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Optical theorem and Aharonov–Bohm scattering

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Abstract. A rigorous derivation of the optical theorem (OT) from the conservation of probability flux (CPF) is presented for scattering on an arbitrary spherically symmetric potential in N-spatial dimensions (ND). The constructed expression for the OT is found to yield the corresponding well-known results for two- and three-dimensional cases in a rather natural way. The Aharonov-Bohm (AB) effect is considered as a scattering event of an electron by a magnetic field confined in an infinitely long shielded solenoid and a similar derivation is attempted for an appropriate optical theorem. Our current understanding of the scattering theory is found to be inadequate for the purpose. The reason for this is discussed in some detail.

Keywords. Optical theorem; AB scattering.

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1. Introduction

In the three dimensional (3D) quantum mechanical scattering the optical theorem (OT) expresses the total cross section in terms of imaginary part of the forward scattering amplitude, $f(\theta = 0)$. In two spatial dimensions the theorem does not look exactly like that found in the 3D case. However, it can be made to appear so by not absorbing a phase factor [1] into the definition of scattering amplitude. Being related to the unitarity of S matrix or t matrix on the energy shell, the OT provides a mathematical statement for the conservation of probability flux [2] (CPF). Therefore, extraction of the OT from CPF should not involve assumptions or approximations that vary with the dimensionality of space for the scattering event. But studies in existing literature [3, 4] indicate that this is not the case with 2D scattering. The derivation of 2DOT does really involve mathematical approximations which are not demanded in a similar logical progression for the 3DOT. More specifically, the 2DOT is constructed by assuming $f(\theta) \simeq f(0)$ or by making explicit use of the stationary phase approximation [5]. The situation thus calls for a more detailed investigation into the causal connection between the OT and CPF. To that end we derive the optical theorem in N-spatial dimensions by using an equivalent mathematical procedure as is followed to deal with the 3DOT. Further, we examine the OT in the context of Aharonov-Bohm (AB) scattering [6, 7]. This is expected to be quite instructive because the AB effect is characterized by a non-trivial geometry [8] that does not fit into the framework of formal scattering theory. Also, regarded as a scattering event the process is characterized by kinetic momentum ($m\mathbf{r} = \mathbf{p} - (e/c)\mathbf{A}$) rather than the canonical momentum ($\mathbf{p}$).

In § 2, we address the proposed generalization of OT for the time-independent Schrödinger equation in an arbitrary number (N) of spatial dimension. As already emphasized, our analysis will be based on the CPF rather than mere identification of the NDOT from partial wave expressions [9]. We shall pay special attention to the subtleties that arise in the course of our derivation. The optical theorem for the AB effect is considered in § 3. We have chosen to work with two types of scattering states of an AB system modelled by simply and non-simply connected spaces and found that neither of the representations permits one to derive the theorem rigorously. However, our analysis of the difficulties encountered followed by appropriate attempts to deal with some of them will be quite illuminating for future researchers in this area of investigation. For example, in view of the apprehension that the AB scattering cross section is strongly beamsize dependent, it may be worthwhile to see if an accurate statement of OT could be made by limiting the spatial extent of the incident beam. Finally, in § 4, we summarize our outlook on the present work and make some concluding remarks.

2. OT in many dimensions

The N dimensional Schrödinger equation for a hypercentral potential $V(r)$ at an energy $E = k^2 > 0$ is given by [10]

$$\left(\frac{1}{2}\nabla_N^2 - V(r) + k^2\right)\psi(\mathbf{r}) = 0 \tag{1}$$

with the hyperradius r written as

$$r = \left(\sum_{i=1}^{N} x_i x^i\right)^{1/2}, \quad x_i = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{i-1}\cos\theta_i. \tag{2}$$

The angular variables θ's satisfy the relation

$$0 \leqslant \theta_i \leqslant \pi, \quad i = 1, 2, \ldots, N-2; \quad 0 \leqslant \theta_{N-1} \leqslant 2\pi \text{ and } \theta_N = 0. \tag{3}$$

The Laplacian ∇_N^2 is

$$\nabla_N^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{L^2(\Omega)}{r^2}. \tag{4}$$

Here $L^2(\Omega)$ stands for the grand orbital operator at the hyperangle $\Omega(=(\theta_1, \theta_2, \ldots, \theta_{N-1}))$ on the hypersphere S^{N-1}. From (1) and (4) we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{L^2(\Omega)}{r^2} - V(r) + k^2\right)\psi(r, \Omega) = 0. \tag{5}$$

Separation of the angular and radial part of (5) yields the radial equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L(L+N-2)}{r^2} - V(r) + k^2\right)R_L(r) = 0. \tag{6}$$

The conservation of probability flux for the ND scattering is expressed as

$$\lim_{r\to\infty} r^{N-1}\int j_r(r, \Omega)\mathrm{d}\Omega = 0 \tag{7}$$

with the radial component of the probability current density

$$j_r(r,\Omega) = \mathrm{Re}\left(\psi^* \frac{\hbar}{im}\frac{\partial}{\partial r}\psi\right). \tag{8}$$

On the unit sphere S^{N-1} the surface element

$$\mathrm{d}\Omega = \prod_{j-1}^{N-1} \sin^{N-j-1}\theta_j \mathrm{d}\theta_j. \tag{9}$$

The physical or outgoing wave solution of (5) behaves asymptotically as [11]

$$\psi(r,\Omega) \sim e^{ikr\cos\theta} + f(\Omega)\frac{e^{ikr}}{r^{(N-1)/2}}. \tag{10}$$

Clearly, θ is the angle between the wave vector $\mathbf{k}$ and $\mathbf{r}$. In terms of the L-th partial wave scattering phase shift δ_L and the Gegenbauer polynomial $C_n^{(\beta)}(x)$, the amplitude $f(\Omega)$ is given by

$$f(\Omega) = -\left(\frac{2}{i}\right)^{(N-5)/2} \frac{\Gamma\left(\frac{N}{2}-1\right)}{\sqrt{\pi}} \frac{1}{k^{(N-1)/2}}$$

$$\sum_{L=0}^{\infty} (2L+N-2)(e^{2i\delta_L}-1)C_L^{((N/2)-1)}(\cos\theta). \tag{11}$$

Substituting (10) in (8), we get

$$j_r(r,\Omega) = \frac{\hbar k}{m}\left[\cos\theta + \frac{1}{r^{N-1}}|f(\Omega)|^2 + \frac{1}{r^{(N-1)/2}}(1+\cos\theta)\,\mathrm{Re}\{f(\Omega)e^{ikr(1-\cos\theta)}\}\right]$$

$$+\frac{\hbar}{m}\frac{N-1}{2}\frac{1}{r^{((N-1)/2)+1}}\mathrm{Im}\{f(\Omega)e^{ikr(1-\cos\theta)}\}. \tag{12}$$

If we take the incident wave along the x_1-axis, (7), (11) and (12) supplemented by the definition

$$\sigma_{\mathrm{tot}} = \int |f(\Omega)|^2 \mathrm{d}\Omega \tag{13}$$

give

$$\sigma_{\mathrm{tot}} = -\lim_{r\to\infty} r^{(N-1)/2}\mathrm{Re}\left[-\left(\frac{2}{i}\right)^{(N-5)/2}\frac{\Gamma((N/2)-1)}{\sqrt{\pi}}\frac{1}{k^{(N-1)/2}}e^{ikr}\right.$$

$$\left.\times \sum_{L=0}^{\infty}(2L+N-2)(e^{2i\delta_L}-1)\left\{I_L^{(N)} - i\frac{N-1}{2}\frac{1}{kr}\tilde{I}_L^{(N)}\right\}\right] \tag{14a}$$

with

$$I_L^{(N)} = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}[1+\cos\theta_1]e^{-ikr\cos\theta_1}C_L^{((N/2)-1)}(\cos\theta_1)$$

$$\times \prod_{j=1}^{N-1}\sin^{N-j-1}\theta_j\mathrm{d}\theta_j. \tag{14b}$$

The integral $\tilde{I}_L^{(N)}$ is obtained from $I_L^{(N)}$ by replacing the quantity in the squared bracket by unity. The expression for $I_L^{(N)}$ is found as

$$I_L^{(N)} = \frac{(2\pi)^{N/2}\Gamma(L+N-2)}{i^L\Gamma(N-2)\Gamma(L+1)(kr)^{((N-2)/2)}}[J_{L+((N-2)/2)}(kr) + iJ'_{L+((N-2)/2)}(kr)]. \tag{15}$$

Here the prime denotes differentiation with respect to the argument. The derivation of this result is given in appendix A. Using the asymptotic form of the $J_\upsilon(z)$ and its derivative in (15) the resulting expression can be used in (14a) to recast σ_{tot} in the form

$$\sigma_{\text{tot}} = \frac{4\pi^{(N-1)/2)}}{\Gamma\left(\dfrac{N-1}{2}\right)}\frac{1}{k^{N-1}}\sum_{L=0}^{\infty}(2L+N-2)\frac{\Gamma(L+N-2)}{\Gamma(L+1)}\sin^2\delta_L. \tag{16}$$

In the asymptotic limit the contribution of $\tilde{I}_L^{(N)}$ to the scattering cross section disappears because of the factor $(kr)^{-1}$ in front of it. From (11) and (16), we get the OT,

$$\sigma_{\text{tot}} = 2\left(\frac{2\pi}{k}\right)^{(N-1)/2}\text{Im}\{i^{(N-3)/2}f(\Omega_{\mathbf{k}})\} \tag{17}$$

for the ND scattering. Clearly $f(\Omega_{\mathbf{k}})$ is the scattering amplitude in the direction $\mathbf{k}$ of the plane wave.

In three dimensional problems with *rotational-reflectional symmetry*, $f(\Omega_{\mathbf{k}})$ will be a function of θ_1 only and by our choice $\theta_1 = 0$. Therefore, for $N = 3$ we recover the well-known optical theorem [2]

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}\, f(0). \tag{18}$$

Similarly for 2D scattering we have [3]

$$\sigma_{\text{tot}} = 2\sqrt{\frac{\pi}{k}}\left[\text{Im}\, f(0) - \text{Re}\, f(0)\right]. \tag{19}$$

We have seen that for all spatial dimensions the optical theorem is a direct consequence of CPF and our derivation for the NDOT does not involve any approximation. However, from (15) it is clear that use of the asymptotic values for J_ν and J'_ν in (14a) through $I_L^{(N)}$ requires $\delta_L(k)$ to vanish for large L. This observation will play a crucial role in the discussion of OT for the AB effect. In wave language, the OT follows from the fact that the total cross section represents the removal of flux from the incident beam and such a removal can only occur as a result of destructive interference between the incident wave and the elastically scattered wave in the forward direction. This view point is followed to visualize the 3D optical theorem in terms of *shadow scattering* [12]. Specific examples are often used to make the situation more transparent. Considering the hard-sphere scattering [13] we have

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}[f_{\text{shadow}}(0)] \tag{20}$$

where

$$f_{\text{shadow}}(\theta) = \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta) \tag{21}$$

with R, the radius of the sphere. The equivalence of (18) and (20) for the present instance can be established by using the fact that $\mathrm{Im}[f_{\text{reflection}}(0)]$ averages to zero due to oscillating phase. Interestingly, by exploiting the orthogonality properties of Gegenbauer polynomials we can also arrive at a similar conclusion for the N-dimensional scattering.

3. OT in AB scattering

The AB effect is a typical quantum mechanical phenomenon which results from the scattering of electrons by a magnetic field confined in an infinite solenoid [6]. It thus constitutes the observable consequence of a hidden field **B**. Here the vector potential **A** plays a crucial role. For example, despite $\mathbf{B} = 0$, the space region outside the solenoid is characterized by a non-vanishing gauge potential together with concentrated singularity along the negative x-axis. The essential singularity in **A** along the line is called Dirac string which, from geometer's point of view, manifest itself in giving rise to a multi-connected space [8]. The AB effect exists for those representations in which the electron wave function is either (i) discontinuous in a simply connected space $\mathscr{R}^2$ or (ii) continuous over a non-simply connected space $(\mathscr{R}^2/\{X_1 \leqslant 0,\ X_2 = 0\})$. Also the long-range behaviour of the vector potential does not permit clear separation of the complete wave function into the incoming and scattered part. In the course of our study we shall see that these features of the AB effect pose problems to write an optical theorem for the process.

Consider the Schrödinger equation

$$\left[\left(\nabla - \frac{ie}{c\hbar}\mathbf{A}\right)^2 + k^2\right]\psi(\mathbf{r}) = 0 \tag{22}$$

for scattering of a charged particle by an impenetrable infinitely long thin cylindrical solenoid of radius R. The vector potential $\mathbf{A} = (0, \Phi/2\pi r)$ outside the solenoid containing the magnetic flux Φ lies on a plane perpendicular to the axis of the cylinder. As already noted, there can be two types of scattering solutions for (22). In the following we use them, one by one to look for an appropriate optical theorem for the AB scattering.

3.1 *Wave function in the simply-connected space*

This wave function has been obtained by making use of the two-potential formula of Gellman and Goldbarger [14]. Asymptotically, $\psi(\mathbf{r})$ behaves as

$$\psi \sim \psi_{\text{AB}} + \psi_C. \tag{23}$$

Here

$$\psi_{\text{AB}} \sim e^{i\alpha(\phi-\pi)+ikr\cos\phi} + \frac{i\sin(\pi\alpha)e^{i(\phi/2)}e^{ikr}}{\sqrt{1-2ikr\sin^2\phi/2}} \tag{24}$$

represents the asymptotic wave function for the non-shielded solenoid [15]. The effect of shielding is taken care of by the second term in (23) and we have

$$\psi_C \sim f_s(\phi)\frac{e^{ikr}}{\sqrt{r}} \tag{25}$$

with

$$f_s(\phi) = \sqrt{\frac{2}{\pi ik}}\sum_m \frac{J_{|m+\alpha|}(kR)}{H_{|m+\alpha|}(kR)} e^{i\pi(|m|-|m+\alpha|)} e^{im\phi}. \tag{26}$$

The marked difference in the structure of (24) from the standard scattering theoretic form shown in (10) may be attributed to the inextricable mixing between the radial and angular variables in the scattered part. However, if we restrict to $kr\sin^2\phi/2 \gg 1$ we can recast it in the form of (10) to get the scattering amplitude

$$f_{AB}(\phi) = \sqrt{\frac{1}{2\pi ik}}\sin(\pi\alpha)\frac{e^{i(\phi/2)}}{\sin(\phi/2)} \tag{27}$$

as obtained by Aharonov and Bohm [16]. It is crucial to observe that the approximation sought here precludes the value $\phi = 0$. This forbids to write an optical theorem for the AB effect by using the wave function in a simply connected space.

3.2 *Wave function in the non-simply connected space*

The wave function in this case is constructed by considering (22) in the cylindrical polar coordinates (r, ϕ) and assuming a general solution in the form

$$\psi(\mathbf{r}) \equiv \psi(r,\phi) = \sum_{m=-\infty}^{\infty} A_m f_m(r)\Phi_m(\phi). \tag{28}$$

This leads us to the differential equations

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(m+\alpha)^2}{r^2}\right] f_m(r) = 0 \tag{29}$$

and

$$\left[\frac{d^2}{d\phi^2} + m^2\right]\Phi_m(\phi) = 0 \tag{30}$$

for the radial and angular parts of (28). Using the Dirichlet and periodic boundary conditions, solutions of (29) and (30) can be obtained to read

$$f_m(r) = A_m[J_{|m+\alpha|}(kr)\,Y_{|m+\alpha|}(kR) - J_{|m+\alpha|}(kR)\,Y_{|m+\alpha|}(kr)] \tag{31}$$

and

$$\Phi_m(\phi) = e^{im\phi}, \quad m\in I. \tag{32}$$

In analogy with (10), we choose the asymptotic form of $\psi(r,\phi)$ as

$$\psi(r,\phi) \sim e^{ikr\cos\phi} + f(\phi)\frac{e^{ikr}}{\sqrt{r}}\cdot \tag{33}$$

From (28), (31) and (33) we get

$$f(\phi) = \frac{1}{\sqrt{2\pi k}} e^{-i(\pi/4)} \sum_{m=-\infty}^{\infty} [e^{2i(\Delta_m + \delta_m(\alpha))} - 1] e^{im\phi} \tag{34}$$

and

$$A_m = \frac{i^{|m|} e^{i(\Delta_m + \delta_m(\alpha))}}{|H^{(1)}_{|m+\alpha|}(kR)|}. \tag{35}$$

Here the quantities $\delta_m(\alpha)$ and Δ_m given by

$$\delta_m(\alpha) = (|m| - |m+\alpha|)\frac{\pi}{2} \tag{36}$$

$$e^{2i\Delta_m} = 1 - \frac{2J_{|m+\alpha|}(kR)}{H^{(1)}_{|m+\alpha|}(kR)} \tag{37}$$

stand for the phases induced by the vector potential **A** and the impenetrable cylinder, respectively.

Equations (7), (12), (13) and (34) can be combined to write

$$\sigma^{\mathrm{AB}}_{\mathrm{tot}} = -\lim_{r\to\infty} \sqrt{r}\frac{1}{\sqrt{2\pi k}} \mathrm{Re}\left[e^{-i(\pi/4)+ikr} \sum_{m=-\infty}^{\infty} (e^{2i(\Delta_m+\delta_m(\alpha))} - 1) I^{(2)}_m \right]. \tag{38}$$

The result for $I^{(2)}_m$ can be obtained from (14b) and we have

$$I^{(2)}_m = \frac{2\pi}{i^m}\left[J_m(kr) + iJ'_m(kr) \right]. \tag{39}$$

It appears that (34), (35) and (39) together with the asymptotic forms

$$J_m(kr) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4}\right) + o\left(\frac{1}{kr}\right) \tag{40a}$$

and

$$J'_m(kr) \sim \sqrt{\frac{2}{\pi kr}}\sin\left(kr - \frac{m\pi}{2} - \frac{\pi}{4}\right) + o\left(\frac{1}{kr}\right) \tag{40b}$$

lead to the optical theorem for AB scattering. Unfortunately, such a derivation involves a serious logical flaw. This can be seen as follows.

The asymptotic forms in (40) are valid only for $m \ll kr$. This inequality is violated for infinitely many of the terms in a Bessel function expansion. Since $\Delta_m + \delta_m(\alpha)$ takes a constant value as $|m| \to \infty$, use of (40) in (38) gives rise to a divergent $\sigma^{\mathrm{AB}}_{\mathrm{tot}}$. This does not happen for the ordinary scattering on short range interactions as discussed in most of the text books. With Henneberger [17] one may attempt to resolve the difficulty associated with the AB effect by controlling of the incident beam ψ_{inc} and study its implication for the optical theorem.

The partial wave expansion for $\psi_{\mathrm{inc}} = e^{ikr\cos\phi}$ is taken in the form

$$\psi_{\mathrm{inc}} = \sum_m i^{|m|} e^{-|m|\varepsilon} J_{|m|}(kr) e^{im\phi}, \tag{41}$$

where the small positive quantity ε satisfies $\varepsilon \gg 1/kd$ and d is the distance from the scatterer to the detector. In this case one obtains the scattering amplitude and cross section as

$$f_\varepsilon(\phi) = \frac{1}{\sqrt{2\pi k}} e^{-i(\pi/4)} \sum_{m=-\infty}^{\infty} e^{-|m|\varepsilon}[e^{2i(\Delta_m+\delta_m(\alpha))} - 1] e^{im\phi} \tag{42}$$

and

$$\sigma_{\rm tot}^{\varepsilon} = -\lim_{r\to\infty}\sqrt{r}\frac{1}{\sqrt{2\pi k}}\mathrm{Re}\left[e^{-i(\pi/4)+ikr}\sum_{m=-\infty}^{\infty}\frac{e^{-|m|\varepsilon}}{i^m}(e^{2i(\Delta_m+\delta_m(\alpha))}-1)\right.$$
$$\left.\times\{J_m(k\rho)+iJ'_m(k\rho)\}\right]. \quad (43)$$

The damping exponential $e^{-|m|\varepsilon}$ tends to remove the awkward divergence problem resulting from the use of (40) and thus, in principle, one could write an optical theorem for the process. But we can see that this claim will also be based on inadequate mathematical treatment of the problem.

For pure AB scattering ($\Delta_m = 0$), (37) and (42) can be combined to write [17]

$$f_\varepsilon(\phi) = \frac{1}{\sqrt{2\pi ik}}\left[e^{-i\pi\alpha} + 2\mathrm{Re}\sum_1^{\infty} e^{-i\pi\alpha}e^{im\phi}e^{-m\varepsilon} - \sum_{-\infty}^{\infty} e^{-im\phi}e^{-|m|\varepsilon}\right]. \quad (44)$$

Interestingly, (44) can be reduced to the standard AB scattering amplitude (27) provided one agrees to drop the third term which vanishes for angles deviating from the forward direction by an amount $\gg \varepsilon$. This is in agreement with the observation of Corinaldesi and Rafeli [18]. However, suppression of this term amounts to the violation of unitarity for the S matrix. Thus, it is not possible to make a statement of the OT by using (42) and yet keep the well known AB scattering amplitude unchanged. In this context another significant point to note is the following.

The two wave functions, one described in the simply-connected space and other, in the non-simply connected space, are locally gauge equivalent. Since the gauge transformation does not change the physics of the problem, the claimed validity of OT, if any will contradict the conclusion drawn by working with the local gauge equivalent multiple valued representation. A more instructive step to dispense with the hope for constructing OT in the present instance is now in order. For example, use of (42) in (13) and (19) leads to

$$\sigma_{\rm tot}^{\varepsilon} = \frac{4}{k}\sum_{m=-\infty}^{\infty} e^{-2|m|\varepsilon}\sin^2(\delta_m(\alpha)+\Delta_m) \quad (45)$$

and

$$\sigma_{\rm tot}^{\varepsilon} = \frac{4}{k}\sum_{m=-\infty}^{\infty} e^{-|m|\varepsilon}\sin^2(\delta_m(\alpha)+\Delta_m). \quad (46)$$

Since $\delta_m(\alpha)$ does not go to zero as $|m|\to\infty$, the results in (45) and (46) do not agree. This disagreement prohibits use of (13) in the derivation of OT from the CPF. Thus, it is not possible to provide a definitive statement for the OT even by working with single valued wave function in our unified approach.

4. Concluding remarks

The optical theorem is embedded in the partial-wave expressions for the scattering quantities. But it is widely believed that to appreciate the physical significance of the theorem, the OT should be derived directly from CPF. We have followed this view point to construct an expression for the OT for N-dimensional potential scattering without using any approximation like that of Boya and Murray [21]. Being mathematically rigorous, the present derivation exhibits the causal connection between the OT and CPF in a more general context. As special instances of our general expression, we have obtained the well-known results for three- and two-dimensional optical theorem. The 3D optical theorem is often physically visualized in terms of shadow scattering. The same is found true for the ND scattering.

We have attempted to provide a statement of the optical theorem for the Aharonov-Bohm scattering. Being related to the observable effects of a hidden field, it is very difficult to treat this process by the methods of formal scattering theory. Despite that, we worked with two types of wave functions in looking for an appropriate OT for the AB process. The scattering amplitude for the wave function in the simply-connected space is not defined in the forward direction and is thus unsuitable for deriving the optical theorem. This is reminiscent of the so-called Coulomb scattering. For the Coulomb problem, divergence in cross section and/or scattering amplitude arises from the long-range nature of the interaction. The speciality of the AB process is that there is no long-range force to cause divergence. However, the long-range vector potential can lead to divergent scattering cross section for an incident beam of infinite extent.

Our next choice of the wave function was that in the non-simply connected space. Here the vector potential is gauged everywhere except on a ray, viz., the negative real axis such that the present wave function is related to that in the simply connected space by unitary transformation. Here we have also chosen to work with an incident beam of finite extent. Even with all these modifications a rigorous derivation of the optical theorem was not possible, presumably because the reflection symmetry of the scattering interaction is broken by the presence of the noted branch-cut. We, therefore, conclude by noting that one requires a thorough modification of the scattering theory to deal with the AB process, but this seems to be far from clear.

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Appendix A

The integral $I_L^{(N)}$ in (14b) can be written as

$$I_L^{(N)} = A(I_L'^{(N)} + I_L''^{(N)}), \tag{A1}$$

where

$$I_L'^{(N)} = \int_0^\pi e^{-ikr\cos\theta_1} C_L^{((N/2)-1)}(\cos\theta_1)\sin^{N-2}\theta_1 \mathrm{d}\theta_1, \tag{A2}$$

$$I_L''^{(N)} = \int_0^\pi e^{-ikr\cos\theta_1} C_L^{((N/2)-1)}(\cos\theta_1)\sin^{N-2}\theta_1 \cos\theta_1 \mathrm{d}\theta_1 \tag{A3}$$

and

$$A = \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \sin^{N-3}\theta_2 \mathrm{d}\theta_2 \int_0^\pi \sin^{N-4}\theta_3 \mathrm{d}\theta_3 \cdots \int_0^\pi \sin\theta_{N-2} \mathrm{d}\theta_{N-2}. \tag{A4}$$

Using the formulae [19]

$$\int_0^\pi e^{iz\cos\theta}\sin^{2\nu}\theta\, C_n^{(\nu)}(\theta)\mathrm{d}\theta = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)\sqrt{\pi}2^\nu}{(-i)^n\Gamma(2\nu)\Gamma(n+1)z^\nu} J_{\nu+n}(z)$$

and

$$J_\nu(e^{i\pi}z) = e^{i\nu\pi}J_\nu(z)$$

in (A2) one obtains

$$I_L^{\prime(N)} = \frac{\Gamma\left(\frac{N-2}{2}+\frac{1}{2}\right)\Gamma(L+N-2)\sqrt{\pi}2^{(N-2)/2}}{i^L\Gamma(N-2)\Gamma(L+1)(kr)^{(N-2)/2}} J_{L+((N-2)/2)}(kr). \tag{A5}$$

Differentiation of (A2) under the sign of integration with respect to kr leads to

$$I_L^{\prime\prime(N)} = i\frac{\mathrm{d}I_L^{\prime(N)}}{\mathrm{d}(kr)}. \tag{A6}$$

The results in (A5) and (A6) give

$$I_L^{\prime\prime(N)} = \frac{\Gamma\left(\frac{N-2}{2}+\frac{1}{2}\right)\Gamma(L+N-2)\sqrt{\pi}2^{(N-2)/2}}{i^L\Gamma(N-2)\Gamma(L+1)(kr)^{(N-2)/2}} \times i\left[J'_{L+((N-2)/2)}(kr) - \left(\frac{N-2}{2}\right)\frac{J_{L+((N-2)/2)}(kr)}{kr}\right]. \tag{A7}$$

Furthermore [20],

$$\begin{aligned}\frac{1}{kr}J_{L+((N-2)/2)}(kr) &= \frac{1}{(2L+N-2)}[J_{L+(N/2)}(kr) + J_{L+(N/2)-2}(kr)]\\ &\sim \frac{1}{(2L+N-2)}\sqrt{\frac{2}{\pi kr}}\left[\cos\left\{kr-\left(L+\frac{N}{2}+\frac{1}{2}\right)\frac{\pi}{2}\right\}\right.\\ &\quad\left.+\cos\left\{kr-\left(L+\frac{N}{2}-\frac{3}{2}\right)\frac{\pi}{2}\right\}\right]\\ &= \frac{2}{(2L+N-2)}\sqrt{\frac{2}{\pi kr}}\cos\left\{kr-\left(L+\frac{N-1}{2}\right)\frac{\pi}{2}\right\}\cos\frac{\pi}{2}\\ &= 0.\end{aligned}$$

Thus, (A7) becomes

$$I_L^{\prime\prime(N)} = \frac{\Gamma\left(\frac{N-2}{2}+\frac{1}{2}\right)\Gamma(L+N-2)\sqrt{\pi}2^{(N-2)/2}}{i^L\Gamma(N-2)\Gamma(L+1)(kr)^{(N-2)/2}} iJ'_{L+((N-2)/2)}(kr). \tag{A8}$$

To evaluate the integrals in (A4), we use the formula

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

and find

$$A = \frac{2\pi^{((N-1)/2)}}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{A9}$$

From (A1), (A5), (A8) and (A9), one obtains the desired expression for $I_L^{(N)}$ given in (15).

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