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BD–FRW models in the framework of Israel–Stewart–Hiscock theory

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Abstract. BD–FRW universe filled with imperfect fluid having bulk viscosity is investigated under the framework of Israel–Stewart–Hiscock causal theory. The field equations have been solved by using the relation $\phi = KR^{\alpha}$ where K and α are constants, between the Brans–Dicke scalar field φ and the scale factor R. This relation, in fact, leads to a constant deceleration parameter q. It is shown that the constancy of the deceleration parameter permits only two possibilities i.e. either $H =$ constant with $m = 1$ or $m = (1 + b - \alpha)/(2(1 + b) - \alpha)$, irrespective of the value of ε.

Keywords. Cosmology; Brans–Dicke theory; Israel–Stewart–Hiscock causal thermodynamics.

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1. Introduction

The evolution of the universe includes a number of dissipative processes. The role of dissipative effects on the evolution of the universe has been investigated by many authors [1]. These investigations are based on the conventional thermodynamic theory of irreversible processes due to Eckart [2]. Hiscock and Lindblom [3] showed that Eckart's theory is non-causal (since it admits dissipative signals with super-luminal velocities) and all its equilibrium states are unstable. These problems arise due to the first order nature of the theory i.e. it considers only first-order deviations from equilibrium. The neglected second-order terms are in fact necessary to prevent non-causal and unstable behaviour. Alternatively, second-order theories [4–6] have been proposed to remedy this drawback. Particularly successful is the so-called 'extended irreversible thermodynamics' (EIT) theory [4,6]. However, Hiscock and co-workers [7,8] have shown that most versions of the causal second-order theories, including EIT, omit certain divergence terms. Hence, the resulting "truncated" evolution equations give rise to very different behaviour than the full equations. Therefore, the best currently available theory for analysing dissipative processes in the universe is the full Israel–Stewart–Hiscock causal thermodynamics.

In this paper we wish to investigate the Brans–Dicke [BD]-FRW universe filled with imperfect fluid having bulk viscosity under the frame-work of the full Israel–Stewart–Hiscock causal theory. Most of the well known BD-FRW models of the universe with curvature parameter $k = 0$ are models with constant deceleration parameter. Hence, in this paper the field equations have been solved by considering the relation $\phi = KR^{\alpha}$ between the scalar field ϕ and the scale factor R. This relation leads to the constancy of the deceleration parameter.

2. Field equations

The field equations of BD theory (with $c = 1$) are

$$G_{ab} = -\frac{8\pi}{\phi}T_{ab} - \frac{\omega}{\phi^2}\left[\phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}\phi_{;c}\phi^{;c}\right] - \frac{1}{\phi}[\phi_{;a;b} - g_{ab}\Box^2\phi] \tag{1}$$

and

$$\Box^2\phi = \frac{8\pi}{3+2\omega}T^a_{.a} \tag{2}$$

where ϕ is the scalar field and ω the BD coupling parameter; T_{ab}, the energy-momentum tensor for a bulk viscous fluid, is given by

$$T_{ab} = [\rho + p + \Pi]u_a u_b + [p + \Pi]g_{ab} \tag{3}$$

where ρ is the energy density, p the thermodynamic pressure, Π the bulk viscous pressure and u_a the four velocity satisfying the condition $u_a u^a = 1$.

Considering the FRW metric, with $k = 0$

$$ds^2 = dt^2 - R^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{4}$$

and an equation of state

$$p = \gamma\rho, \quad 0 \leqslant \gamma \leqslant 1 \tag{5}$$

the field eqs (1) and (2) reduce to

$$3\left[\frac{\dot R}{R}\right]^2 + 3\frac{\dot\phi\dot R}{\phi R} - \frac{\omega}{2}\left[\frac{\dot\phi}{\phi}\right]^2 = \frac{8\pi\rho}{\phi} \tag{6}$$

$$2\frac{\ddot R}{R} + \left[\frac{\dot R}{R}\right]^2 + 2\frac{\dot\phi\dot R}{\phi R} + \frac{\omega}{2}\left[\frac{\dot\phi}{\phi}\right]^2 + \frac{\ddot\phi}{\phi} = -\frac{8\pi\gamma\rho}{\phi} - \frac{8\pi\Pi}{\phi} \tag{7}$$

$$\ddot\phi + 3\frac{\dot R}{R}\dot\phi = \frac{8\pi}{3+2\omega}[(1-3\gamma)\rho - 3\Pi] \tag{8}$$

while the continuity equation $T^{ab}_{;b} = 0$ gives

$$\dot\rho + 3(1+\gamma)H\rho = -3\Pi H \tag{9}$$

where $H \equiv \dot R/R$ is the Hubble's constant.

The causal evolution equation for the bulk viscous pressure is given by

$$\Pi + \tau\dot\Pi = -3\xi H - \frac{\varepsilon}{2}\tau H\left(3H + \frac{\dot\tau}{\tau} - \frac{\dot\xi}{\xi} - \frac{\dot T}{T}\right) \tag{10}$$

where T is the temperature, ξ the bulk viscosity co-efficient and τ the relaxation time. For $\tau = 0$, (10) reduces to the non-causal equation. In (10), $\varepsilon = 0$ gives the truncated theory (extended irreversible thermodynamics [EIT]), while $\varepsilon = 1$ gives the full Israel–Stewart–Hiscock theory.

3. The general solution

In view of the complex non-linear character of the BD field equations, to solve these equations we assume a functional relation between ϕ and R to be of the form

$$\phi = KR^{\alpha} \tag{11}$$

where K and α are constants. With the choice (11), (6) to (8) reduce to

$$\frac{8\pi\rho}{\phi} = \left[3 + 3\alpha - \frac{\omega\alpha^2}{2}\right]H^2 \tag{12}$$

$$-\frac{8\pi\gamma\rho}{\phi} - \frac{8\pi\Pi}{\phi} = (2+\alpha)\dot{H} + \left[3 + \frac{\omega\alpha^2}{2} + 2\alpha + \alpha^2\right]H^2 \tag{13}$$

$$\frac{8\pi}{\phi}[(1-3\gamma)\rho - 3\Pi] = (3+2\omega)[\alpha(\alpha+3)H^2 + \alpha\dot{H}]. \tag{14}$$

Eliminating ρ and Π from the above equations, we have

$$\dot{H} = -(1+b)H^2 \tag{15}$$

i.e.,

$$\frac{\ddot{R}}{R} + b\left[\frac{\dot{R}}{R}\right]^2 = 0 \tag{16}$$

where

$$b = \frac{\omega\alpha^2 + 4\omega\alpha - 6}{2(\omega\alpha - 3)}; \quad \omega\alpha - 3 \neq 0. \tag{17}$$

Equation (15) shows that these models have a constant deceleration parameter b. Incidentally, most of the well-known perfect fluid models in general relativity and BD theory belong to this category.

Equations (12–14), (9) and (10) imply the dynamical equation for the Hubble parameter

$$\frac{\tau}{8\pi}[(2+\alpha)\ddot{H}\phi + [6(1+\gamma) + \omega\alpha^2(1-\gamma) + 2\alpha(2+3\gamma) + 2\alpha^2]H\dot{H}\phi] \tag{18}$$

$$= 3\xi H + \frac{\varepsilon}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right) + \Pi.$$

On integration, (16) gives the exact solution

$$R(t) = \begin{cases} [D + Ct]^{1/(1+b)} & b \neq -1 \\ R_0 e^{\chi t} & b = -1 \end{cases} \tag{19}$$

where C, D and χ are constants of integration.

Case (i): $b \neq -1$

In this case it can be seen that the Hubble parameter can be written in terms of the scale factor as

$$H = \frac{C}{(1+b)}R^{-(1+b)}. \tag{20}$$

Using (20) in (11) we have

$$\phi = \phi_0 H^{-\alpha/(1+b)}. \tag{21}$$

From (12) and (13) we have

$$\Pi = \frac{-\phi}{8\pi}\left[(2+\alpha)\dot{H} + \left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\frac{\omega\alpha^2}{2} + \alpha^2\right]H^2\right]. \tag{22}$$

Now using (21) and (22) in (18) yields

$$\begin{aligned}
&\frac{\phi_0}{8\pi}\left[\tau(2+\alpha)\ddot{H}H^{-\alpha/(1+b)} + 2\left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\frac{\omega\alpha^2}{2} + \alpha^2\right]\right.\\
&\quad \times \tau\dot{H}H^{1-(\alpha/(1+b))} + (2+\alpha)\dot{H}H^{-\alpha/(1+b)}\\
&\quad \left. + \left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\frac{\omega\alpha^2}{2} + \alpha^2\right]H^{2-(\alpha/(1+b))}\right]\\
&= 3\xi H - \frac{\varepsilon}{2}\frac{\tau}{8\pi}\phi_0 H^{-(\alpha/(1+b))}\left[(2+\alpha)\dot{H} + \left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\right.\right.\\
&\qquad \left.\left. \times \frac{\omega\alpha^2}{2} + \alpha^2\right]H^2\right]\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right).
\end{aligned} \tag{23}$$

Further, if we assume

$$\xi = \alpha'\rho^m, \quad \tau = \frac{\xi}{\rho} = \alpha'\rho^{m-1}, \quad T = \beta\rho^r \tag{24}$$

where $\alpha'(\geqslant 0)$, $m(\geqslant 0)$, $\beta(\geqslant 0)$ and $r(\geqslant 0)$ are constants, then we have

$$\frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} = -\left[2 - \frac{\alpha}{1+b}\right](1+r)\frac{\dot{H}}{H}. \tag{25}$$

Using (25) in (23) leads to

$$\begin{aligned}
&\frac{\phi_0}{8\pi}\left[\tau(2+\alpha)\ddot{H}H^{1-(\alpha/(1+b))} + 2\left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\frac{\omega\alpha^2}{2} + \alpha^2\right]\right.\\
&\quad \times \tau\dot{H}H^{2-(\alpha/(1+b))} + (2+\alpha)\dot{H}H^{1-(\alpha/(1+b))}\\
&\quad + \left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\frac{\omega\alpha^2}{2} + \alpha^2\right]H^{3-(\alpha/(1+b))}\\
&\quad + \frac{\varepsilon}{2}\tau H^{-(\alpha/(1+b))}\left[(2+\alpha)\dot{H} + \left[3(1+\gamma) + (2+3\gamma)\alpha + (1-\gamma)\right.\right.\\
&\qquad \left.\left.\left. \times \frac{\omega\alpha^2}{2} + \alpha^2\right]H^2\right]\left(3H^2 - \left[2 - \frac{\alpha}{1+b}\right](1+r)\dot{H}\right)\right] - 3\xi H^2 = 0
\end{aligned} \tag{26}$$

Now using (24) together with (12) and (15) in (26) we have

$$\frac{\phi_0}{8\pi}\Bigg[\Bigg[2(2+\alpha)(1+b)^2B-2B(1+b)\left[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)\frac{\omega\alpha^2}{2}+\alpha^2\right]$$
$$-\frac{3\varepsilon}{2}(2+\alpha)(1+b)B-3A\Bigg]H^{(2(1+b)+m(2(1+b)-\alpha))/(1+b)}$$
$$+\left[\left[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)\frac{\omega\alpha^2}{2}+\alpha^2\right]-(1+b)(2+\alpha)\right]$$
$$\times H^{3-(\alpha/1+b)}+B\Bigg[\frac{\varepsilon}{2}\left(2-\frac{\alpha}{1+b}\right)(1+r)(2+\alpha)(1+b)^2$$
$$-\frac{3\varepsilon}{2}\left[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)\frac{\omega\alpha^2}{2}+\alpha^2\right]$$
$$-\frac{\varepsilon}{2}\left(2-\frac{\alpha}{1+b}\right)(1+r)(2+\alpha)(1+b)\Bigg[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)$$
$$\times\frac{\omega\alpha^2}{2}+\alpha^2\Bigg]\Bigg]H^{(2(1+b)+m(2(1+b)-\alpha)/(1+b)}\Bigg]=0 \qquad (27)$$

where $A=\alpha'\left[\dfrac{3+3\alpha-(\omega\alpha^2/2)}{8\pi}\right]^m\phi_0^m$ and $B=\alpha'\left[\dfrac{3+3\alpha-(\omega\alpha^2/2)}{8\pi}\right]^{m-1}\phi_0^{m-1}$.

From (27) it can be seen that m can only take the value

$$m=\frac{1+b-\alpha}{2(1+b)-\alpha} \qquad (28)$$

Now (28) can be rewritten as

$$1+b=\frac{(m-1)\alpha}{(2m-1)}. \qquad (29)$$

For models of the expanding universe with ϕ increasing we require $\alpha>0$ and $1+\beta>0$; this constrains m not to lie in the range $\frac{1}{2}\leqslant m\leqslant 1$.

For singular models, since $R(0)=0$, (19) leads to

$$R=R_0t^{1/(1+b)}. \qquad (30)$$

From (30) and (11) we have

$$\phi=\phi_0t^{\alpha/(1+b)}. \qquad (31)$$

Using (30) and (31) in (12) yields

$$\rho=\frac{\phi_0}{8\pi}\left[3+3\alpha-\frac{\omega\alpha^2}{2}\right]t^{(\alpha/1+b)-2}. \qquad (32)$$

From (32) it can be seen that ρ would increase or decrease with time according as

$$\alpha>2(1+b)\quad\text{or}\quad\alpha<2(1+b).$$

Now using (30), (31) and (32) in (13), we have

$$\Pi = -\frac{\phi_0}{8\pi}\left[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)\frac{\omega\alpha^2}{2}+\alpha^2-(2+\alpha)(1+b)\right]t^{(\alpha/1+b)-2}. \tag{33}$$

From (24) and (32) with $m = (1+b-\alpha)/(2(1+b)-\alpha)$ we get

$$\xi = At^{(\alpha-(1+b))/(1+b)} \tag{34}$$

$$\tau = Bt. \tag{35}$$

From (34) it can be seen that ξ would increase or decrease according as

$$\alpha > (1+b) \quad \text{or} \quad \alpha < (1+b).$$

From (35) it can be seen that the behaviour of the relaxation time τ is independent of the scalar-field ϕ (i.e., the power-index α) and it increases with time.

Case (ii): $b = -1$

In this case H is a constant (as can be seen from (15)) given by

$$H = \chi = \text{constant}. \tag{36}$$

From (11) and (19) we have

$$\phi = \phi_0 \exp(\alpha\chi t). \tag{37}$$

Using (36) and (37) in (12) leads to

$$\rho = \frac{\phi_0}{8\pi}\left[3+3\alpha-\frac{\omega\alpha^2}{2}\right]\chi^2\exp(\alpha\chi t). \tag{38}$$

Now using (36), (37) and (38) in (13) yields

$$\Pi = -\frac{\phi_0}{8\pi}\left[3(1+\gamma)+(2+3\gamma)\alpha+(1-\gamma)\frac{\omega\alpha^2}{2}+\alpha^2-(2+\alpha)(1+b)\right]\chi^2\exp(\alpha\chi t). \tag{39}$$

Equation (18) now reduces to

$$3\xi\chi + \frac{\varepsilon}{2}\tau\Pi\left(3H+\frac{\dot{\tau}}{\tau}-\frac{\dot{\xi}}{\xi}-\frac{\dot{T}}{T}\right)+\Pi = 0. \tag{40}$$

Using (38) in (24) we have

$$\xi = A'\exp(\alpha m\chi t) \tag{41}$$

$$\tau = B'\exp(\alpha(m-1)\chi t) \tag{42}$$

where

$$A' = \alpha'\left[\frac{3+3\alpha-(\omega\alpha^2/2)}{8\pi}\right]^m \phi_0^m\chi^{2m}$$

$$B' = \alpha'\left[\frac{3+3\alpha-(\omega\alpha^2/2)}{8\pi}\right]^{m-1} \phi_0^{m-1}\chi^{2(m-1)}.$$

From (24) we have

$$\frac{\dot{\tau}}{\tau}-\frac{\dot{\xi}}{\xi}-\frac{\dot{T}}{T}=-(1+r)\frac{\dot{\rho}}{\rho}. \tag{43}$$

Using (38) in (43) leads to

$$\frac{\dot{\tau}}{\tau}-\frac{\dot{\xi}}{\xi}-\frac{\dot{T}}{T}=-(1+r)\alpha\chi. \tag{44}$$

Using (44) in (40) yields

$$3\xi\chi+\frac{\varepsilon}{2}\tau\Pi\chi(3-(1+r)\alpha)+\Pi=0. \tag{45}$$

From (39), (41), (42) and (45) we see that m can only take the value 1.

4. Conclusion

Constancy of the deceleration parameter permits only two possibilities i.e., either $H=\text{constant}$ with $m=1$ or $m=(1+b-\alpha)/(2(1+b)-\alpha)$, irrespective of the value of ε. Incidentally, most of the well known perfect fluid models in general relativity and BD theory are models with constant deceleration parameter. It is interesting to note that in the presence of bulk viscous pressure there could be exponential expansion in BD theory.

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