles which are intermediate between singlet state and mixture of product states having spin zero configuration along a particular direction, we shall find out how much interference term can be allowed so that Bell's inequalities are always satisfied for any set of three spin-1/2 observables. This will be done in § 3.

In § 2 the two parameter family of states will be described and in these states probabilities for spin-1/2 observables will be calculated. In § 4, the maximum amount of interference term which allows the CHSH inequalities to be valid for the chosen sets of four observables which give maximal violation in the extreme pure states will be investigated. In this case the whole set of states described in § 2 will be considered.

2. Two parameter family of states and probabilities

We consider a system of two spin-1/2 particles which is associated to a 4-dimensional Hilbert space ($\mathscr{C}^2 \otimes \mathscr{C}^2$). States are represented by positive, self adjoint and trace class operator of trace one.

We consider the family of states of the form

$$W = \frac{1+\lambda}{2} P[|\phi^1_+(\mathbf{n})\rangle \otimes |\phi^2_-(\mathbf{n})\rangle + \frac{1-\lambda}{2} P[|\phi^1_-(\mathbf{n})\rangle \otimes |\phi^2_+(\mathbf{n})\rangle]$$

$$- \frac{r}{2} |\phi^1_+(\mathbf{n})\rangle\langle\phi^1_-(\mathbf{n})| \otimes |\phi^2_-(\mathbf{n})\rangle\langle\phi^2_+(\mathbf{n})|$$

$$- \frac{r}{2} |\phi^1_-(\mathbf{n})\rangle\langle\phi^1_+(\mathbf{n})| \otimes |\phi^2_+(\mathbf{n})\rangle\langle\phi^2_-(\mathbf{n})| \tag{1}$$

where $|\phi^\alpha_\pm(\mathbf{n})\rangle (\alpha = 1, 2)$ are eigenstates of $\mathbf{n}\cdot\boldsymbol{\sigma}$($\boldsymbol{\sigma}$ being Pauli matrices and $\mathbf{n}$ is a unit vector) for the α-th spin-1/2 particle. $P[\cdot]$ is one-dimensional projection operator corresponding to the vector state in the third bracket and $-1 \leqslant r \leqslant 1$ and $-1 \leqslant \lambda \leqslant 1$.

For $r = 1$ and $\lambda = 0$, W corresponds to singlet state and for $r = -1$ and $\lambda = 0$, W corresponds to a pure entangled state given by

$$\psi_n = \tfrac{1}{2}[|\phi^1_+(\mathbf{n})\rangle \otimes |\phi^2_-(\mathbf{n})\rangle + |\phi^1_-(\mathbf{n})\rangle \otimes |\phi^2_+(\mathbf{n})\rangle]. \tag{2}$$

The spin observables are represented by projection operators. Let $P(\mathbf{n}_i)$ represent the projection operator corresponding to the spin observable which measures spin along the direction $\mathbf{n}_i$.

The probability and joint probability distributions that measurements of $\mathbf{n}_i\cdot\boldsymbol{\sigma}$ on one subsystem, $\mathbf{n}_j\cdot\boldsymbol{\sigma}$ on the other, and $\mathbf{n}_i\cdot\boldsymbol{\sigma}$ and $\mathbf{n}_j\cdot\boldsymbol{\sigma}$ jointly, will give $+1$ result for the subsystems as well as jointly for both systems in the state W, are respectively given by

$$p_i = \mathrm{Tr}[WP(\mathbf{n}_i) \otimes I] = \tfrac{1}{2}[1 + \lambda(\mathbf{n}\cdot\mathbf{n}_i)]$$

$$p_j = \mathrm{Tr}[WI \otimes P(\mathbf{n}_j)] = \tfrac{1}{2}[1 - \lambda(\mathbf{n}\cdot\mathbf{n}_j)]$$

$$p_{ij} = \mathrm{Tr}[WP(\mathbf{n}_i) \otimes P(\mathbf{n}_j)] = \frac{r}{4}(1 - \mathbf{n}_i\cdot\mathbf{n}_j)$$

$$+ \frac{\lambda}{4}[\mathbf{n}\cdot\mathbf{n}_i - \mathbf{n}\cdot\mathbf{n}_j] + \frac{1-r}{4}[1 - (\mathbf{n}\cdot\mathbf{n}_i)(\mathbf{n}\cdot\mathbf{n}_j)]. \tag{3}$$

3. Weak objectification and classical representation of probability sequence

When the two particle system is in the state W, the reduced mixed states of the subsystems are

$$W_1 = \mathrm{Tr}_2[W] = \frac{1+\lambda}{2} P[|\phi^1_+(\mathbf{n})\rangle] + \frac{1-\lambda}{2} P[|\phi^1_-(\mathbf{n})\rangle]$$

$$W_2 = \mathrm{Tr}_1[W] = \frac{1-\lambda}{2} P[|\phi^2_+(\mathbf{n})\rangle] + \frac{1+\lambda}{2} P[|\phi^2_-(\mathbf{n})\rangle]. \tag{4}$$

Tr_α denotes trace operation on W with respect to the Hilbert space associated with the α-th particle.

From the last expressions apparently it seems that the subsystems are in a mixture of eigenstates where ignorance interpretation can be applied. To check that, let us now assume that the spin observables $\mathbf{n}\cdot\boldsymbol{\sigma}$ is weakly objectified with respect to the subsystem S_1 in the state W. This implies that the observable $\mathbf{n}\cdot\boldsymbol{\sigma}\otimes I$ is weakly objectified with respect to the compound system in the state W. Now weak objectification entails that [2]

$$\mathrm{Tr}[WP(\mathbf{n}_1)\otimes P(\mathbf{n}_2)] = \mathrm{Tr}[L(\mathbf{n})\, WP(\mathbf{n}_1)\otimes P(\mathbf{n}_2)] \tag{5}$$

where $P(\mathbf{n}_1)$ and $P(\mathbf{n}_2)$ are arbitrary test observables and the Luder operation $L(\mathbf{n})$ is given by

$$L(\mathbf{n})\, W = P(\mathbf{n})\, WP(\mathbf{n}) + (I - P(\mathbf{n}))\, W(I - P(\mathbf{n}))$$

$$= \tfrac{1}{2}P[|\phi^1_+(\mathbf{n})\rangle \otimes |\phi^2_-(\mathbf{n})\rangle] + \tfrac{1}{2}P[|\phi^1_-(\mathbf{n})\rangle \otimes |\phi^2_+(\mathbf{n})\rangle]. \tag{6}$$

Now (5) gives

$$r[(\mathbf{n}\times\mathbf{n}_1)(\mathbf{n}\times\mathbf{n}_2)] = 0. \tag{7}$$

Equation (7) is the condition under which the observable $n\cdot\boldsymbol{\sigma}$ can be weakly objectified with respect to the state W if the observable $\mathbf{n}_1\cdot\boldsymbol{\sigma}\otimes\mathbf{n}_2\cdot\boldsymbol{\sigma}$ is used as test observable. Equation (7) is fulfilled if one of the following conditions are satisfied:

i) one of the factors $(\mathbf{n}\times\mathbf{n}_1)$ or $(\mathbf{n}\times\mathbf{n}_2)$ or both are zero.
ii) the planes spanned by $(\mathbf{n},\mathbf{n}_1)$ and $(\mathbf{n},\mathbf{n}_2)$ are orthogonal.
iii) $r = 0$ i.e. there will be no interference term in W.

From the above observation we conclude that for $r \neq 0$, in the case of arbitrary choice of test observables weak objectification fails.

We shall now find the restriction on r for the set of states $W(0 \leqslant r \leqslant 1)$ so that the probability sequence $\{p_1, p_2, p_3, p_{12}, p_{32}, p_{11}\}$ can have classical representation.

One should take care that p_{11} has been included in the probability sequence as in the case of Bell's inequalities one of the three spin observables has to pertain to both the system. Now in the case of singlet state, $p_{11} = 0$, and so Bell-Wigner inequality does not contain such term but for the state $W(r \neq 0)$

$$p_{11} = \frac{1-r}{4}[1 - (\mathbf{n}\cdot\mathbf{n}_1)]. \tag{8}$$

Let us now write down one of the CHSH inequalities [6] concerning four observables $P(\mathbf{n}_1)$ and $P(\mathbf{n}_3)$ for one of the particles and $P(\mathbf{n}_2)$ and $P(\mathbf{n}_4)$ for the other particle in the state W.

$$0 \leqslant p_1 + p_2 - p_{12} - p_{14} - p_{23} + p_{34} \leqslant 1. \tag{9}$$

Now making two observables $P(\mathbf{n}_1)$ and $P(\mathbf{n}_4)$ same i.e. $\mathbf{n}_1 = \mathbf{n}_4$ and putting all the probabilities from (3) we get

$$2 - r[1 + \mathbf{n}_1\cdot\mathbf{n}_2 + \mathbf{n}_2\cdot\mathbf{n}_3 - \mathbf{n}_1\cdot\mathbf{n}_3] + (1-r)[(\mathbf{n}_1\cdot\mathbf{n})(\mathbf{n}_2\cdot\mathbf{n}) + (\mathbf{n}_2\cdot\mathbf{n})(\mathbf{n}_3\cdot\mathbf{n}) + (\mathbf{n}_1\cdot\mathbf{n})^2 - (\mathbf{n}_1\cdot\mathbf{n})(\mathbf{n}_3\cdot\mathbf{n})] \geqslant 0. \tag{10}$$

To find the maximum value of r for which there will be no violation of the inequality (10), let us choose the unit vectors for which maximal violation of Bell's inequality occurs. The choice is given by

$$\mathbf{n}_1 = \hat{i}, \quad \mathbf{n}_2 = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}, \quad \mathbf{n}_3 = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}.$$

Let the polar and azimuthal angles of the unit vector $\mathbf{n}$ are θ and ϕ respectively. Then

$$\mathbf{n} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along co-ordinate axes.

Then calculating all the scalar product and putting them in the inequality (10) we get

$$2 - \tfrac{5}{2}r - \tfrac{1}{2}(1-r)\sin^2\theta[3\sin^2\phi - \sqrt{3}\sin\phi\cos\phi] \geqslant 0. \tag{11}$$

The maximum value of $\sin^2\theta[3\sin^2\phi - \sqrt{3}\sin\phi\cos\phi]$ will occur at $\theta = \pi/2$ and $\phi = \pi/2 + 15°$. Putting this value of θ and ϕ we get

$$\tfrac{5}{2}r + \tfrac{1}{4}(1-r)[3 + 2\sqrt{3}] - 2 \leqslant 0 \tag{12}$$

which gives

$$r \leqslant \frac{5 - 2\sqrt{3}}{7 - 2\sqrt{3}} = 0{\cdot}433 = r_B. \tag{13}$$

For this choice of unit vectors other three inequalities [6] are not violated even for singlet state.

So as long as $r \leqslant r_B$, there will be no violation of Bell's inequalities which implies that the above probability sequence, obtained from three spin-1/2 observables giving maximal violation for singlet state, in a state W, will have classical representation.

4. Non-violation of CHSH inequalities and bound on interference term

In this section we shall study CHSH inequality concerning four different observables for the whole set of two parameter family of states W given in (1).

Let $P(\mathbf{n}_1)$, $P(\mathbf{n}_2)$ pertain to one particle and $P(\mathbf{n}_3)$, $P(\mathbf{n}_4)$ to the other. We consider the CHSH inequality

$$0 \leqslant p_1 + p_4 - p_{13} - p_{14} - p_{24} + p_{23} \leqslant 1. \tag{14}$$

Using the expression for probabilities in (3) we get

$$r[\mathbf{n}_1\cdot\mathbf{n}_3+\mathbf{n}_1\cdot\mathbf{n}_4+\mathbf{n}_2\cdot\mathbf{n}_4-\mathbf{n}_2\cdot\mathbf{n}_3]+(1-r)[(\mathbf{n}_1\cdot n)(n_3\cdot n)+ (\mathbf{n}_1\cdot n)(\mathbf{n}_4\cdot n)+(\mathbf{n}_2\cdot n)(\mathbf{n}_4\cdot n)-(\mathbf{n}_2\cdot n)(\mathbf{n}_3\cdot n)]\leqslant 2 \tag{15}$$

which can be written as

$$r[(\mathbf{n}_1\times\mathbf{n})(\mathbf{n}_3\times\mathbf{n})+(\mathbf{n}_1\times\mathbf{n})(\mathbf{n}_4\times\mathbf{n})+(\mathbf{n}_2\times\mathbf{n})(\mathbf{n}_4\times\mathbf{n}) -(\mathbf{n}_2\times\mathbf{n})(\mathbf{n}_3\times\mathbf{n})]+[(\mathbf{n}_1\cdot\mathbf{n})(\mathbf{n}_3\cdot\mathbf{n})+(\mathbf{n}_1\cdot\mathbf{n})(\mathbf{n}_4\cdot\mathbf{n})+(\mathbf{n}_2\cdot\mathbf{n})(\mathbf{n}_4\cdot\mathbf{n}) -(\mathbf{n}_2\cdot\mathbf{n})(\mathbf{n}_3\cdot\mathbf{n})]\leqslant 2. \tag{16}$$

From (16) it is clear that if the observable $\mathbf{n}\cdot\boldsymbol{\sigma}$ is weakly objectified in the state W for all the pair of test observables $\mathbf{n}_1\otimes\mathbf{n}_3\ldots$ etc, the first term in the right hand side of (16) vanishes and there will be no violation of CHSH inequalities.

Now to the restriction on r, let us first check whether there is any violation for the state W with $r=-1$. Putting $r=-1$ in (15) we get

$$-[\mathbf{n}_1\cdot\mathbf{n}_3+\mathbf{n}_1\cdot\mathbf{n}_4+\mathbf{n}_2\cdot\mathbf{n}_4-\mathbf{n}_2\cdot\mathbf{n}_3]+2[(\mathbf{n}_1\cdot n)(\mathbf{n}_3\cdot n)+ (\mathbf{n}_1\cdot\mathbf{n})(\mathbf{n}_4\cdot\mathbf{n})+(\mathbf{n}_2\cdot\mathbf{n})(\mathbf{n}_4\cdot\mathbf{n})-(\mathbf{n}_2\cdot\mathbf{n})(\mathbf{n}_3\cdot\mathbf{n})]\leqslant 2. \tag{17}$$

Now with the choice,

$$\mathbf{n}_4=\hat{i},\quad \mathbf{n}_3=\hat{j},$$

$$\mathbf{n}_2=\frac{1}{\sqrt{2}}(-\hat{i}+\hat{j}),\quad \mathbf{n}_1=\frac{1}{\sqrt{2}}(-\hat{i}-\hat{j}) \tag{18}$$

and

$$\mathbf{n}=\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j}+\cos\theta\hat{k} \tag{19}$$

inequality (17) gives

$$2\sqrt{2}-2\sqrt{2}\sin^2\theta\leqslant 2. \tag{20}$$

So from inequality (20) there will be violation as long as $\theta<\sin_{-1}[(1-1/\sqrt{2})^{1/2}]$ and the maximal violation will occur at $\theta=0°$. To find out the restriction on r in general, we shall apply the choice of vectors of (18) for the states W with $-1\leqslant r<0$ and for those with $0<r\leqslant 1$ we shall apply the standard choice giving maximal violation of singlet state and which is given by

$$\mathbf{n}_2=\hat{i},\quad \mathbf{n}_4=\hat{j}$$

$$\mathbf{n}_3=\frac{1}{\sqrt{2}}(-\hat{i}+\hat{j}),\quad \mathbf{n}_1=\frac{1}{\sqrt{2}}(\hat{i}+\hat{j}). \tag{21}$$

Case 1

For $0<r<1$, and for the choice of the unit vectors of (21), inequality (15) gives

$$-2+2\sqrt{2}r+(1-r)\sqrt{2}\sin^2\theta=f(r,\theta)\leqslant 0. \tag{22}$$

$f(r,\theta)$ is maximum for $\theta=90°$, and then (22) gives $r\leqslant\sqrt{2}-1$. For $\theta=0°$, there will be no violation of CHSH inequality as long as $r\leqslant 1/\sqrt{2}$.

So we conclude that for $r \leqslant \sqrt{2} - 1$, there will be no violation of CHSH inequality whichever choice the vector $\mathbf{n}$ takes.

Case 2

For $-1 < r < 0$, and for the choice of unit vectors given in (18) and putting $r = -r$, inequality (15) gives

$$-2 + 2\sqrt{2}r - (1 + r)\sqrt{2}\sin^2\theta = f(r,\theta) \leqslant 0. \tag{23}$$

Here $f(r,\theta)$ is maximum for $\theta = 0^\circ$. So as long as $r \geqslant -1/\sqrt{2}$, there will be no violation for the choice of observables which give maximal violation in the state W with $r = -1$.

For both the above choices of observables the remaining CHSH inequalities are not violated even for singlet state. So in general we can say that for the density operators in (1), there will be no violation of CHSH inequalities if

$$-\frac{1}{\sqrt{2}} \leqslant r \leqslant (\sqrt{2} - 1)$$

for the above choices of spin observables.

5. Discussion

We see from the above result that for correlated state of two spin-1/2 particles with spin zero configuration along any direction, whether Bell/CHSH inequalities will be satisfied for spin observables depend on the value of the parameter r, be it a pure or a non-pure state. But this result in no way forbids from getting four dichotomic observables violating CHSH inequality for any correlated pure state [7].

One more important thing should be noted. In case where dimension of Hilbert space is three or more, and if we consider all the observables, no state allows classical representation of probabilities [8]. So classical representation of probabilities has very limited scope in quantum mechanics.

Comment

The study of Bell/CHSH inequalities for more general state than (1) can be studied in the same way and it will be published elsewhere in the near future.

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