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On the Dirac equation with anomalous magnetic moment term in a beam of electromagnetic wave

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Abstract. The object of the paper is to obtain the solution of the Dirac equation with the Pauli-term in an electromagnetic field depending on the single variable $(ct - \mathbf{nr})$ along the direction $\mathbf{n}$.

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1. Introduction

Recently, in a short paper by Salamin [1], an attempt was made to solve the Dirac equation with anomalous magnetic moment in the presence of a plane electromagnetic wave. Apart from the fact that the exact nature of the electromagnetic field is not stated clearly, the solution presented in the paper does not satisfy the equation which is proposed to be solved. The object of the present paper is to investigate the same problem for a general class of electromagnetic field considered in [1].

The field we consider is a function of a single variable

$$\theta = R^{\mu} x_{\mu} = Rx, \quad x^{\mu} \equiv (x^0 = ct, \mathbf{r}).$$

We use the notation $ab = a^{\mu} b_{\mu}$ for any two vectors a and b. The Lorentz potential for the external electromagnetic field is $A^{\mu} = A^{\mu}(\theta)$. For the wave field,

$$RR = R^{\mu} R_{\mu} = 0. \tag{1}$$

So that, without any loss of generality, with the choice of the gauge and the Lorentz condition,

$$A_0 = 0, \quad \frac{\partial A^{\mu}}{\partial x^{\mu}} = R\frac{dA}{d\theta} = 0. \tag{1'}$$

In general the electric and magnetic fields in these cases are always in the plane normal to the fixed direction $\mathbf{R}/R_0$, and the most general form of this class of field may be written as,

$$\mathbf{A} = \int_{-\infty}^{\infty} \{\mathbf{e}_1 a_1(R_0) f_1(\theta) + \mathbf{e}_2 a_2(R_0) f_2(\theta)\} dR_0 \tag{2}$$

where

$$\mathbf{e}_i \cdot \mathbf{R} = 0, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (i,j = 1,2). \tag{2'}$$

In the case of radiation, it is not necessarily monochromatic, R_0 is proportional to the frequency. The Dirac equation without the Pauli-term for some special class of fields belonging to this category was first solved by Volkov [2] and later by others [3–5] for the investigation of the scattering of electromagnetic radiation and other physical problems.

In the following sections we proceed to construct the solution. The problem is reduced to the solution of a second order ordinary differential equation. It may be emphasized that the problem, in general, cannot be reduced to quadrature without further specification of the nature of the functions $f_1(\theta)$ and $f_2(\theta)$. The last section is devoted to discussion of nature of the solution for various special cases.

2. The equation and the solution

The Dirac equation with the Pauli-term may be written as

$$\left\{i\gamma^\mu \frac{\partial}{\partial x^\mu} + \varepsilon\gamma^\mu A_\mu + M + i\frac{\lambda}{2}\gamma^\mu\gamma^\nu \frac{\partial A_\mu}{\partial x^\nu}\right\}\Psi = 0, \tag{3}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{3'}$$

$g^{\mu\nu} \equiv (1, -1, -1, -1)$ and $\varepsilon = e/\hbar c$, $M = mc/\hbar$, e and m are the charge and mass of the particle and λ is the parameter depending on the strength of the anomalous magnetic moment of the particle. Since $\partial/\partial v$, $\partial/\partial v_1$ and $\partial/\partial v_2$, where $v = R_0 x_0 + \mathbf{R}\cdot\mathbf{r}$ and $v_i = \mathbf{e}_i\cdot\mathbf{r}$ commute with the operator on Ψ (eq. (3)) they are constants. Hence, one can write the solution in the form,

$$\Psi(x_0, \mathbf{r}) = \mathrm{e}^{-i(p.x + S(\theta))}\Phi(\theta), \tag{4}$$

where p^μ is an arbitrary vector and $S(\theta)$ is a scalar function of θ. Substituting this in (3) one gets,

$$\{\gamma(p + \varepsilon A) + M\}\Phi + \left\{\frac{\mathrm{d}S}{\mathrm{d}\theta} + i\frac{\lambda}{2}\gamma\frac{\mathrm{d}A}{\mathrm{d}\theta} + i\frac{\mathrm{d}}{\mathrm{d}\theta}\right\}\gamma R\Phi = 0. \tag{5}$$

By virtue of eq. (1')

$$(\gamma R)^2 = 0. \tag{6}$$

Again fron eqs (1') and (2)

$$(RA) = 0 \text{ and } \gamma R\gamma A + \gamma A\gamma R = 0. \tag{7}$$

Hence multiplying eq. (5) by γR from left,

$$2(pR)\Phi = \{\gamma(p + \varepsilon A) - M\}\gamma R\Phi, \tag{8}$$

i.e.,

$$2(pR)\{\gamma(p + \varepsilon A) + M\}\Phi = \{(p + \varepsilon A)(p + \varepsilon A) - M^2\}\gamma R\Phi. \tag{8'}$$

We choose p and S such that

$$pp = M^2, \tag{9}$$

$$2(pR)\frac{\mathrm{d}S}{\mathrm{d}\theta} + 2\varepsilon pA + \varepsilon^2 AA = 0 \tag{10}$$

Finally from (5) and (8),

$$\left(\frac{d}{d\theta} + \frac{\lambda}{2}\gamma\frac{dA}{d\theta}\right)(\gamma R)\Phi = 0. \tag{11}$$

In the absence of the Pauli-term i.e., for $\lambda = 0$, Φ is constant and the problem reduces to that of the usual Dirac equation which has been already mentioned. From (9) $\hbar p$ is the free particle momentum. It is important to note that eq. (10), hence the expression for S is independent of λ.

3. Special cases

Although eq. (11) is linear, the determination of Φ cannot be, in general, reduced to quadrature, because it contains at least two mutually anti-commuting matrices. We consider special cases:

a) One of the functions f_1 and f_2 in (2) is zero, or equivalently the ratio f_1/f_2 is a constant, i.e.,

$$\mathbf{A} = \mathbf{e} f(\theta). \tag{12}$$

In this case the electric and magnetic fields **E** and **H** are given by,

$$\mathbf{E} = \mathbf{e}\cdot R_0 \frac{df}{d\theta}$$

and

$$\mathbf{H} = \mathbf{e} \times \mathbf{R}\frac{df}{d\theta} \tag{13}$$

(*i*) If f is a linear function of θ, the field consists of constant mutually perpendicular electric and magnetic field of same magnitude.

and

(*ii*) If f is a periodic function, e.g., $f = F\cos(\theta + \delta)$ the field is due to plane polarized radiation, not necessarily monochromatic, along a fixed direction $\mathbf{R}/R_0$. Both these types of fields are important for applications in physical problems. From (11), one gets directly

$$(\gamma R)\Phi = \exp - \frac{\lambda}{2}(\gamma A)\Phi_0 \tag{14}$$

where Φ_0 is a constant spinor. This case has been discussed by Chakrabarti [6].

b) In the general case eq. (11) cannot be directly integrated. But the solution can be expressed in terms of solutions of second order linear differential equations. We may write

$$\mathbf{A} = F(\mathbf{e}_1 \cos\mu + \mathbf{e}_2 \sin\mu)$$

where

$$F^2 = f_1^2 + f_2^2 \quad \text{and} \quad \tan\mu = f_2/f_1 \tag{15}$$

Let $u_\pm$ be eigenspinors of $i(\mathbf{e}_1\cdot\gamma)(\mathbf{e}_2\cdot\gamma)$ with eigenvalues ± 1, i.e.,

$$i(\mathbf{e}_1\cdot\gamma)(\mathbf{e}_2\cdot\gamma)u_\pm = \pm u_\pm$$

$$(\mathbf{e}_2\cdot\gamma)u_\pm = \mp i(\mathbf{e}_1\cdot\gamma)u_\pm,$$

and (16)

$$i(\mathbf{e}_1\cdot\gamma)u_\pm \equiv u_\mp,$$

$$\tilde{u}_\pm\cdot u_\mp = 0, \quad \tilde{u}_\pm\cdot u_\pm = 1.$$

Let

$$\gamma\cdot R\Phi = g_+(\theta)u_+ + g_-(\theta)u_-. \tag{17}$$

Substituting these in eq. (11), one is lead to two simultaneous first order linear equations,

$$\begin{aligned}\frac{\mathrm{d}g_+}{\mathrm{d}\rho} &= i\lambda \mathrm{e}^{-i\mu} g_- \\ \frac{\mathrm{d}g_-}{\mathrm{d}\rho} &= -i\lambda \mathrm{e}^{i\mu} g_+\end{aligned} \tag{18}$$

where $d\rho = Fd\theta$. It is clear that g_+ and g_- are mutually complex conjugate of each other. It follows from above that

$$\begin{aligned}\frac{\mathrm{d}^2 g}{\mathrm{d}\rho^2} + i\frac{\mathrm{d}\mu}{\mathrm{d}\rho}\frac{\mathrm{d}g_+}{\mathrm{d}\rho} - \lambda^2 g_+ &= 0 \\ \frac{\mathrm{d}^2 g_-}{\mathrm{d}\rho^2} - i\frac{\mathrm{d}\mu}{\mathrm{d}\rho}\frac{\mathrm{d}g_-}{\mathrm{d}\rho} - \lambda^2 g_- &= 0.\end{aligned} \tag{19}$$

Thus the problem ultimately reduces to solving the second order linear differential equation. It must be emphasized that one has to restrict to only those solutions of (19), which satisfies the first order (18). One cannot proceed further without knowing explicitly the nature of functions f_1 and f_2. We can consider some of the particular cases of physical importance; namely the case of circular and elliptic polarized radiation field. For monochromatic radiation

$$\mathbf{A} = \mathbf{e}_1 a_1 \cos(\theta+\delta_1) + \mathbf{e}_2 a_2 \sin(\theta+\delta_2).$$

Introducing suitable rotation in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ one can write

$$\mathbf{A} = \mathbf{e}'_1 a'_1 \cos\theta + \mathbf{e}'_2 a'_2 \sin\theta,$$

where $\mathbf{e}'_1, \mathbf{e}'_1$ are the new mutually orthogonal unit vectors in the same plane and a'_1, and a'_2 depend on a_1, a_2, δ_1 and δ_2. The radiation is, in general, elliptically polarized, and in particular for $a'_1 = a'_2 = a$ (say) the polarization is circular and the expressions for g_+ and g_- take the simple form,

$$\begin{aligned} g_+ &= \mathrm{e}^{-ia(\theta/2)}\{B_+ \mathrm{e}^{i(1-\lambda^2/2)^{1/2} a_\theta} + B_- \mathrm{e}^{-i(1-\lambda^2/2)a_\theta}\} \\ &= g^*_- \end{aligned} \tag{20}$$

where B_+ and B_- depend on λ and a. They are to be so choosen that the linear simultaneous equation for g_+ and g_- (eqs (18)) are satisfied.

Finally, once the solution of eq. (11) is obtained, (for the special cases considered they are given by eqs (14) and (20)), one can finally obtain Φ from (8). Thus

$$\Phi(p,x) = \frac{1}{2(pR)}\{\gamma(p+\varepsilon A) - M\}\Phi_0', \tag{21}$$

where Φ_0' stands for the solution eq. (11), so that

$$\Psi(p,x) = \mathrm{e}^{-i(p\cdot x + S(\theta))}\Phi(p,x). \tag{22}$$

It should be noted that each of the spinors $u_\pm$ introduced by (18) is two dimensional and $u_\pm$ arbitrary unit spinor in the respective subspaces. This degeneracy is removed by the operator (γA) which also changes $u_\pm$ to $u_\mp$. Again the part $(\gamma \mathbf{p}_1)$ of (γp) where $\mathbf{p}_1$ is a vector in the plane perpendicular to $\mathbf{R}$, changes $u_\pm$ to $u_\mp$. The subspace is invariant with respect to the remaining part of γp, namely $\gamma_0^0 p_0 - \gamma\cdot\mathbf{p}_\parallel$ ($\mathbf{p}_\parallel$ along $\mathbf{R}$), but the degeneracy may be removed.

4. Remarks

Since, eq. (2) is linear the most general solution is obtained by suitable superposition of the solutions given by (22). The most general solution is obtained in the form

$$\Psi(x) = \int_{-\infty}^{\infty} K(\mathbf{p})\Psi(\mathbf{p},x)\mathrm{d}\mathbf{p}. \tag{23}$$

The function $K(\mathbf{p})$ is to be obtained by the given initial and/or boundary condition. There is no need of appeal for the field to vanish at infinity. Again, we want to emphasize that the radiation field need not be monochromatic but it should be unidirectional. One can also similarly investigate the class of fields for which R is not a null vector, $R, R \neq 0$. This encompases a wider class of fields which we hope to consider in future.

It can be easily checked in the absence of the anomalous magnetic moment, i.e., $\lambda = 0$, the solution given by eq. (22) reduced to known solution [3–5]. On the other hand for particles with no charge but anomalous magnetic moment, i.e., $e = 0$ but $\lambda \neq 0$, which may be of importance in astrophysical problem is obtained from (21) and (22) putting $\varepsilon = 0$ and hence $S = 0$.

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