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Polarized light

G RAMACHANDRAN, A R USHA DEVI and
N S SAANDEEP KANNADAVARDHANA
Department of Studies in Physics, University of Mysore, Manasagangotri, Mysore 570006, India

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Abstract. Following the recent work of Chandler *et al* on quasi probability distributions for spin-1/2 particles, we show that polarized light can be interpreted in terms of trivariate probability distributions in two different ways by choosing the variates to correspond to (i) the co-ordinates on the Poincare sphere, (ii) the components of the spin operator of the photon. In either case, it is shown that the Margenau–Hill procedure leads to probability mass functions while the Wigner–Weyl approach leads to probability density functions and the well-known Stokes parameters are also realised as appropriate averages with respect to these distribution functions.

Keywords. Density matrix; polarization; Stokes parameters; quasi-probability distributions.

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1. Introduction

The subject of polarization of light [1] is an ancient one going back, according to Born and Wolf [2], to Huygens [3] of the seventeenth century. The state of polarization was characterized by Stokes [4] in terms of a set of four parameters s_0, s_1, s_2, s_3, named after him. Poincare [5] developed a geometrical representation for all states of polarization in terms of points on a sphere known by his name. It is well-known [6] that photons, the quanta associated with light, are spin-1 particles which are massless and hence deprived of the $|10\rangle$ or longitudinal state. The review articles by Fano [7] and McMaster [8] show how a partially polarized state of light can be represented in terms of a 2×2 hermitian and positive semidefinite matrix ρ employing the concept of the density matrix [9]. What is the composition in a partially polarized beam of light? If ρ has eigen values λ_1 and λ_2 with corresponding eigen states $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ respectively, one can say that the beam consists of $N\lambda_1$ photons with polarization $|\varepsilon_1\rangle$ and $N\lambda_2$ photons with polarization $|\varepsilon_2\rangle$, where N denotes the total number of photons which is proportional to the Stokes parameter $s_0 \geqslant (s_1^2 + s_2^2 + s_3^2)^{1/2}$ representing the total intensity I of the beam. The equality corresponds to completely polarized or pure states characterized by one of the eigenvalues being zero. The recent work of Chandler *et al* [10] on spin-1/2 particles shows that one can interpret the contents of a 2×2 density matrix for spin-1/2 particles in probabilistic ways based on the correspondence rules of Margenau and Hill [11] on one hand and Weyl [12] and Wigner [13] on the other. More recently Usha Devi *et al* [14] have adduced arguments following Margenau and Hill to show that an aligned spin-1 system can be viewed in terms of bivariate probability distribution by identifying the associated characteristic function. It is interesting to note that a beam of photons is akin to an aligned spin-1 system when the

Stokes parameter s_3 is zero. It is interesting to investigate whether a joint probability distribution interpretation can be given to an arbitrarily polarized beam of light. In this paper, we extend the Margenau–Hill procedure to provide a probabilistic interpretation for partially polarized light, in terms of a trivariate probability distribution even in the general case when $s_3 \neq 0$. We find, in fact, that there are two convenient ways in which this extension can be carried out viz., using the operators representing

(i) the co-ordinates on the Poincare sphere
(ii) the components of the spin of the photon

as the variates. This leads to a probability mass function where the variates assume discrete values. On the other hand a probability density function can be obtained using the Wigner-Weyl correspondence rule where once again the variates may be chosen in two alternative ways (i) and (ii).

2. States of polarization and the density matrix for photons

Our notations are as follows: Using a right handed co-ordinate system where the z-axis is chosen parallel to the momentum vector $\mathbf{k}$ of the photon, all possible states of polarization can be represented by [15]

$$|\varepsilon_{\alpha\beta}\rangle = (\cos\beta\cos\alpha - i\sin\beta\sin\alpha)|\varepsilon_x\rangle + (\cos\beta\sin\alpha + i\sin\beta\cos\alpha)|\varepsilon_y\rangle \quad (1)$$

where $|\varepsilon_x\rangle$, $|\varepsilon_y\rangle$ denote linear states of polarization along x, y axes respectively. Here $0 \leqslant \alpha < \pi$ and $-\pi/4 \leqslant \beta \leqslant \pi/4$ are angles such that $\theta = \pi/2 - 2\beta$ and $\phi = 2\alpha$ designate points on the Poincare sphere where the north pole corresponds to right circular polarization and the south pole corresponds to left circular polarization. Clearly, $|\varepsilon_{\alpha\beta}\rangle$ and $|\varepsilon_{\alpha+\pi/2-\beta}\rangle$ are orthogonal states in a two dimensional complex vector space and correspond geometrically to diametrically opposite points on the Poincare sphere. We choose the circular polarisation states

$$\begin{aligned} |\varepsilon_L\rangle &= |\varepsilon_{0\,-\pi/4}\rangle = (|\varepsilon_x\rangle - i|\varepsilon_y\rangle)/\sqrt{2} \\ |\varepsilon_R\rangle &= |\varepsilon_{0\,\pi/4}\rangle = (|\varepsilon_x\rangle + i|\varepsilon_y\rangle)/\sqrt{2} \end{aligned} \quad (2)$$

as our basis with respect to which the density matrix ρ assumes the form

$$\rho = \frac{\mathrm{Tr}\rho}{2}[1 + \boldsymbol{\sigma}\cdot\mathbf{P}] \quad (3)$$

where $\sigma_1, \sigma_2, \sigma_3$ denote the standard Pauli matrices, $I = \mathrm{Tr}\,\rho = s_0$ denotes the intensity of the light beam. Moreover,

$$\begin{aligned} s_0 P_1 &= s_1 = \mathrm{Tr}(\rho\sigma_1) = I_x - I_y \\ s_0 P_2 &= s_2 = \mathrm{Tr}(\rho\sigma_2) = I_{x'} - I_{y'} \\ s_0 P_3 &= s_3 = \mathrm{Tr}(\rho\sigma_3) = I_L - I_R \end{aligned} \quad (4)$$

where x', y' denote linearly polarized states with $\beta = 0$ and $\alpha = \pi/4$ and $3\pi/4$ respectively. Note that s_1, s_2, s_3 are expectation values of the operators $\sigma_1, \sigma_2, \sigma_3$ respectively and hence we could identify $\sigma_1, \sigma_2, \sigma_3$ as operators representing the co-ordinates of a point on the Poincare sphere. On the other hand, considering photon as a spin-1 particle, the density matrix can be expressed in terms of the Fano statistical tensors [7] t^k_q through

$$\rho = \frac{\mathrm{Tr}\rho}{3} \sum_{k=0}^{2} \sum_{q=-k}^{k} (-1)^q t^k_{-q} \tau^k_q(\mathbf{J}) \tag{5}$$

where $\tau^k_q(\mathbf{J})$ are spherical tensor operators constructed out of the spin operators $\mathbf{J}$ of the photon and are normalized such that

$$\langle 1m' | \tau^k_q(\mathbf{J}) | 1m \rangle = c(1k1; mqm')\sqrt{2k+1} \tag{6}$$

where $c(1k1; mqm')$ are Clebsch–Gordan coefficients. An explicit 3×3 matrix form for ρ is given by (1) of [16]. Observing that the basis states $|1m\rangle$ are such that $|11\rangle = -|\varepsilon_R\rangle$ and $|1-1\rangle = |\varepsilon_L\rangle$ leads to the identification,

$$\begin{aligned} t^0_0 &= 1 \\ t^1_0 &= -\left(\frac{3}{2}\right)^{1/2} \frac{s_3}{s_0} \\ t^2_0 &= \frac{1}{\sqrt{2}} \\ \mathrm{Re}(t^2_2) &= -\frac{\sqrt{3}}{2}\frac{s_1}{s_0} \\ \mathrm{Im}(t^2_2) &= \frac{\sqrt{3}}{2}\frac{s_2}{s_0} \\ t^2_{\pm 1} &= t^1_{\pm 1} = 0. \end{aligned} \tag{7}$$

Clearly the system is aligned [17] if $s_3 = 0$ and an unitary transformation

$$U = \begin{pmatrix} e^{-i\theta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{+i\theta/2} \end{pmatrix} \tag{8}$$

with $\tan\theta = s_2/s_1$ takes the photon density matrix into its principal axes of alignment frame (PAAF) [17],

$$\rho_{\mathrm{PAAF}} = \frac{1}{2} \begin{pmatrix} s_0 & 0 & (s_1^2 + s_2^2)^{1/2} \\ 0 & 0 & 0 \\ (s_1^2 + s_2^2)^{1/2} & 0 & s_0 \end{pmatrix}. \tag{9}$$

The conclusions drawn by Usha Devi *et al* [14] apply in toto here, together with the identification,

$$\begin{aligned} \Pi_x &= \frac{1}{2}[s_0 - (s_1^2 + s_2^2)^{1/2}] \\ \Pi_y &= \frac{1}{2}[s_0 + (s_1^2 + s_2^2)^{1/2}] \\ \Pi_z &= 0 \end{aligned} \tag{10}$$

and we can explain the polarization in terms of the bivariate joint probabilities [14] with any two of the spin components of the photon as variates. To describe the more general situation i.e., when $s_3 \neq 0$ we need a trivariate probability distribution where we have to treat all the three components of photon spin as variates.

3. Margenau–Hill probabilities for photons

The Margenau–Hill trivariate characteristic function for three non-commuting operators $\hat{X}_1, \hat{X}_2, \hat{X}_3$ is defined by,

$$\begin{aligned}\phi_{\mathrm{MH}}(I_1, I_2, I_3) = \frac{1}{3!}\mathrm{Tr}[\rho\{&\exp(i\hat{X}_1 I_1)\exp(i\hat{X}_2 I_2)\exp(i\hat{X}_3 I_3)\\ &+ \exp(i\hat{X}_1 I_1)\exp(i\hat{X}_3 I_3)\exp(i\hat{X}_2 I_2)\\ &+ \exp(i\hat{X}_2 I_2)\exp(i\hat{X}_3 I_3)\exp(i\hat{X}_1 I_1)\\ &+ \exp(i\hat{X}_2 I_2)\exp(i\hat{X}_1 I_1)\exp(i\hat{X}_3 I_3)\\ &+ \exp(i\hat{X}_3 I_3)\exp(i\hat{X}_1 I_1)\exp(i\hat{X}_2 I_2)\\ &+ \exp(i\hat{X}_3 I_3)\exp(i\hat{X}_2 I_2)\exp(i\hat{X}_1 I_1)\}]. \end{aligned} \tag{11}$$

Choosing $\hat{X}_1, \hat{X}_2, \hat{X}_3$ to be $\sigma_1, \sigma_2, \sigma_3$ and ρ as given in (3), we can simplify (11) to obtain

$$\begin{aligned}\phi^{(\sigma)}_{\mathrm{MH}}(I_1, I_2, I_3) &= s_0 \cos I_1 \cos I_2 \cos I_3 + i s_1 \sin I_1 \cos I_2 \cos I_3\\ &\quad + i s_2 \cos I_1 \sin I_2 \cos I_3 + i s_3 \cos I_1 \cos I_2 \sin I_3. \end{aligned} \tag{12}$$

Simplifying (12) further, $\phi^{(\sigma)}_{\mathrm{MH}}(I_1, I_2, I_3)$ could be cast into the form

$$\phi^{(\sigma)}_{\mathrm{MH}}(I_1, I_2, I_3) = \sum_{m_1, m_2, m_3 = -1, +1} f^{(\sigma)}_{\mathrm{MH}}(m_1, m_2, m_3)\exp(i(m_1 I_1 + m_2 I_2 + m_3 I_3)) \tag{13}$$

where $f^{(\sigma)}_{\mathrm{MH}}(m_1, m_2, m_3)$ could be identified as the Margenau–Hill trivariate probability mass function for photons and is given by

$$f^{(\sigma)}_{\mathrm{MH}}(m_1, m_2, m_3) = \frac{1}{8}[s_0 + m_1 s_1 + m_2 s_2 + m_3 s_3]. \tag{14}$$

m_1, m_2, m_3 are the classical random variables representing the operators $\sigma_1, \sigma_2, \sigma_3$ and the Stokes parameters could be identified as first moments of the distribution $f^{(\sigma)}_{\mathrm{MH}}(m_1, m_2, m_3)$ i.e.,

$$s_k = \mathrm{Tr}(\rho\sigma_k) = \sum_{m_1, m_2, m_3 = -1, +1} m_k f^{(\sigma)}_{\mathrm{MH}}(m_1, m_2, m_3), \quad k = 1, 2, 3. \tag{15}$$

On the other hand $\hat{X}_1, \hat{X}_2, \hat{X}_3$ in (11) could be chosen as J_1, J_2, J_3, the photon spin components and the density matrix ρ is then given by (5). To simplify the characteristic function $\phi^{(J)}_{\mathrm{MH}}(I_1, I_2, I_3)$ we follow the approach outlined in [14] where we identify that the unitary matrices

$$U_1 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad U_2 = \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & i\sqrt{2} & -1 \end{pmatrix} \tag{16}$$

diagonalise J_1 and J_2 respectively. We then replace

$$\exp(iJ_1 I_1) = U_1^\dagger \exp(iJ_1^d I_1) U_1, \exp(iJ_2 I_2) = U_2^\dagger \exp(iJ_2^d I_2) U_2 \tag{17}$$

in (11) where J_1^d and J_2^d are diagonal matrices with diagonal elements 1, 0, − 1. We can now simplify the characteristic function further to yield

$$\phi_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) = \sum_{m_1, m_2, m_3 = -1, 0, +1} f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) \times \exp(i(m_1 I_1 + m_2 I_2 + m_3 I_3)) \tag{18}$$

where

$$f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) = \frac{1}{3}\mathrm{Re}[(\rho U_1^\dagger)_{m_3 m_1}(U_1 U_2^\dagger)_{m_1 m_2}(U_2)_{m_2 m_3} + (U_2 \rho)_{m_2 m_3}(U_1^\dagger)_{m_3 m_1}(U_1 U_2^\dagger)_{m_1 m_2} + (U_2 \rho U_1^\dagger)_{m_2 m_1}(U_2^\dagger)_{m_3 m_2}(U_1)_{m_1 m_3}]. \tag{19}$$

Explicitly it may be seen that $f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3)$ assumes the simple form

$$f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) = \frac{1}{48}\{2m_1 m_2 s_2 - m_3 s_3\} \tag{20}$$

if $m_1, m_2, m_3 = \pm 1$;

$$f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) = \frac{1}{24}\{m_1 m_2 s_2 + 3m_3^2[s_0 - (m_1^2 - m_2^2)s_1] - 2m_3 s_3\} \tag{21}$$

if one of the variables m_1, m_2, m_3 assumes the value 0 and

$$f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) = -\frac{1}{12} m_3 s_3 \tag{22}$$

if more than one of m_1, m_2, m_3 take the value 0. Consequently the Stokes parameters could now be realised as the classical averages,

$$s_1 = \mathrm{Tr}[\rho(J_2^2 - J_1^2)] = \sum_{m_1, m_2, m_3 = -1}^{1} f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3)(m_2^2 - m_1^2)$$

$$s_2 = \mathrm{Tr}[\rho(J_1 J_2 + J_2 J_1)] = \sum_{m_1, m_2, m_3 = -1}^{1} f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3)(2m_1 m_2)$$

$$s_3 = -\mathrm{Tr}[\rho J_3] = -\sum_{m_1, m_2, m_3 = -1}^{1} f_{\mathrm{MH}}^{(\mathrm{J})}(m_1, m_2, m_3) m_3 \tag{23}$$

4. Wigner–Weyl probabilities for photons

The Wigner–Weyl characteristic function is defined by

$$\phi_{\mathrm{WW}}(I_1, I_2, I_3) = \mathrm{Tr}[\rho \exp(i(\hat{X}_1 I_1 + \hat{X}_2 I_2 + \hat{X}_3 I_3))]. \tag{24}$$

Substituting $\hat{X}_1 = \sigma_1, \hat{X}_2 = \sigma_2, \hat{X}_3 = \sigma_3$ and ρ given by (3) in (24), we get

$$\phi_{\mathrm{WW}}^{(\sigma)}(I_1, I_2, I_3) = s_0\left[\cos I + \mathbf{P}\cdot\mathbf{I}\frac{\sin I}{I}\right] \tag{25}$$

where $I = (I_1^2 + I_2^2 + I_3^2)^{1/2}$.

Fourier transform of the characteristic function $\phi^{(\sigma)}_{WW}(I_1, I_2, I_3)$ yields the Wigner–Weyl trivariate probability density function $f^{(\sigma)}_{WW}(x_1, x_2, x_3)$:

$$f^{(\sigma)}_{WW}(x_1, x_2, x_3) = (2\pi)^{-3} \iiint d^3 I \phi^{(\sigma)}_{WW}(I_1, I_2, I_3) \exp(-i(x_1 I_1 + x_2 I_2 + x_3 I_3))$$
$$= (2\pi)^{-3} s_0 \int_0^\infty \int_0^\pi \int_0^{2\pi} I^2 \mathrm{d}I \sin\theta_I \mathrm{d}\theta_I \mathrm{d}\phi_I \left[\cos I + i(\mathbf{P}\cdot\mathbf{I})\frac{\sin I}{I}\right]$$
$$\times \exp(-i(x_1 I_1 + x_2 I_2 + x_3 I_3)). \quad (26)$$

Carrying out the integration over the solid angle $\mathrm{d}\Omega_I = \sin\theta_I \mathrm{d}\theta_I \mathrm{d}\phi_I$, we obtain

$$f^{(\sigma)}_{WW}(x_1, x_2, x_3) = (2\pi^2 x)^{-1} \left[\int_0^\infty \mathrm{d}I\, I \cos I \sin Ix - \mathbf{P}\cdot\mathbf{\nabla}_x \int_0^\infty \mathrm{d}I \sin I \sin Ix\right] \quad (27)$$

where $\mathbf{\nabla}_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, $x = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We can simplify (27) further to get

$$f^{(\sigma)}_{WW}(x_1, x_2, x_3) = s_0[F_0(x) + F_1(x)\mathbf{P}\cdot\mathbf{x}] \quad (28)$$

where

$$F_0(x) = -\frac{1}{4\pi x}\frac{\mathrm{d}}{\mathrm{d}x}[\delta(1+x) + \delta(1-x)]$$
$$F_1(x) = -\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\delta(1+x) - \delta(1-x)}{x}\right]. \quad (29)$$

Now the quantum mechanical expectation values could be obtained in terms of $f^{(\sigma)}_{WW}(x_1, x_2, x_3)$ as,

$$s_k = \mathrm{Tr}(\rho\sigma_k) = \iiint \mathrm{d}^3 x f^{(\sigma)}_{WW}(x_1, x_2, x_3) x_k, \quad k = 1, 2, 3. \quad (30)$$

Alternatively we can replace $\hat{X}_1, \hat{X}_2, \hat{X}_3$ in (20) by J_1, J_2, J_3 in which case

$$\phi^{(J)}_{WW}(I_1, I_2, I_3) = \mathrm{Tr}(\rho \exp(i\mathbf{J}\cdot\mathbf{I})). \quad (31)$$

We now express

$$\exp(i\mathbf{J}\cdot\mathbf{I}) = R^\dagger(\phi_I, \theta_I, 0)\exp(iJ_3 I)R(\phi_I, \theta_I, 0) \quad (32)$$

where $R(\alpha, \beta, \gamma)$ denote rotations through Euler angles α, β, γ. Making use of (5) and (6) and the transformation properties of the spherical tensor operators $\tau^k_q(\mathbf{J})$ under rotations viz.,

$$R(\alpha, \beta, \gamma)\tau^k_q(\mathbf{J})R^\dagger(\alpha, \beta, \gamma) = \sum_{q'=-k}^{k} \tau^k_{q'}(\mathbf{J}) D^k_{q'q}(\alpha, \beta, \gamma) \quad (33)$$

where D^k denotes the standard $(2k+1)$ dimensional irreducible representation of rotations [18], we obtain

$$\phi^{(J)}_{WW}(I_1,I_2,I_3)=\left[s_0-\frac{3}{2}\frac{s_3I_3}{I}\sum_{m=-1,0,1}m\exp(imI)\right.$$
$$+\frac{1}{4I^2}\{s_0(3I_3^2-I^2)+3s_1(I_2^2-I_1^2)-6s_2I_1I_2\}$$
$$\left.\times\sum_{m=-1,0,1}(3m^2-2)\exp(imI)\right]. \tag{34}$$

Fourier transform of $\phi^{(J)}_{WW}(I_1,I_2,I_3)$ could now be carried out by generalising the techniques used in (26) to (29) to yield the Wigner–Weyl probability density function,

$$f^{(J)}_{WW}(x_1,x_2,x_3)=\left[s_0\mathscr{F}_0(x)-\mathscr{F}_1(x)s_3x_3+\mathscr{F}_2(x)\right.$$
$$\left.\times\left\{\frac{s_0}{3}(3x_3^2-x^2)+s_1(x_2^2-x_1^2)+2s_2x_1x_2\right\}\right] \tag{35}$$

where

$$\mathscr{F}_0(x)=-\frac{1}{6\pi x}\sum_m\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-m)$$
$$\mathscr{F}_1(x)=\frac{1}{4\pi x}\sum_m m\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\delta(x-m)}{x}\right]$$
$$\mathscr{F}_2(x)=\frac{1}{24\pi}\sum_m(3m^2-1)\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varepsilon(m-x)}{x}\right)\right\}\right] \tag{36}$$

Here $\varepsilon(m-x)$ denotes the step function

$$\varepsilon(a)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iat}}{t}\mathrm{d}t=\begin{cases}1 & \text{if } a>0\\ 0 & \text{if } a<0\end{cases} \tag{37}$$

Making use of the Weyl correspondence rule

$$x_1^l x_2^m x_3^n \to S\{J_1^l J_2^m J_3^n\} \tag{38}$$

where S stands for the symmetrizer, the Stokes parameters could be realised as the classical averages,

$$s_1=\mathrm{Tr}[\rho(J_2^2-J_1^2)]=\iiint f^{(J)}_{WW}(x_1,x_2,x_3)(x_2^2-x_1^2)\mathrm{d}^3x$$
$$s_2=\mathrm{Tr}[\rho(J_1J_2+J_2J_1)]=\iiint f^{(J)}_{WW}(x_1,x_2,x_3)(2x_1x_2)\mathrm{d}^3x$$
$$s_3=-\mathrm{Tr}[\rho J_3]=-\iiint f^{(J)}_{WW}(x_1,x_2,x_3)x_3\mathrm{d}^3x \tag{39}$$

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